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Investigation of Spectrum for a Sturm–Liouville problem with Two-Point Nonlocal Boundary Conditions

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Contents

Glossary of Notation

$\square$	–	end of proof
$:=$	–	attribution, definition marker
$=$	–	equality marker
$\equiv$	–	identity mark
$x \in X$	–	x is an element of X, x belongs to a set X
$X \cap Y$	–	is the intersection of sets X and Y
$X \cup Y$	–	is the union of sets X and Y
$\varnothing$	–	empty set
$X \times Y$	–	Cartesian products of sets X and Y
$\mathbb{N}$	–	$\{1, 2, ...\}$-the set of positive integer numbers
$\mathbb{N}_0$	–	$\{0, 1, 2, ...\}$-the set of nonnegative integer numbers
$\mathbb{N}_e$	–	$\{2, 4, 6, ...\}$-the set of an even positive integer numbers
$\mathbb{N}_o$	–	$\{1, 3, 5, ...\}$-the set of an odd positive integer numbers
$\mathbb{N}_k$	–	$k\mathbb{N} := \{n \in \mathbb{N} : n = km, m \in \mathbb{N}\}$, $k \in \mathbb{N}$
$\gcd(m; n)$	–	the greatest common divisor of two m, $n \in \mathbb{N}$
$\mathbb{Z}$	–	the set of integer numbers
$\mathbb{Q}$	–	the set of rational numbers
$\mathbb{I}$	–	the set of irrational numbers
$\mathbb{R}$	–	the set of real numbers
$\overline{\mathbb{R}}$	–	extended set of real numbers $\mathbb{R} \cup \{\infty\} = \mathbb{R}P^1$
$\mathbb{R}P^1$	–	projective line
$\mathbb{R}_-$	–	the set of negative real numbers
$\mathbb{R}_+$	–	the set of positive real numbers
$\mathbb{R}_{0+}$	–	the set of not negative real numbers
$\mathbb{R}_{0-}$	–	the set of not positive real numbers
$\mathbb{R}^2$	–	$\mathbb{R} \times \mathbb{R}$ the plame of real numbers
$\mathbb{R}^2_+$	–	$\{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, y \geqslant 0\}$
ι	–	imaginary unit $\sqrt{-1}$
$\mathbb{C}$	–	the set of complex numbers
$\overline{\mathbb{C}}$	–	extended set of complex numbers $\mathbb{C} \cup \{\infty\}$
BC	–	Boundary Condition
NBC	–	Nonlocal Boundary Condition
BVP	–	Boundary Value Problem
NC	–	Nonlocal Condition
FDS	–	Finite Difference Scheme
SLP	–	Sturm–Liouville Problem
dSLP	–	discrete Sturm–Liouville Problem

$\mathbb{C}_q$	–	$\{q \in \mathbb{C}\colon -\pi/2 < \arg q \leq \pi/2$ or $q = 0\}$, 17
$\mathbb{R}_q$	–	$\{q \in \mathbb{C}_q\colon \lambda = (\pi q)^2 \in \mathbb{R}\}$, **36**
$\gamma_c(q)$	–	Complex Characteristic Function, Complex CF, 18, **41**, 57
$\gamma(q)$, CF	–	(Complex-Real) Characteristic Function, 19, **42**
$\mathcal{D}_\xi$	–	Domain of CF $:= \{q \in \mathbb{C}_q\colon \operatorname{Im}\gamma_c(q) = 0\} \subset \mathbb{C}_q$, 19, **42**
$\gamma_r(x)$	–	Real Characteristic Function, Real CF, 20, **43**, 57
NEP	–	Nonconstant Eigenvalue Point, 18, **41**
$\mathcal{C}_\xi$	–	set of all CEPs, 18, **37**
$\mathcal{P}_\xi$	–	set of all PPs of CF, 18, **41**
$\mathcal{Z}$	–	set of all zerous, 16, **37**
$\mathcal{K}_\xi$	–	set of all Critical Points of CF, **45**
$\mathcal{N}_\xi$	–	Spectrum Domain $:= \mathcal{D}_\xi \cup \mathcal{C}_\xi$, 33, **42**
$\mathcal{N}_l$	–	Spectrum Curve, 33, **48**
EP	–	Eigenvalue Point, 18, **36**, 56
CE	–	Constant Eigenvalue, 17, **37**, 56
kCP	–	the k-order Critical Point, **45**
●, CP	–	Critical Point, 19, **45**
○, BP	–	Branch Point, **36**
◎	–	Critical Point at Branch Eigenvalue Point, **50**
○, BEP	–	Branch Eigenvalue Point, **36**
○, RP	–	Ramification Point, **36**
●, CEP	–	Constant Eigenvalue Point, 18, **37**, 56
○, PP	–	Pole Point, 18, **41**
◎	–	Pole Point of the second order, **41**
●, ZP	–	Zero Point, **38**
◎	–	Constant Eigenvalue Point at Ramification Point, **65**
β_{ZP}	–	Zero and Pole bifurcation, **50**
β_{2B}	–	the second order CP bifurcation, **50**
β_{2B}^{-1}	–	the inverse β_{ZP} bifurcation, **63**
β_{2B}^{-1}	–	the inverse β_{2B} bifurcation, **63**
β_{ZP}^{0}	–	symmetric Zero and Pole bifurcation, **64**

Introduction

1 Formulation of the problem

During the last two decades a lot of attention has been paid to the problems of differential equations with different types of *Boundary Conditions* (BC). Problem of this type arise in mathematical models of various processes in physics, biology, biotechnology, chemistry, etc. Theoretical investigation of differential problems with various types of *Nonlocal Boundary Conditions* (NBC) is a topical problem and recently has been paid much attention to it in the scientific literature. NBC appears when we can not measure data directly at the moment. In this case, the problem is formulated where the value of a solution (or value of its derivative) of the differential or the discrete problem on the boundary is related to the values inside the domain.

J.R. Cannon was the first investigator of the parabolic problems with integral BC [14, 1963]. Later, Bitsadze and Samarskii formulated and investigated a *Boundary Value Problem* (BVP) for an elliptic equation with NBC [10, Bitsadze and Samarskii 1969]. One of the most important problems is to find eigenvalues of differential problem with NBCs [162, Štikonas 2014]. An eigenvalue problem for the second-order differential operator with *Nonlocal Condition* (NC) was formulated in [54, Ionkin 1977]. The eigenvalue problem with NBCs is the part of the general nonselfadjoint operator theory [48, Il'in 1994]. In the article [149, Shkalikov 1982] Shkalikov described the result of investigation of the properties of eigenfunctions for integral BC. *Finite Difference Schemes* (FDS) for parabolic problems with two point (Samarskii–Ionkin type) NBC were analysed in the papers [38,42, Gulin *et.al.* 2001, 2003], where the properties of the spectrum and stability of FDS were obtained.

Problems with an integral NBC arise in various fields of mathematical physics, biology, biotechnology, etc. Nowadays the investigation of problems with various types of NBCs is a relevant problem. L.I. Kamynin began to investigate parabolic equations with nonlocal integral BCs [69, 1964]. The problems with integral NBCs were studied in many papers, such as [34, Gordeziani and Avalishvili 2001] [174, Zikirov 2007] (NBCs for hyperbolic equations), [46, Gushchin and Mikhailov 1994] [170, Wang 2002] (NBCs for elliptic equations), etc. Integral BCs are the special case of a more general nonlocal BC for stationary BVP [112, 115, 165, Štikonas and Roman 2009–2010].

M. Sapagovas with co-authors began to analyse *Sturm–Liouville Problem* (SLP) for one-dimensional differential operator with Bitsadze–Samarskii and integral type NBCs [23, 137, 138]. They showed that eigenvalues, which do not depend on some NBCs parameters, can exist. Problems with NBC are not self-adjoint and spectrum for such problems may be not positive (or real, too). So, negative, multiple and complex eigenvalues for some values of NBC parameters can exist [102,140,152,160, Štikonas *et.al.* 2005–2010]. Spectrum of *discrete Sturm–Liouville Problem* (dSLP) with NBCs was analyzed and the stability of FDS was investigated in [62, Jachimavičienė *et.al.* 2009]. See M. Sapagovas scientific group'ss results of problems with NBC's in [162, Štikonas 20014]. The structure of eigenvalues for some nonlocal boundary conditions presented in [77, Leonavičienė *et.al.* 2016].

Investigation of the spectrum of a differential equation with nonlocal conditions is quite a new area related to the problems of this type. Problems with NBCs are not self-adjoint and the spectrum of such problems may be not positive or real. This work is focused on investigation of the spectrum of SLP with one classical BC and another nonlocal two-points BC. We conclude that general properties of the Characteristic Function and Spectrum Curves for such a problem.

2 Topicality of the problem

The fundamental theory and basic concepts of differential equations field are based on the research of the problems of classical mathematical physics. However, contemporary problems encourage to formulate and investigate a new class differential problems, for example, a class of various type Nonlocal Problems. Problems with a NBCs arise in various fields of mathematical physics [25, Day 1982], [35, Gordeziani 2000], [54,57,58, Ionkin 1977], biology and biotechnology [84, Nakhushev 1995], [147, Schuegerl 1987], chemistry etc. The initial boundary or nonlocal boundary value problems formulated for equations of mathematical physics are nonclassical problems. In these problems, instead of the initial or boundary conditions a particular dependence of the values of the unknown function on the boundary on its values in inner points of the considered domain are given. Nonclassical problems with NBCs are used for mathematical modeling problems of pollution processes, caused by sewage, in rivers and seas. Also NBCs can also be used in simulation of decreasing of pollution under the influence of natural factors of filtration and settling that cause self-purification of the environment. In nonclassical problems with nonlocal initial conditions instead of classical initial conditions combinations of the initial values of the unknown function and it is given values at later times. Nonlocal in time problems are obtained when modeling problems of the processes of radionuclides propagation in Stokes fluid, diffusion and flow in porous media.

1964 R.W. Beals wrote PhD thesis Nonlocal Elliptic BVPs. He investigated elliptic type differential equations with NBCs [4, 5, 1964]. L.I. Kamynin began to investigate parabolic equations with nonlocal integral BCs [69, 1964]:

$$\int_{x_1(t)}^{x_2(t)} g(x,t)u(x,t)\,\mathrm{d}t = E(t).$$

Parabolic problems with nonlocal integral BCs were also analyzed in [13, Bouziani 1999], [21, 22, Čiegis *et.al.* 2001, 2002], [58, Ionkin 1977], [121, Sagwon 2000] [171, Yin 1994] [172, Yurchuk 1986].

The problems with integral NBCs were investigated in many papers. Nonlocal initial BVPs for some hyperbolic equations formulated and investigated by [34, Gordeziani and Avalishvili 2001], [174, Zikirov 2007] elliptic equations with NBC were analyzed in articles [46, Gushchin and Mikhailov 1994], [170, Wang 2002]. Integral BCs are the special case of a more general NBC for stationary BVP [114, Štikonienė and Sapagovas 2010]. [112,113,115, Roman and Štikonas 2009–2010]

A. Samarskii and A. Bitsadze have formulated and investigated a new problem for uniformly elliptic equations and, in case of rectangular space domain and Laplace operator, applying methods of the theory of integral equations. These boundary conditions connect values of the desired solution on the boundary with inner points of the domain. A. Samarskii and A. Bitsadze proved the existence and uniqueness of solution in [10, 11, 1964]. The problem considered by A. Bitsadze and A. Samarskii and others. This problem generalizations were studied by D. Gordeziani applying different approach [33, 1970]. D. Gordeziani suggested rather general iteration procedure, that proved the existence of the solution. He also provided an opportunity to construct algorithm for numerical solution, since on each step of the procedure classical BVPs were considered. Now some one-dimensional

NCs

$$u(a) = \gamma_0 u(\xi_0) + \mu_0, \qquad a < \xi_0 < b,$$
$$u(b) = \gamma_1 u(\xi_1) + \mu_1, \qquad a < \xi_1 < b$$

are called as Bitsadze–Samarskii type NBCs. A. Samarskii and A. Bitsadze had influence for appearance of many others mathematical scientists research direction and articles concepts [2, Ashyralyev 2008], [52, Infante 2003], [137, Sapagovas 2000], [162, Štikonas 2014], [166, Štikonas and Štikonienė 2009], [132, Sapagovas and Štikonienė 2009].

Differential equations for ordinary, elliptic, parabolic etc. with various types of NBCs were investigated by many scientists. The clasical BVP was formulated and investigated by two German mathematician Dirichlet ($u(x) = 0$) and Neumann ($u'(x) = 0$), and now it known as the BC of the first and the second kind. This type BC appears in many mechanical and civil engineering, thermodynamic, electrostatics and fluid dynamics problems and ect. [116, Rutkauskas 2014]. The Dirichlet and Neumann BVPs play an important role in the theory of harmonic functions and have been widely studied in literature. Another important problem, called periodic BVP, arises when one considers the problem in a one-dimensional case or in a multidimensional parallelepiped. The first time a new class of the BVPs for the Poisson equation in a multidimensional ball were studied in [118, 119, Sadybekov 2012, 2014]. V.A. Il'in [47, 1976] obtained necessary and sufficient properties of a subsystem of eigenfunctions and adjoint functions (as a basis) for Keldysh's bundle of ordinary differential operators. The multi-point BVP for the second-order ordinary differential equations with multipoints NBCs

$$u(0) = \sum_{i=0}^{m} \alpha_i u(\xi_i) \qquad u(1) = \sum_{i=0}^{m} \beta_i u(\xi_i)$$

where all α_i, β_i are real numbers and $\xi_i \in [0, 1]$. was initiated by V.A. Il'in and E.I. Moiseev [49, 50, 1987]. A multi-point nonlocal BVP for the second-order ordinary differential equation was also investigated in [15, Cao and Ma 2000], [78, Ma 1998], [102, Pečiulytė and Štikonas 2006].

Then one of the most popular nonclassical problems problems is the Samarskii–Ionkin problem, arisen in connection with the study of the processes occurring in the plasma in the 70s of the last century by physicists (see e.g. [54, 57, 58, Ionkin 1977]). Problems of Samarskii–Ionkin type are known classes of problems that represent a generalization of classical ones. At the same time they are obtained in a natural way by constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology, etc. Considerable interest in the study of problems of the Samarskii–Ionkin type emerged after the well-known classical work of N.I. Ionkin [58]. Two-points NBCs we can find in [59, Ionkin and Moiseev 1979]. N.I. Ionkin [57] considered parabolic problem with two types BCs:

$$u(0, t) = v(t) \qquad \int_0^1 u(1, t)\,\mathrm{d}t = \mu(t)$$

In [55, Ionkin and Morozova 2000], he investigate stability FDS for a parabolic equation with NBCs

$$u(0, t) = 0 \qquad u_x(1, t) = u_x(0, t).$$

Such problems differ from the classical ones, as corresponding spatial differential operator is nonself-adjoint and hence the system of eigenfunctions is not complete. From this conditions arise the problems of studying completeness and basicity of such systems, which play an important role in research of boundary value problems. Nowadays there is an extensive list of scientific papers pertaining to study of such problems and all of the works relate mainly to partial differential equations of the second order [80, Moiseev 1999], [6, 7, Berdyshev 2011, 2016], [55, Ionkin and Morozova 2000], [73, Kirane 2013]. The new one article where is presented the similar problems for fourth-order partial differential equations with three variables [71, Kerbal 2018]. Direct problem is addressed with a time-dependent source term, while a time-independent one is used to address an inverse problem. So, the authors investigate two nonlocal problems for fractional parabolic equations in 2D. The theory of boundary value problems for fractional differential equations has grown for the past years in applications in real-life problems, including viscoelastic, dynamic processes, biosciences, signal processing, system control theory, electrochemistry, diffusion processes, and many others [72, Kilbas 2006], [74, Koeller 1984], [16, Caputo 2001], [29, Fonseca 2008], [67, Jesus 2008], [108, Podlubny 1999], [79, Magin 2006] and for its intensive contribution in the general theory of differential equations. It has motivate more scientist to research and investigate fractional differential equations problems for different fields that appear in real-life, for instance in quantum physics (inverse problems in quantum theory of scattering), geophysics (inverse problems of electric prospecting, seismology, and theory of potentials), biology, medicine, quality checking programs, and others [1, Aleroev *et.al.* 2013], [28, Feng *et.al.* 2015], [30, Fragnelli 2015], [31, Furati *et.al.* 2014], [68, Kabanikhin 1999], [117, Sabitov 2010], [148, Shadan 1989], [120, Sadybekov 2015]. Nonlinear higher order BVPs with nonhomogenous multipoints BCs were also investigated by L. Kong and Q. Kong in [75, Kong and Kong 2010].

In 1836-1837 C. Sturm and J. Liouville published a series of papers on the second order linear ordinary differential equations including BVPs [168]. The influence of their work was such that this subject became known as Sturm–Liouville theory. A standard form of the Sturm-–Liouville differential equation is given as

$$-\frac{\mathrm{d}}{\mathrm{d}x}(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}) + q(x)y = \lambda w(x)y,$$

where $p(x)$, $q(x)$ and $w(x)$ are specified functions, which, depending on the studies considered, are required to satisfy additional conditions [3, Atkinson 1964], [173, Zettl 2005]. The above differential equation, in conjunction with the separated boundary conditions of the form,

$$\begin{aligned} c_1 y(a) + c_2 y'(a) = 0, \quad (c_1^2 + c_2^2 > 0), \\ d_1 y(b) + d_2 y'(b) = 0, \quad (d_1^2 + d_2^2 > 0), \end{aligned}$$

is known as a regular SLP if $p(x)$,$w(x) > 0$, and $p'(x)$, $q(x)$, and $w(x)$ are continuous functions over the finite interval $[a, b]$. A λ for which the above problem has a non-trivial solution is called an eigenvalue, and the corresponding solution, an eigenfunction. One of the key results in this field is that the eigenvalues of the above problem are real, and the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

SLP is very important for studying classical stationary problems and investigation of existence and uniqueness of this problems solutions. Problem of this type with NBCs are not self-adjoint and very complicated because spectrum for such problems may be negative or complex. Ionkin et. all. [54–57, Ionkin *et.al.* 1977–2000] deal with the stability analysis of difference schemes for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = u_0(x), \quad 0 < x < 1, \quad t > 0,$$

with the NBCs

$$u(0, t) = 0 \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t).$$

Eigenvalues are found analytically and the question of the existence of a basis of eigenfunctions and adjudicate functions is studied in a special case. Gulin et al. investigated a spectrum for one or two-dimentional parabolic equation with Samarskii—Ionkin and others types NBCs and proved stability for FDS

[36–39, 41–45, Gulin *et.al.* 2001–2013] Gulin and Mokin analyze Stability of a family of difference schemes with NBCs or Samarskii—Ionkin problem with variable coefficient [40, Gulin and Mokin 2009], [81–83, Mokin 2010–2014]. Recently, the authors studied in [27, El-Shahed 2017] the existence of eigenvalues and eigenfunctions of the BVP of the Sturm–Liouville differential equation

$$-u'' + q(x)u = \lambda^2 u \quad 0 \leqslant x \leqslant \pi$$

with each one of the two NCs

$$u(0) = 0, \quad u(\xi) = 0, \quad 0 < \xi \leqslant \pi,$$

and

$$u(\eta) = 0, \quad u(\eta) = 0, \quad 0 \leqslant \eta < \pi,$$

The author proved that the eigenvalues λ_n, $n = 0,\ 1,\ 2,\ \ldots$ of this problems are real and the corresponding eigenfunctions $\varphi(x; \lambda)$, $\psi(x; \lambda)$ are orthogonal. In [26, El-Shahed 2016] the authors study the existence and some general properties of the eigenvalues and eigenfunctions of of the same problem with the non-local boundary value problems:

$$\begin{aligned} &u(\eta) = 0, \quad u(\xi) = 0, \quad \text{or} \\ &u(0) = 0, \quad u(\pi) = 0, \quad 0 \leqslant \eta < \pi, \quad 0 < \xi \leqslant \pi. \end{aligned}$$

SLPs with integral conditions constitute a very interesting class of problems since they include as special cases two-, three- and multi-point boundary conditions.

B. Chanane use the regularized sampling method introduced recently to compute the eigenvalues of SLPs with NCs [18, Chanane 2009]:

$$\begin{cases} -u'' + q(x)u = \lambda u, & 0 \leqslant x \leqslant 1 \\ x_0(u) = 0,\ x_1(u) = 0, \end{cases}$$

where $q \in L^1$ and x_0 and x_1 are continuous linear functionals defined by:

$$\begin{aligned} x_0(u) &= \int_0^1 [u(t)\,\mathrm{d}\psi_1(t) + u'(t)\,\mathrm{d}\psi_2(t)] \\ x_1(u) &= \int_0^1 [u(t)\,\mathrm{d}\phi_1(t) + u'(t)\,\mathrm{d}\phi_2(t)] \end{aligned}$$

where x_0 and x_1 are independent, and ψ_1, ψ_2, ϕ_1 and ϕ_2 are functions of bounded variations, integration is in the sense of Riemann—Stieltjes. The regularized sampling method is an offshoot of the sampling method and it introduced in [12, Boumenir 1996]. B. Chanane also studied SLP with parameter dependent potential, complex valued functions belonging to L^1 and BCs [17, Chanane 2008].

In Lithuania, the problems with BCs started to be analyzed in 1965 [76, Kvedaras 1965]. Majority of the studies was motivated not only by the applied problems but also by the internal needs of mathematics. The first differential equation with BCs studies related only with mathematical theory. M. Sapagovas and T. Veidaitė, the two scientists of the Institute of Mathematics and Cybernetics start investigate differential problem with NC [169]. Prof. M. Sapagovas was pioneer in the study of such problems and found the scientific school. The main field of investigations at this school are investigation of the problem with NBC. The first problem with NBCs came from the applications. It was investigated problem of a mercury drop of liquid, in electric contact by the given droplet volume. In the papers [124, 134–136, 142, Sapagovas 1978–1984] solving problems with nonlinear or nonselfadjoined operators. Difference scheme for two-dimensional elliptic problem with an integral condition was constructed and studied in [123, Sapagovas 1983]. Later scientists R. Čiegis and professor Sapagovas investigated elliptic and parabolic problems with integral and Bitsadze—Samarskii type NBCs and finite-difference schemes for them and get new results about numerical solutions for problems with NBCs.

[19, 20, Čiegis 1984, 1988], [143, Sapagovas 1984], [144, 145, Sapagovas and Čiegis 1987]. In papers were analysed eigenvalues for Bitsadze–Samarskii and integral type NBCs

$$
\begin{aligned}
&u(0) = 0, \quad u(1) = \gamma u(\xi), \quad 0 < \xi < 1, \\
&u(0) = \gamma_0 \int_0^1 \alpha_0(x) u(x)\,\mathrm{d}x, \quad u(1) = \gamma_1 \int_0^1 \alpha_1(x) u(x)\,\mathrm{d}x.
\end{aligned}
$$

Sapagovas with co-authors [23, Sapagovas *et.al.* 2004] [125,137,138, Sapagovas 2000–2007], [140, 146, Sapagovas and Štikonas 2005] showed that there may exists not only eigenvalues, which do not depend on parameters γ_0 and γ_1 in NBCs, but also exist complex eigenvalues. The eigenvalue problems, investigation of the spectra, analysis of nonnegative solutions and similar problems for the operators with NBCs of Bitsadze—Samarskii or of integral-type are given in the papers [23]. Some results of complex eigenvalues for differential operators problems with NBCs are published in [162, Štikonas 2014], [166, Štikonas and Štikonienė 2009], [140, Sapagovas and Štikonas 2005], [151, A. Skučaitė *et.al.* 2009], [152, A. Skučaitė, K. Skučaitė-Bingelė *et.al.* 2010], [153–156, A. Skučaitė and Štikonas 2011–2015], [157–159, K. Skučaitė-Bingelė *et.al.* 2009–2013]. The authors presents some new results on a spectrum in a complex plane for the second order stationary differential equation with one Bitsadze–Samarskii type NBC. They survey the Characteristic Function Method for investigation of the spectrum of this problem and analyzed the spectrum depending on NBC parameters.

The SLP has played an important role in modeling many physical problems. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. Many numerical algorithms have been produced to look for approximate solutions. Sapagovas with co-authors investigated the spectrum of discrete SLP, too. These results can be applied to prove the stability of FDS for nonstationary problems and the convergence of iterative methods.

Numerical methods were proposed for parabolic and iterative methods for solving two-dimensional elliptic equation with Bitsadze-–Samarskii or integral type NBCs. The scientist in Lithuania presents some results on Alternating Direction Method (ADM) for a two-dimensional parabolic equation with NBCs [129, Sapagovas *et.al.* 2007], FDS of increased order of accuracy for the Poisson equation with NCs [139, Sapagovas 2008], FDS for two-dimensional elliptic equation with NC [64, 65, Jakubelienė 2009, 2013], the fourth-order ADM for FDS with NC [132, Sapagovas and Štikonienė 2009], ADM for the Poisson or two-dimensional parabolic equation with variable weight coefficients in an integral condition [128, 131, Sapagovas *et.al.* 2011, 2012]; ADM for a mildly nonlinear elliptic equation with integral type NCs [133, Sapagovas and Štikonienė 2011], FDS for nonlinear elliptic equation with NC [24,167, Štikonienė *et.al.* 2013, 2014]. Spectral analysis was applied for two- and three-layer FDS for parabolic or pseudoparabolic equations with NBCs: FDS for one-dimensional differential operator with integral type NCs [122, Sapagovas and Sajavičius 2009], [127, 130, 141, Sapagovas *et.al.* 2012], [61, Jachimavičienė 2013]. Stability analysis was done for FDS in the case of one- and two-dimensional parabolic or pseudoparabolic equation with NBCs [62,63, Jachimavičienė *et.al.* 2009, 2014], [86,87, Novickij and Štikonas 2014], [60, Ivanauskas *et.al.* 2009], [66, Jesevičiūtė 2009], [126, Sapagovas 2008]. J. Novickij and A. Štikonas consider the stability of a weighted FDS for a linear hyperbolic equation with integral NBCs [85–87, 89]. The authors also studied the equivalence of dSLP with NBCs to the algebraic eigenvalue problem [88].

One of the main task of the group of researchers in Vilnius University was The investigations of Green's functions and relations between them.

The first results were obtained for the second-order differential problem in [111–113, Roman and Štikonas 2009]. These results were generalised in [165, Štikonas and Roman 2009] for BVP

$$
\begin{aligned}
-(p(x)u')' + q(x)u &= f(x), \\
\langle L_1, u\rangle = g_1 \qquad \langle L_2, u\rangle &= g_2.
\end{aligned}
$$

The authors studied the second-order linear differential equation with two additional conditions was investigated and constructed Green's function. The relation between two Green's functions for two such problems with different additional conditions was derived and general formulae were applied to a stationary problem with NBCs. Green's function for a problem with NBCs can be expressed per Green's function for a problem with classical BCs [109, 110, Roman 2011]. For investigation of solutions can be constructed generalized Green's function. Most authors have developed a theory of generalized Green's matrix for systems of differential equations with two-points BCs. A. Štikonas and G. Paukštaitė investigated a generalized Green's function that describes the minimum norm least squares solution of every second-order discrete problem with two nonlocal conditions. In [90–93] properties of generalized Green's function that are analogous to properties of ordinary Green's function were proved. The authors investigated the nullity of discrete problem and presented its classifications. Null spaces of discrete problem and its adjoint problem were also analysed, explicit formulas of bases of null spaces were found. Moreover, [94–99,163,164, Paukštaitė and Štikonas 2015–2017] the necessary and sufficient existence conditions of exact solutions were given.

3 Aims and problems

The main aim of the dissertation is the analysis of the differential or the discrete SLP with two-point NBC and investigation of the spectrum of SLPs.

- In the case of the differential Sturm–Liouville Problem we investigate the following NBC:

$$u(1) = \gamma u(\xi), \qquad u(1) = \gamma u'(\xi),$$
$$u'(1) = \gamma u(\xi), \qquad u'(1) = \gamma u'(\xi),$$
$$u(\xi) = \gamma u(1-\xi),$$

  where $\gamma \in \mathbb{R}$ ir $\xi \in [0,1]$. Main problems:

  - find Constant Eigenvalues, which do not depend on parameter γ;
  - find Zeroes, Poles and Critical Points of Characteristic Function;
  - describe Spectrum Curves and investigate their properties;
  - investigate the dependence of Spectrum Domain on parameter ξ in NBC, find bifurcation points and types.

- In the case of discrete Sturm–Liouville Problem we investigate the following NBC:

$$U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}, \qquad U_n = \gamma U_m.$$

  At the left side of interval one of the conditions was selected:

$$U_0 = 0, \qquad U_1 = U_0.$$

  The discrete problem was obtained by approximating the differential problem by a finite difference scheme. Main problems:

  - find Constant Eigenvalues, which do not depend on parameter γ;
  - find Zeroes, Poles and Critical Points of Characteristic Function;
  - determine the dependence of these points on the number of grid points;
  - investigate the behavior of Spectral Curves in the neighborhood of special special points;
  - find the quantitative relationships between the numbers of points mentioned.

4 Methods

Characteristic Function (CF) analysis is used for investigation of the Spectrum Curves for differential and discrete SLP with two-point NBCs [166, Štikonas and Štikonienė 2009], [102, 104, Pečiulytė and Štikonas 2006, 2007]. The properties of the Spectrum Curves for such type problems depend on Constant Eigenvalue Points (CEP) and zeros, poles, Critical Points (CP) of CF. Investigations of real and complex parts of the spectrum are provided with the results of numerical experiments. Some results are given as graphs of CF Spectrum Curves.

5 Actuality and novelty

Most of the results of studying SLP with two-point NBC and presented in this work are quite new and have not appeared before in the scientific literature. Although the results do not embrace all the possible variants of the spectrum but this thesis contributes to a better understanding of the spectrum for SLP with two-point NBCs. Real part of the Spectrum Curves for SLP with two-point NBC were investigated in [101–105, 107, Pečiulytė and Štikonas 2006–2008]. The eigenvalue problems, investigation of the real part of the Spectrum Curves, analysis of nonnegative solutions and similar problems for the operators with integral NBCs are given in the papers [106, 107, Pečiulytė and Štikonas 2005, 2008]. Complex eigenvalues for differential operators with integral NBCs are less investigated than the real case. Some results of Spectrum Curves in a complex plane are published [151–156, A.Skučaitė *et.al.* 2009–2015].

6 Dissemination of results

The results of this thesis were presented in the following international conferences:

- *MMA2018*, Sigulda, Latvia, May 29-June 1, 2018
  "Sturm–Liouville problem with two-point nonlocal boundary condition";
- *MMA2017*, Druskininkai, Lithuania, May 30-June 2, 2017,
  "Investigation of spectrum curves for Sturm–Liouville problem with two-point nonlocal boundary condition";
- *MMA2014*, Druskininkai, Lithuania, May 26–29, 2014;
  "Investigation of critical points for Strum–Liouville problem with two-point nonlocal boundary condition";
- *MMA2013*, Tartu, Estonia, May 27–30, 2013;
  "Investigation of eigenvalues for stationary problem with two Bicadze–Samarskii type nonlocal boundary conditions";
- *MMA2012*, Tallinn, Estonia, June 6–9, 2012;
  "Investigation of complex eigenvalues for stationary problems with two-point nonlocal boundary condition";
- *MMA2011*, Sigulda, Latvia, May 25–28, 2011;
  "Investigation of Complex Eigenvalues for Stationary Problem with Two-Point Nonlocal Boundary Conditions";
- *MMA2010*, Druskininkai, Lithuania, May 26–29, 2010;
  "Investigation of Complex Eigenvalues for Stationary Problem with Nonlocal Boundary Conditions";
  "Investigation Discrete Sturm–Liouville problem with Nonlocal Boundary Condition";

and other results were presented in the national conference of the Lithuanian Mathematical Society (LMD) and Mathematic and Mathematical Modeling (MMM):

- *LMD*, Kaunas, Lithuania, June 18-19, 2018;
  "Investigation of Spectrum Curves for a Sturm–Liouville Problem with Two-Point Nonlocal Boundary Conditions";
- *LMD*, Vilnius, Lithuania, June 21-22, 2017;
  "Investigation of spectrum curves for Sturm–Liouville problem with two-point nonlocal boundary condition";
- *LMD*, Vilnius, Lithuania, June 26–27, 2014;
  "The dynamics of Sturm—Liouville problem's with integral BCs bifurcation points";

- *LMD*, Vilnius, Lithuania, June 19–20, 2013;
  "Inverstigation of the spectrum for Sturm–Liouville problems with a nonlocal boundary condition";
- *LMD*, Vilnius, Lithuania, June 16-17, 2011;
  "Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition";
- *LMD*, Šiauliai, Lithuania, June 17-18, 2010;
  "Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition";
- *MMM2010*, Kaunas, Lithuania, April 8–9, 2010; "Investigation Sturm—Liouville problems with two-point nonlocal boundary condition";
- *LMD*, Vilnius, Lithuania, June 18–19, 2009;
  "Investigation of complex eigenvalues for stationary problems with nonlocal boundary condition";
- *MMM2009*, Kaunas, Lithuania, April 2–3, 2009;
  "Investigation of complex eigenvalues for Sturm—Liouville problems with nonlocal two-point boundary condition";

7 Publications

Results of the research were published in 6 scientific papers.

Main publications:

1. **Kristina Bingelė**, Agnė Bankauskienė, Artūras Štikonas. Investigation of spectrum curves for the Sturm–Liouville problem with two-point nonlocal boundary condition. *Math. Model. Anal.*, **25**(1):53–70, 2020. `https://doi.org/10.3846/mma.2020.10787`.

2. **Kristina Bingelė**, Agnė Bankauskienė, Artūras Štikonas. Spectrum Curves for a discrete Sturm–Liouville problem with one integral boundary condition. *Nonlinear Anal. Model. Control*, **24**(5):755–774, 2019. `https://doi.org/10.15388/NA.2019.5.5`.

3. Agnė Skučaitė, **Kristina Skučaitė-Bingelė**, Sigita Pečiulytė, Artūras Štikonas. Investigation of the spectrum for the Sturm–Liouville problem with one integral boundary condition. *Nonlinear Anal. Model. Control*, **15**(4):501–512, 2010. `https://doi.org/10.15388/NA.15.4.14321`.

Other publications:

4. **Kristina Skučaitė-Bingelė**, Artūras Štikonas. Inverstigation of the spectrum for Sturm-Liouville problems with a nonlocal boundary condition. *Liet. Matem. Rink. Proc. LMR*, Ser. A, **54**:73–78, 2013. `https://doi.org/10.15388/LMR.A.2013.16`.

5. **Kristina Skučaitė-Bingelė**, Artūras Štikonas. Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition. *Liet. Matem. Rink. LMD darbai*, **52**:303–308, 2011. `https://doi.org/10.15388/LMR.2011.sm04`.

6. **Kristina Skučaitė-Bingelė**, Sigita Pečiulytė, Artūras Štikonas. Investigation of Complex Eigenvalues for Stationary Problem with Two-Points Nonlocal Boundary Condition. *Mathematics and Mathematical Modelling*, 5:24–32, 2009.

8 Traineeships

During the doctoral studies were made one week research visit to Barcelona, Spain, JISD2014, Universitat Polit‘ecnica de Catalunya, June 16–20, 2014.

9 Structure of the dissertation and main results

This dissertation is composed of the introduction, three chapters, general conclusions and the bibliography. Each chapter begins with the introduction and ends with the conclusions as well. Below we formulate main results of each section.

In Chapter 1 we investigates the SLP with one classical and another nonlocal two-point BC. The qualitative study of the Spectral Curves was done. Some new properties of CF were found. We analyze zeroes, poles and critical points of the characteristic function and how the properties of this function depend on parameters in NBC. Classification of such points is done. Properties of the Spectrum Curves are formulated and illustrated in Figures for various values of the parameter ξ.

In the first section of Chapter 2 we analyze a complex eigenvalues for a stationary differential problem with one classical BC $u(0) = 0$ and another two-point NBC of Samarskii–Bitsadze type: $u'(1) = \gamma u(\xi)$ or $u'(1) = \gamma u'(\xi)$. We investigate how the spectrum in the complex plane of these problems depends on the two point NBCs parameters γ and ξ.

In the second section of Chapter 2 we investigates the SLP with one classical Neumann BC $u'(0) = 0$ and another NBC.

These problems with NBC are not self-adjoint. So, the investigation of the Spectrum Curves are complicated. We investigate how the spectrum in the complex plane of these problems depends on parameters of the NBCs.

In the third section of Chapter 2 we analyze the SLP with one classical BC $u(0) = 0$ and another two-point NBC of Samarskii–Bitsadze type

$$u(\xi) = \gamma u(1 - \xi),$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$. In NBC both sides of condition depends on parameter ξ. As the theoretical investigation of the complex spectrum is more complicated, because in this cases CF zeroes depend on NBC parameter ξ too. We present the result of modelling and illustrate the existing situation in graphs.

In Chapter 3 we analyzed *a discrete Sturm–Liouville Problems* (dSLP) corresponding to the problem in the second chapter. We investigate how the spectrum depends on the number of grid points. The behavior of Spectrum Curves in the neighbourhood of special points ($q = 0$, $q = n$ and $q = \infty$) was analyzed.

10 A Method of Characteristic Function

Investigation of the spectrum (and complex part of the spectrum particularly) of differential equations with NBCs is quite a new, but important area related to the problems in this field. The eigenvalue problems, investigation of the spectra, analysis of nonnegative solutions and similar problems for the operators with NBCs of Bitsadze–Samarskii or of integral-type are given in the papers [23, 138, 140, Sapagovas *et.al.* 2002–2005], [52, 53, Infante 2003, 2005]. Complex eigenvalues for differential operators with NBCs are less investigated than the real case. Some results of these eigenvalues for a problem with one Samarskii–Bitsadze NBC are published [140, Sapagovas and Štikonas 2005], [166, Štikonas and Štikonienė 2009].

In [166], authors analyze a complex eigenvalue problem for a stationary differential operator with two cases of nonlocal integral NBC. Authors investigate how the complex eigenvalues of these problems depend on the parameters γ and ξ of the nonlocal integral BCs. As the theoretical investigation of the complex spectrum is a very difficult problem, they present the results of modelling and computational analysis and illustrate the existing situation in graphs. Zeroes, poles and CPs of the CF are important for investigating complex eigenvalues. Real eigenvalues of the SLP are also important [166].

Let us consider the SLP with Bitsadze–Samarskii type NBC:

$$-u'' = \lambda u, \quad t \in (0,1), \tag{10.1}$$

$$u(0) = 0, \quad u(1) = \gamma u(\xi), \tag{10.2}$$

with the parameters $\gamma \in \overline{\mathbb{C}}$, $\xi \in [0,1]$ and $\lambda \in \mathbb{C}$.

Remark 1. The case of NBC (10.2) is important in the investigation of multidimensional and non-stationary problems, and numerical methods. CF other types of NBCs were investigated by Pečiulytė in her PhD Thesis and in [102,104, Pečiulytė and Štikonas 2006, 2007]. Theoretical results on the real spectrum are presented in [160, Štikonas 2007].

If $\gamma = 0$, then we have the classical SLP. In this case, all eigenvalues of problem (10.1)–(10.2) are positive and algebraically simple:

$$\lambda_k = (\pi z_k)^2, \quad u_k(t) = \sin(\pi z_k t), \quad z_k = k \in \mathbb{N}. \tag{10.3}$$

In the case of $\gamma \neq \infty$ and $\xi = 0$, or $\xi = 1$ and $\gamma \neq 1$ we have the classical case, as well. For $\gamma = \infty$ we have a "boundary" condition $u(\xi) = 0$ instead of (10.2). So, this case is similar to the classical one for $\xi > 0$:

$$\lambda_l = (\pi p_l)^2, \quad u_l(t) = \sin(\pi z_l t), \quad p_l = l/\xi,\ l \in \mathbb{N}. \tag{10.4}$$

We denote sets $\hat{\mathcal{Z}} := \{z_k\}_{k=1}^{\infty}$, $\overline{\mathcal{Z}}_\xi := \{p_l\}_{l=1}^{\infty}$.

The case with one classical boundary condition. If $\gamma = \infty$ and $\xi = 0$, then the second BC (10.2) is the same as the first BC in (10.2). If $\gamma = 1$ and $\xi = 1$, then the second BC (10.2) becomes the identity $0 \equiv 0$. So, these two cases correspond to the problem with one classical BC:

$$-u'' = \lambda u, \quad t \in (0, 1), \tag{10.5}$$

$$u(0) = 0. \tag{10.6}$$

The characteristic equation for the second order ordinary equation (10.5) is $-\mu^2 = \lambda$. If $\lambda = 0$, then the general solution of this equation is $u = Ct + C_1$, and functions $u = Ct$ satisfy problem (10.5)–(10.6). If $\lambda \neq 0$, then the characteristic equation has two different roots $\mu = \pi q_{1,2}$ (see Figure 1). Let q be the root with a positive real part when λ is a nonnegative real number, and the root with a positive imaginary part for negative λ. Thus, we have a bijection $\lambda = (\pi q)^2$ between the domain $\mathbb{C}_q = \{z \in \mathbb{C}\colon\ -\pi/2 < \arg z \leq \pi/2 \text{ or } z = 0\}$ and the whole complex plane $\mathbb{C}_\lambda = \mathbb{C}$ (see Figure 2).

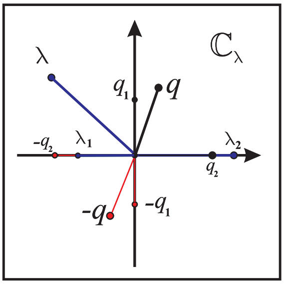


Fig. 1. Roots of the equation $(\pi q)^2 = \lambda$ [166].

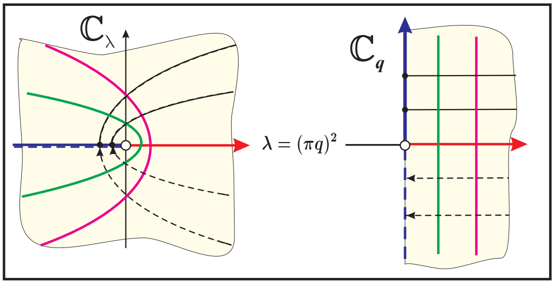


Fig. 2. Bijective mapping $\lambda = (\pi q)^2$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$ [166].

If we find such q, then the corresponding eigenvalue is defined as $\lambda = (\pi q)^2$. Then the roots $q_{1,2} = \pm\iota\pi\, q$, and the general solution are given in the following form

$$u = C_1 \mathrm{e}^{\iota\pi\, q\, t} + C_2 \mathrm{e}^{-\iota\pi\, q\, t} = C_3 \sin(\pi q t) + C_4 \cos(\pi q t).$$

All functions $u = C \sin(\pi q t)$ satisfy equation (10.5) and BC (10.6). In both cases ($q = 0$ and $q \neq 0$), we can write a formula for the solution

$$u = C \frac{\sin(\pi q t)}{\pi q}, \quad q \in \mathbb{C}_q, \tag{10.7}$$

where $C \in \mathbb{C}$ is an arbitrary constant [160, Štikonas 2007]. So all $\lambda \in \mathbb{C}$ are eigenvalues of problem (10.5)–(10.6).

10.1 Constant Eigenvalues

Let us return to problem (10.1)–(10.2). If we substitute function (10.7) into the second BC (10.2), then we get the equality

$$C\Big(\frac{\sin(\pi q)}{\pi q} - \gamma\frac{\sin(\xi\pi q)}{\pi q}\Big) = 0. \tag{10.8}$$

There exists a nontrivial solution (eigenfunction) if q is the root of the function

$$f(q) := \gamma\frac{\sin(\xi\pi q)}{\pi q} - \frac{\sin(\pi q)}{\pi q}. \tag{10.9}$$

Such q we call Eigenvalue Point (EP). In the case $\lambda = q = 0$, we get the equality $\gamma\xi - 1 = 0$. So, the eigenvalue $\lambda = 0$ exists if and only if $\gamma = 1/\xi$ [140, Sapagovas and Štikonas], [160, Štikonas].

Let us consider the case $q \neq 0$. If

$$\begin{cases} \sin(\pi q) = 0, \\ \sin(\xi\pi q) = 0, \end{cases} \tag{10.10}$$

then equality (10.9) is valid for all γ. In this case, we have Constant Eigenvalues (that do not depend on the parameter γ) $\lambda = (\pi q)^2$, where q is a *Constant Eigenvalue Point* (CEP), see [140, 160].

We suppose that m and n ($n > m > 0$) are positive coprime integer numbers. *Constant Eigenvalues* (CE) exist only for rational $\xi = r = \frac{m}{n} \in (0, 1)$, and those eigenvalues are equal to $\lambda_k = (\pi c_k)^2$, $c_k = k \in \mathbb{N}_n$ [160], and $c_k \in \mathcal{C}_\xi := \hat{\mathcal{Z}} \cap \overline{\mathcal{Z}}_\xi$ are positive real numbers.

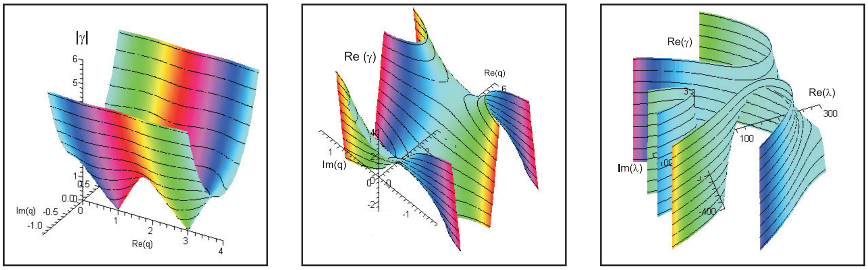
10.2 Complex Characteristic Function

All Nonconstant Eigenvalue Points (NEP) (it depends on the parameter γ) are γ-points of the meromorphic function

$$\gamma = \gamma_c(q) := \frac{\sin(\pi q)}{\sin(\xi\pi q)}, \quad \gamma_c\colon \mathbb{C}_q \to \overline{\mathbb{C}}. \tag{10.11}$$

Note that, if γ is fixed, then the γ-point is the root of the equation $\gamma_c(q) = \gamma$. So, nonconstant eigenvalue is defined as $\lambda = (\pi q(\gamma))^2$. We call this function a *Complex Characteristic Function* (Complex CF). The graphs of the functions $|\gamma_c(q)|$, $\operatorname{Re}\gamma_c(q)$, and $\operatorname{Re}\gamma_c(\sqrt{\lambda})$ are presented in Figure 3 for $\xi = 1/2$. We can investigate these functions only for $\operatorname{Im} q \geq 0$ and $\operatorname{Re} q \geq 0$ because $\bar{\gamma}_c(q) = \gamma_c(\bar{q})$ [160].

All zeroes $z \in \mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$ and *Pole Points* (PP) $p \in \mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$ of the function $\gamma_c(q)$, lie in the positive part of the real axis [160]. We also

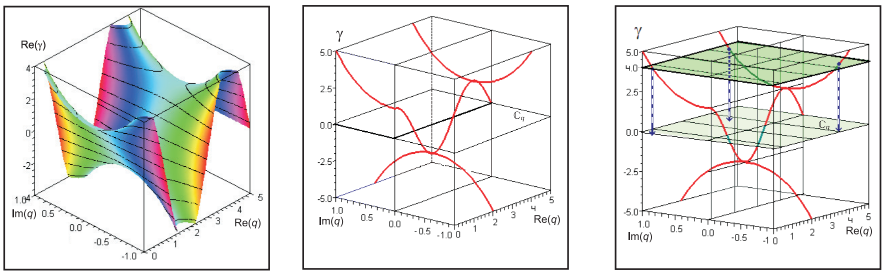


a) $|\gamma_c(q)|,\ q \in \mathbb{C}_q$ b) $\operatorname{Re}\gamma_c(q),\ q \in \mathbb{C}_q$ c) $\operatorname{Re}\gamma_c(\sqrt{\lambda}),\ \lambda \in \mathbb{C}$

Fig. 3. The Complex Characteristic Function $\gamma_c(q)$ for $\xi = 1/2$ [166].

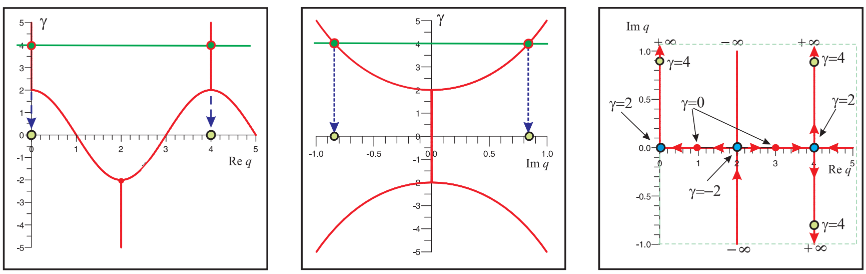
use the notation $\overline{\mathcal{P}}_\xi = \mathcal{P}_\xi \cup \{p_0 := 0, p_\infty := +\infty\}$. Note that, for $\xi = 1/n$, $n \in \mathbb{N}$, there are no poles, i.e., the Complex CF $\gamma_c(q)$ is an entire function. So, there are no poles in the whole interval $(p_0; p_\infty) = (0; +\infty)$ in this case, while in the other cases we have an infinite sequence of poles [160].

We call *Complex-Real Characteristic Function* (CF) the restriction of the Complex CF on subset $\mathcal{D} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q \colon \operatorname{Im}\gamma_c(q) = 0\} \subset \mathbb{C}_q$ and $\gamma = \gamma_c|_{\mathcal{D}} : \mathcal{D} \to \mathbb{R}$.



a) Complex CF b) CF c) Eigenvalue Points, $\gamma = 4$

Fig. 4. Characteristic Functions for $\xi = 1/2$ in $\mathbb{C}_q$ [166].



a) $(\operatorname{Re} q, \gamma)$ b) $(\operatorname{Im} q, \gamma)$ c) $\mathcal{D} \subset \mathbb{C}_q$

Fig. 5. Projections of Characteristic Function $\gamma(q)$ for $\xi = 1/2$ [166].

We get the EPs as γ-values of the CF $\gamma(q)$ (see, Figure 4). Projections of the CF are shown in Figure 5. The domain $\mathcal{D}$ is a projection into the domain $\mathbb{C}_q$. We add arrows that show how the EPs are moving, i.e., the direction in which the parameter γ is growing. In Figure 7, we see the parts (on three planes) of the CF graph where there exist complex EPs. The case of a positive imaginary axis ($\operatorname{Re} z = 0$, $\operatorname{Im} z > 0$) corresponds to negative eigenvalues. Projections into plane $\mathbb{C}_q$ may give various Spectrum Curves. We call the point $q_c \in \mathbb{C}_q$ such that $\gamma'(q_c) = 0$, a Critical Point (CP).

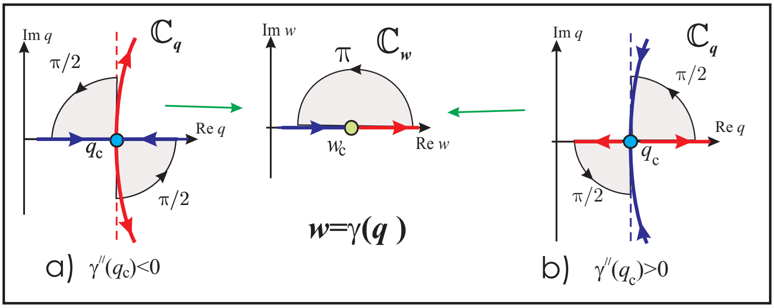


Fig. 6. Projection of a Characteristic Function to $\mathbb{C}_q$ near the Critical Point q_c; $w = \gamma_c(q)$ is Characteristic Function, w_c is the critical value [166].

A few Spectrum Curves of the domain $\mathcal{D}$ can intersect only at the CP q_c (see, Figure 6). It is valid that $\gamma''(q_c) \neq 0$ for problem (10.1)–(10.2) (see, [160, Corollary 7]). So, two symmetrical "complex" curves are orthogonal with the real axis and EPs leave or enter the real axis orthogonally in this problem.

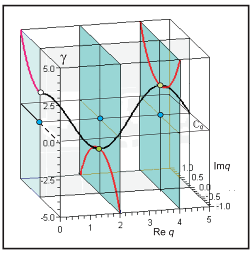
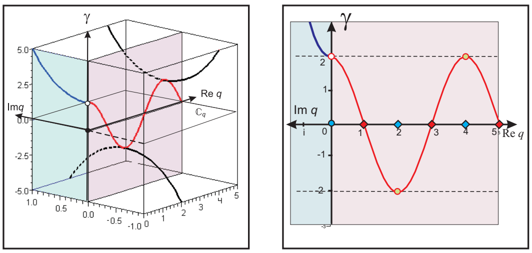


Fig. 7. Characteristic Function for complex and negative eigenvalues in the case $\xi = 1/2$ [166].



a) b)

Fig. 8. a) Characteristic Function for real eigenvalues in the case $\xi = 1/2$. b) The graph of the Real Characteristic Function [166].

10.3 Real Characteristic Function

If we take q only in the rays $q = x \geq 0$ and $q = -\iota x$, $x \leq 0$ (see, Figure 8) instead of $q \in \mathbb{C}_q$, then we get positive eigenvalues in case of the ray $q =$

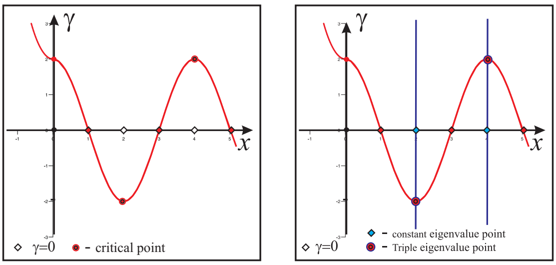
$x > 0$, and we get negative eigenvalues for the ray $q = -\imath x$, $x < 0$. The point $q = x = 0$ corresponds to $\lambda = 0$. For the function $\gamma_c : \mathbb{C}_q \to \overline{\mathbb{R}}$ we obtain its two restrictions on those rays: $\gamma_+(x) = \gamma_c(x + \imath 0)$ for $x \geq 0$, and $\gamma_-(x) = \gamma_c(0 - \imath x)$ for $x \leq 0$. The function γ_+ corresponds to the case of nonnegative eigenvalues, while the function γ_- corresponds to that of nonpositive eigenvalues. Let us use the notation

$$\{f_1(x); f_2(x)\} := \begin{cases} f_1(x), & \text{for } x \leq 0 \\ f_2(x), & \text{for } x \geq 0. \end{cases}$$

All the real eigenvalues $\lambda = \{-(\pi x)^2; (\pi x)^2\}$ are investigated using *Real Characteristic Function* (Real CF) $\gamma_r(x) := \{\gamma_-(x); \gamma_+(x)\}$ (see, Figure 8). Such a restriction of CF is very useful for investigating real eigenvalues. For the Complex CF (10.11), the Real CF can be written as:

$$\gamma_r(x) = \begin{cases} \dfrac{\sinh(\pi x)}{\sinh(\pi \xi x)}, & x \leqslant 0; \\ \dfrac{\sin(\pi x)}{\sin(\pi \xi x)}, & x \geqslant 0. \end{cases}$$

The cases $\xi = 1/2$ and $\xi = 1/4$ were described in [140].



a) Real CF b) Generalized Real CF

Fig. 9. Real Characteristic Functions, $\xi = 1/2$ [166].

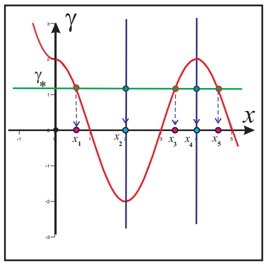


Fig. 10. Generalized Real Characteristic Function and real Eigenvalue Points x_i [166].

We draw vertical lines which intersect with the x-axis at the CEPs. We call the union of the graph of the Real CF and all CE lines a Generalized Real CF (see, Figure 9). Then, for each γ_*, the constant function $\gamma \equiv \gamma_*$ (horizontal line) intersects the graph of the Real CF $\gamma_r(x)$ or CE lines at some points. Now we have all real EPs x_i for this γ_* (see, Figure 10). Usually, we enumerate the eigenvalues in such a way: $x_k(0) = k \in \mathbb{N}$, i.e., using the classical case. Eigenvalues (and $x_k(\gamma)$) are continuously dependent on the parameter γ and we get Spectrum Curves.

Remark 2. We can define a Generalized Characteristic Function analogously, i.e., add vertical lines at CEPs (see, Figure 10, $\xi = 0.5$) to the graph of CF.

Real eigenvalues for SLPs were investigated in [102, 104, Pečiulytė and Štikonas 2006, 2007], [160, Štikonas 2007]. Graphs of the Generalized Real CF $\gamma_r(x)$ for various ξ are shown in the Figure 11.

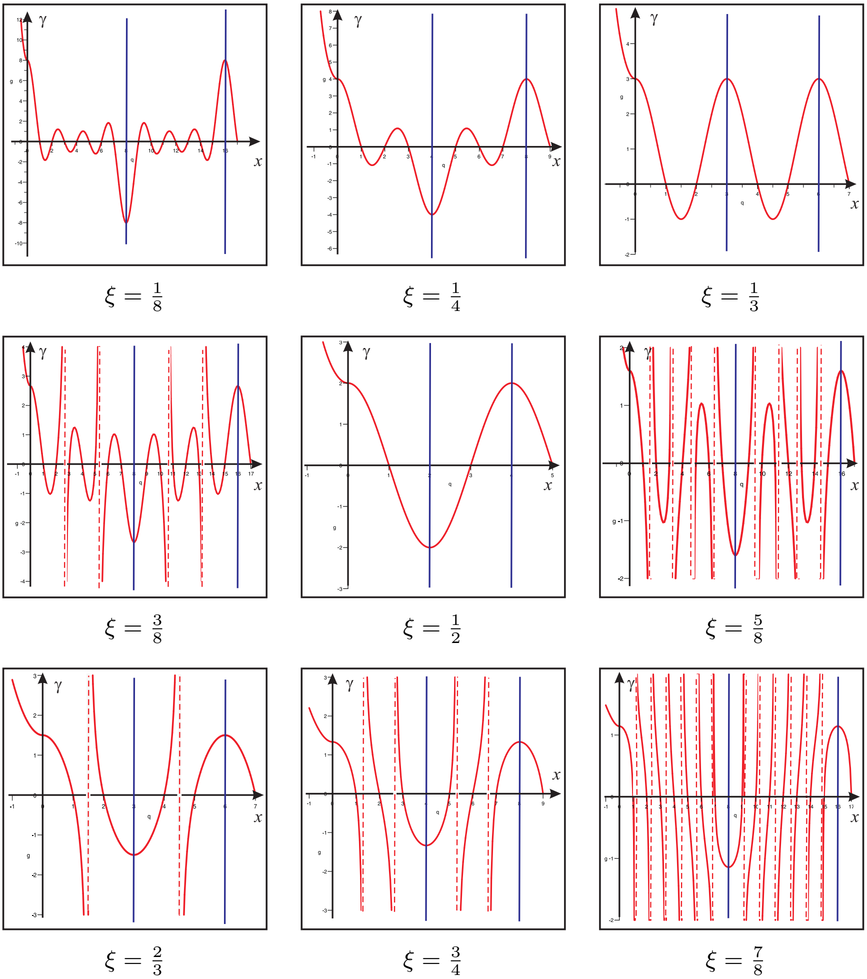
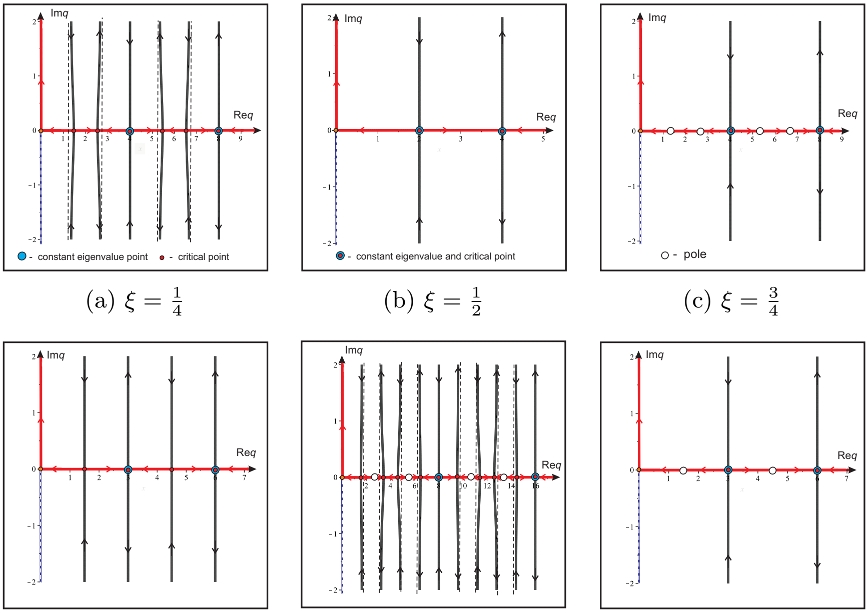


Fig. 11. Generalized Real CF $\gamma_r(x)$ for various ξ [166].

10.4 Complex eigenvalues

Theoretical investigation of complex eigenvalues is a very difficult problem even for simple SLP with NBCs such as (10.1)–(10.2). Therefore, we present only simple properties of a complex part of the spectrum for this problem. Spectrum Curves for various ξ are shown in Figure 12.



(a) $\xi = \frac{1}{4}$ (b) $\xi = \frac{1}{2}$ (c) $\xi = \frac{3}{4}$

(d) $\xi = \frac{1}{3}$ (e) $\xi = \frac{3}{8}$ (f) $\xi = \frac{2}{3}$

Fig. 12. Spectrum Curves for various ξ [166].

As shown in the previous subsection, the behavior of EPs is quite simple for fixed ξ. Zeroes of the CF are fixed for all ξ. The poles depend on ξ: $p_l(\xi) = l/\xi$. A qualitative view of the domain $\mathcal{D}$ with respect to ξ changes when the pole and zero meet at the CEP. If ξ is growing, then the pole moves to the left.

10.5 The case of more General Conditions

Let us consider a SLP with the following NBCs:

$$-(p(t)u')' + q(t)u = \lambda u, \quad t \in (0,1), \tag{10.12}$$

$$\langle l_0, u(t)\rangle = 0, \tag{10.13}$$

$$\langle l_1, u(t)\rangle = \gamma\langle k, u(t)\rangle, \tag{10.14}$$

where $p(t) \geqslant p_0 > 0$, $p \in C^1[0,1]$, $q \in C[0,1]$, l_0, l_1 and k are linear functionals. For example, the functional k can describe multi-point or integral NBCs:

$$\langle k, u(t)\rangle = \sum_{j=1}^{n} \left(\varkappa_j u(\xi_j) + \kappa_j u'(x_j)\right), \quad \langle k, u(t)\rangle = \int_0^1 \varkappa(t)u(t)\,\mathrm{d}t,$$

and the functionals l_i, $i = 0, 1$ can describe local (classical) BCs

$$\langle l_0, u(t)\rangle = \alpha_0 u(0) + \beta_0 u'(0), \quad \langle l_1, u(t)\rangle = \alpha_1 u(1) + \beta_1 u'(1),$$

where the parameters $|\alpha_i| + |\beta_i| > 0$, $i = 0, 1$.

Let $\varphi_0(t;\lambda)$ and $\varphi_1(t;\lambda)$ be two independent solutions of equation (10.12). For example, we can find such a solution by solving initial value problems with the conditions: $u(0) = 1$, $u'(0) = 0$ and $u(0) = 0$, $u'(0) = 1$. Let us denote

$$D_s^t(\lambda) := D_s^t[\varphi_0, \varphi_1](\lambda) = \begin{vmatrix} \varphi_0(t;\lambda) & \varphi_1(t;\lambda) \\ \varphi_0(s;\lambda) & \varphi_1(s;\lambda) \end{vmatrix},$$

$$\langle k_1 \cdot k_2, D_s^t(\lambda)\rangle := \begin{vmatrix} \langle k_1, \varphi_0(t;\lambda)\rangle & \langle k_1, \varphi_1(t;\lambda)\rangle \\ \langle k_2, \varphi_0(s;\lambda)\rangle & \langle k_2, \varphi_1(s;\lambda)\rangle \end{vmatrix}.$$

All solutions of equation (10.12) are of the form $u{=}C_0\varphi_0(t;\lambda){+}C_0\varphi_1(t;\lambda)$. There exists a nontrivial solution of problem (10.12)–(10.14) if and only if $\Psi(\lambda)\gamma{=}\Phi(\lambda)$, where $\Psi := \langle l_0 \cdot k, D_s^t(\lambda)\rangle$, $\Phi := \langle l_0 \cdot l_1, D_s^t(\lambda)\rangle$. Both functions $\Psi(\lambda)$ and $\Phi(\lambda)$ are entire functions for $\lambda \in \mathbb{C}$.

We can find CEs for problem (10.12)–(10.14) as the roots of the system

$$\begin{cases} \Psi(\lambda) = 0, \\ \Phi(\lambda) = 0. \end{cases}$$

The Complex Characteristic Function

$$\tilde{\gamma}_c := \Phi(\lambda)/\Psi(\lambda), \quad \tilde{\gamma}_c \colon \mathbb{C}_\lambda \to \overline{\mathbb{C}} \tag{10.15}$$

is a meromorphic function and describes nonconstant eigenvalues.

11 Sturm–Liouville problem with integral type NBC [152]

Now, in Section 11, we formulate the problem and present the earlier obtained main results on CFs, zeroes, poles, and CPs in all cases. In Section 11.1, a short review of real eigenvalues properties of the analyzed problem is given. These results are wider discussed in the previous papers [106,107, Pečiulytė *et.al.* 2005, 2008], [160, Štikonas 2007] and they are useful for investigating complex eigenvalues.

Let us consider a SLP with one classical BC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{11.1}$$

$$u(0) = 0, \tag{11.2}$$

and another integral NBC:

$$u(1) = \gamma \int_{\xi}^{1} u(t)\ \mathrm{d}t \quad \text{(Case 1)}, \tag{11.3$_1$}$$

$$u(1) = \gamma \int_{0}^{\xi} u(t)\ \mathrm{d}t \quad \text{(Case 2)}, \tag{11.3$_2$}$$

with parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$. Also we analyze the SLP with the BC

$$u'(0) = 0 \tag{11.4}$$

on the left side, and with integral NBC (11.3) on the right side of the interval. We enumerate these cases from Case 1′ to Case 2′.

We get all the CEPs as the roots of the systems:

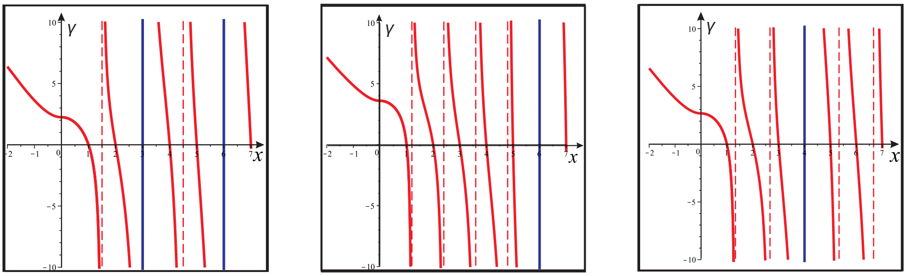
$$\begin{cases} \sin(\pi q) = 0, \\ \cos(\xi \pi q) - \cos q = 0, \end{cases} \qquad \begin{cases} \sin(\pi q) = 0, \\ 1 - \cos(\xi \pi q) = 0, \end{cases} \tag{11.5$_{1,2}$}$$

$$\begin{cases} \cos(\pi q) = 0, \\ \sin(\pi q) - \sin(\xi \pi q) = 0, \end{cases} \qquad \begin{cases} \cos(\pi q) = 0, \\ \sin(\xi \pi q) = 0. \end{cases} \tag{11.5$_{1',2'}$}$$

CEs exist only for rational $\xi = r = m/n \in (0, 1)$, those eigenvalues are equal to $\lambda_k = (\pi c_k)^2$, $k \in \mathbb{N}$, where CEPs c_k are given by formulae shown in Table 1. We used notation for the sets $\mathbb{N}_o := \{1, 3, 5, \ldots\}$, $\mathbb{N}_e := \{2, 4, 6, \ldots\}$ and $\mathbb{N}_l := \{lk\colon k \in \mathbb{N}\}$, $l \in \mathbb{N}$.

Table 1. Constant Eigenvalue Points c_k, $k \in \mathbb{N}$ [152].

Case	$n - m \in \mathbb{N}_e$	$n - m \in \mathbb{N}_o$	$m \in \mathbb{N}_e$	$m \in \mathbb{N}_o$
Case 1	nk	$2nk$	–	–
Case 1′	$n(k - 1/2)$	$2n(k - 1/2)$	–	–
Case 2	–	–	nk	$2nk$
Case 2′	–	–	$n(k - 1/2)$	$2n(k - 1/2)$



$\xi = 1/3$ $\xi = 2/3$ $\xi = 1/2$

Fig. 13. Real CF $\gamma_r(x)$ for various ξ in Case 1 [152].

All nonconstant eigenvalues are γ-points of CF

$$\gamma(q) := \frac{\pi q \sin(\pi q)}{2\sin((1+\xi)\pi q/2)\sin((1-\xi)\pi q/2)}, \tag{11.6$_1$}$$

$$\gamma(q) := \frac{\pi q \sin(\pi q)}{2\sin^2(\xi\pi q/2)}, \tag{11.6$_2$}$$

$$\gamma(q) := \frac{\pi q \cos(\pi q)}{2\cos((1+\xi)\pi q/2)\sin((1-\xi)\pi q/2)}, \tag{11.6$_{1'}$}$$

$$\gamma(q) := \frac{\pi q \cos(\pi q)}{\sin(\xi\pi q)}. \tag{11.6$_{2'}$}$$

Proposition 3. *[152] Zero points z of the functions $\gamma(q)$ are of the first order. These positive zeroes are equal to:*

$$z_k := k, \qquad k \in \mathbb{N}, \tag{11.7$_{1,2}$}$$

$$z_k := (k - 1/2), \quad k \in \mathbb{N}. \tag{11.7$_{1',2'}$}$$

Proposition 4. *[152] Points $p_k = 2k/\xi, k \in \mathbb{N}$ are poles of the second order for the function $\gamma(q)$ in Case 2 and there are no first order poles in this case. Other poles are of the first order, and they are equal to:*

$$p_k := \frac{2k}{1+\xi}, \quad k \in \mathbb{N} \quad \textit{and} \quad \tilde{p}_l := \frac{2l}{1-\xi}, \quad l \in \mathbb{N}, \tag{11.8$_1$}$$

$$p_k := \frac{2(k-1/2)}{1+\xi}, \quad k \in \mathbb{N} \quad \textit{and} \quad \tilde{p}_l := \frac{2l}{1-\xi}, \quad l \in \mathbb{N}, \tag{11.8$_{1'}$}$$

$$p_k := \frac{k}{\xi}, \quad k \in \mathbb{N}. \tag{11.8$_{2'}$}$$

If $\xi = r = m/n \in \mathbb{Q}$, then a part of zeroes z_j of the function $\gamma(q)$ are coincident with the poles p_k or $\tilde{p}_l$. We have two families of poles: $p_k = k/(n-m)$, $k \in \mathbb{N}_{2n}$, and $\tilde{p}_l = l/(n+m)$, $l \in \mathbb{N}_{2n}$. The poles from the first family coincide with the poles from the second family at the points

$q_j = j \in \mathbb{N}_n$, $n-m \in \mathbb{N}_e$ or $q_j = j \in \mathbb{N}_{2n}$, $n-m \in \mathbb{N}_o$. These points are zeroes of the sinus function as well, therefore they are coincident in the case of CEPs. Thus, in Case 1 all the points p_k, $k \in \mathbb{N}_{2n}$ or $\tilde{p}_l$, $l \in \mathbb{N}_{2n}$ are poles of the first order.

Points $p_k = k/m$, $k \in \mathbb{N}_{2n}$ are poles of Case 2. They are poles of the second order, except $k \in \mathbb{N}_{mn}$, $m \in \mathbb{N}_e$ and $k \in \mathbb{N}_{2mn}$, $m \in \mathbb{N}_o$ (coincident with the case of CEs), which are the first order poles. When $m = 1$ and $m = 2$, there are no poles of the second order.

In Case 1′, we also have two families of poles $p_k = k/(n-m)$, $k \in \mathbb{N}_{2n}$ and $p_l = (l-1/2)/(n+m)$, $l \in \mathbb{N}_{2n}$. All the positive poles of this problem are of the first order. These poles of two families are coincident with CEPs $c_k = n(k-1/2)$, $n-m \in \mathbb{N}_e$, $k \in \mathbb{N}$.

In Case 2′, the points $p_k = k/m$, $k \in \mathbb{N}_n$ are poles of the first order. If these PP are coincident with zeroes of the cosine function at the points $z_k = (k-1/2)n$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, we have CEPs.

11.1 Real eigenvalues of the Sturm–Liouville Problem

For complex functions (11.6) the Real CFs are:

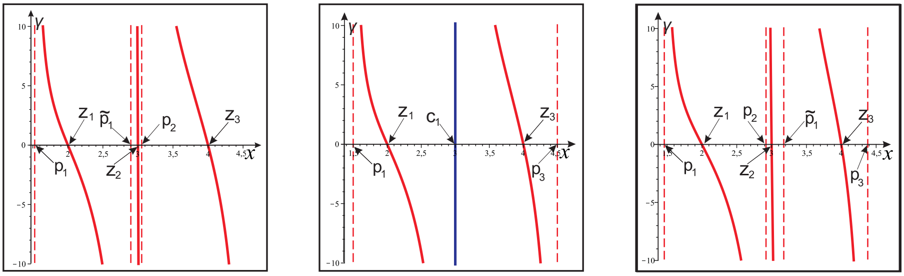
$$\gamma_r(x) := \begin{cases} \dfrac{\pi x \sinh(\pi x)}{2\sinh((1+\xi)\pi x/2)\sinh((1-\xi)\pi x/2)}, & x \leqslant 0, \\ \dfrac{\pi x \sin(\pi x)}{2\sin((1+\xi)\pi x/2)\sin((1-\xi)\pi x/2)}, & x \geqslant 0, \end{cases} \tag{11.9$_1$}$$

$$\gamma_r(x) := \begin{cases} \dfrac{\pi x \sinh(\pi x)}{2\sinh^2(\xi\pi x/2)}, & x \leqslant 0, \\ \dfrac{\pi x \sin(\pi x)}{2\sin^2(\xi\pi x/2)}, & x \geqslant 0, \end{cases} \tag{11.9$_2$}$$

$$\gamma_r(x) := \begin{cases} \dfrac{\pi x \cosh(\pi x)}{2\cosh((1+\xi)\pi x/2)\sinh((1-\xi)\pi x/2)}, & x \leqslant 0, \\ \dfrac{\pi x \cos(\pi x)}{2\cos((1+\xi)\pi x/2)\sin((1-\xi)\pi x/2)}, & x \geqslant 0, \end{cases} \tag{11.9$_{1'}$}$$

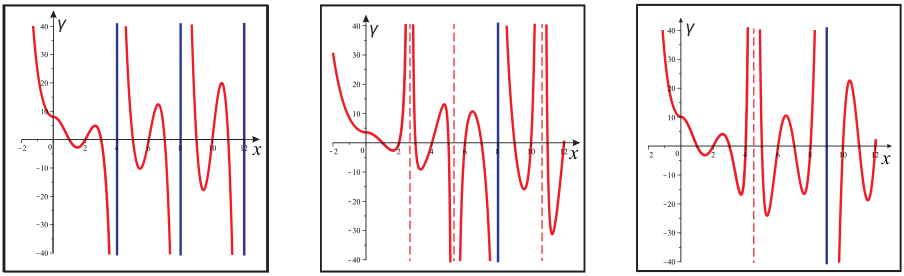
$$\gamma_r(x) := \begin{cases} \dfrac{\pi x \cosh(\pi x)}{\sinh(\xi\pi x)}, & x \leqslant 0, \\ \dfrac{\pi x \cos(\pi x)}{\sin(\xi\pi x)}, & x \geqslant 0. \end{cases} \tag{11.9$_{2'}$}$$

The graphs of these Real CFs for some parameter ξ values are presented in Figs. 13, 14, and 15. For some cases, the vertical line of the CEP is coincident with the vertical asymptotic line at the point of a pole. More properties of the Real CF and real spectrum in each case are investigated in [107, Pečiulytė *et.al.* 2008].



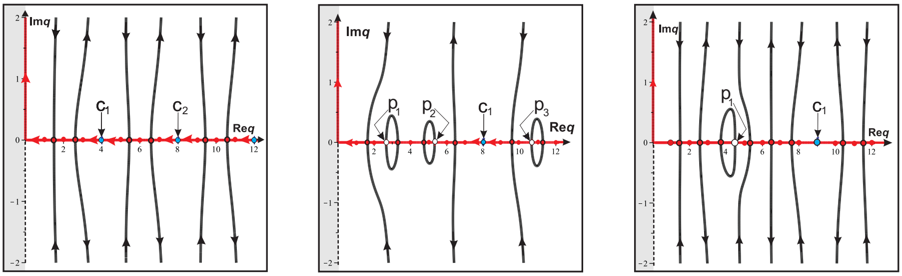
$\xi = 0.31$ $\xi = 1/3$ $\xi = 0.37$

Fig. 14. Real CF $\gamma_r(x)$ in the neighborhood of the Constant Eigenvalue Point in Case 1 [152].



$\xi = 1/2$ $\xi = 3/4$ $\xi = 4/9$

Fig. 15. Real CF $\gamma_r(x)$ for various ξ in Case 2 [152].



$\xi = 1/2$ $\xi = 3/4$ $\xi = 4/9$

Fig. 16. Spectrum Domain $\mathcal{N}$ with various ξ for the CF $\gamma(q)$ in Case 2 [152].

The spectra in Cases 1, 1′. The spectra for problems (11.1), (11.2), (11.3$_1$) and (11.1), (11.4), (11.3$_1$) lie on the real axis as shown in papers [106, 107, Pečiulytė *et.al.* 2005, 2008].

The function $\gamma_r(x)$ is a monotone decreasing function in each interval (α, β), where α and β are the points of the first order poles. In Fig. 13, we see CEPs and poles in Case 1 where the value of the parameter ξ is changing. For example, if $\xi = \xi_c = 1/3$ in Fig. 13, we see two CEPs. In this case two poles are coincident with CEP. If we change $\xi < \xi_c$ or $\xi > \xi_c$, then poles are moving from each other. Such a situation is shown in Fig. 14. We have the same situation with the Spectrum in Case $1'$. So, if the poles p_k and $\tilde{p}_l$ move toward the zero point z_r, then a part of the graph of the CF, that was in $(\tilde{p}_l, p_k)$, becomes a vertical line, i.e., we have a CEP $c_s = p_k = \tilde{p}_l = z_r$ for $\xi = \xi_c$. For $\xi > \xi_c$ we have the interval $(p_k, \tilde{p}_l)$, i.e., the poles change places with each other.

11.2 Complex eigenvalues of the Sturm–Liouville Problem

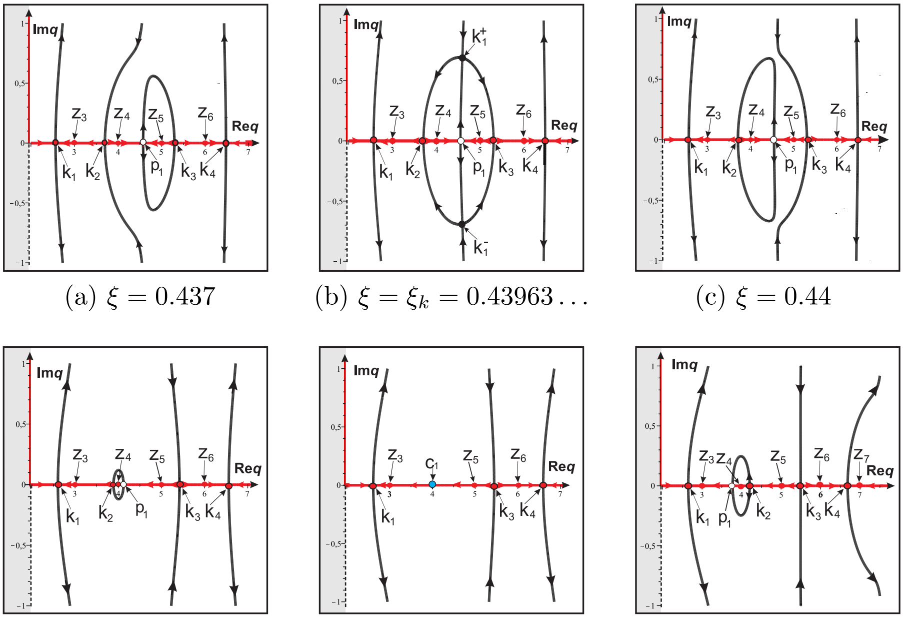
In the recent scientific literature there are many papers, in which real eigenvalues of the SLP are analysed. However, a complex spectrum of this problem is considerably less investigated [140, Sapagovas and Štikonas 2005], [166, Štikonas and Štikonienė 2009].

It is important to investigate complex eigenvalues of the SLP (11.1), (11.2), (11.3) and (11.1), (11.4), (11.3) with $\gamma \in \mathbb{R}$.

The poles of the function $\gamma(q)$ are eigenvalues of the problems (11.1)–(11.3) and (11.1), (11.4), (11.3) in the case $\gamma = \infty$. All zeros and poles of the Complex CF $\gamma(q)$ lie on the positive part of the real axis. From (11.6) and from the properties of sine and cosine functions, we have that all zeros of this function are real numbers $q = k \in \mathbb{N}$ in Cases 1, 2 and $q = (k - 1/2)$, $k \in \mathbb{N}$ in Cases $1'$,$2'$. So, only positive zeroes and poles exist in $\mathbb{C}_q$.

Dynamics of complex eigenvalues in Case 2. In this case, the spectrum of complex eigenvalues is more complicated. By changing the value of the parameter ξ we get various types of the domain $\mathcal{N} = \mathcal{D} \cup \mathcal{C}$.

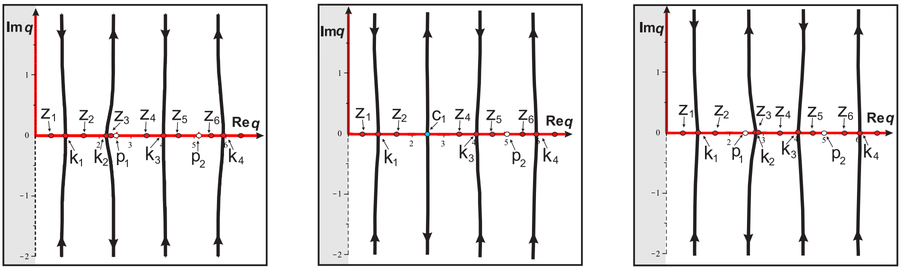
We can see a qualitative view of dependence of Spectrum Curves on the parameter ξ in Fig. 16. In Case 2, there are two types of bifurcation. The first type is where two different Spectrum Curves join at CP. We get the second type bifurcation when pole of the second order coincident with zero and CP, and we get CEP and pole (of the first order). Fig. 17 shows, how the domain $\mathcal{N}$ is changing dependent on the parameter ξ value near to $\xi_k = 0.43963\ldots$ (we have a CP in the complex part of $\mathbb{C}_q$) and $\xi_c = 0.5$ (we have CEP and pole of the first order) points. Spectrum Curves make a loop. In this example, the value of ξ is increasing from 0.437 to 0.53. When $\xi \lesssim \xi_k$, two complex eigenvalues become close, and when $\xi = \xi_k$, those different curves join each other at the CPs k_1^+ and k_1^- (see Fig. 17(a), (b), (c)). As $\xi \in (\xi_k; \xi_c)$, the loop tightens and intersects the real axis at the pole and CPs. Zero is inside the loop. When $\xi \lesssim \xi_c$ and $\xi \gtrsim \xi_c$ we have pole of the second order. If $\xi = \xi_c$ pole of the first order coincide with CEP.



(a) $\xi = 0.437$ (b) $\xi = \xi_k = 0.43963\ldots$ (c) $\xi = 0.44$

(d) $\xi = 0.485$ (e) $\xi = \xi_c = 0.5$ (f) $\xi = 0.53$

Fig. 17. Spectrum Domain $\mathcal{N}$ (Spectrum Curves) with various ξ for the CF $\gamma(q)$ in Case 2 [152].

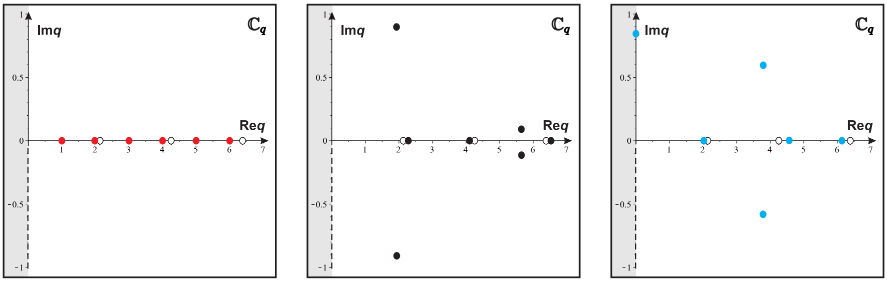


$\xi = 0.39$ $\xi = \xi_c = 0.4$ $\xi = 0.41$

Fig. 18. Spectrum Domain $\mathcal{N}$ (Spectrum Curves) with various ξ for the CF $\gamma(q)$ in Case 2′ [152].

Dynamics of complex eigenvalues in Case 2′. The complex spectrum in Case 2′ is not so complicated as in Case 2. In Fig. 18 it is shown how the spectrum of complex eigenvalues is approaching to CEP ξ_c.

If $\xi < \xi_c = 2/5$, the pole moves toward zero from right side and the curve of the complex eigenvalue moves toward zero from the left. When the pole and zero meet, we have CEP which coincide with CP. If ξ is growing, the pole moves to the left.



(a) ● – zero ($\gamma = 0$); ○ – pole (b) ● $\gamma = -17$ (c) ● $\gamma = +17$

Fig. 19. A part of the spectrum for SLP (12.1)–(12.3), $\boldsymbol{\xi} = (0.32, 0.61)$ [155].

12 Spectrum Curves for SLP with not full integral NBC [155]

A. Skučaitė in her doctoral thesis was investigated SLP

$$-u'' = \lambda u, \quad t \in (0, 1), \tag{12.1}$$

$\lambda \in \mathbb{C}$, with one classical BC

$$u(0) = 0, \tag{12.2}$$

and another integral type NBC:

$$u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\, \mathrm{d}t \tag{12.3}$$

with parameters $\gamma \in \mathbb{R}$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in S_{\boldsymbol{\xi}} := [0, 1] \times [0, 1]$.

For the case $\gamma = 0$ (classical one) eigenvalues and eigenfunctions are well known:

$$\lambda_k = (k\pi)^2, \quad v_k(t) = \sin(k\pi t), \quad k \in \mathbb{N}.$$

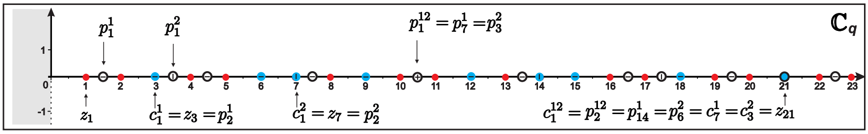
Note that the same classical problem is obtained in the limit case $\xi_1 = \xi_2$.

A nontrivial solution of the problem (12.1)–(12.3) exists if q is a root of the equation

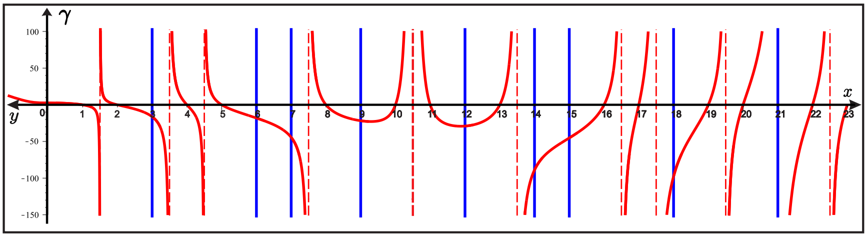
$$u_q(1) = \gamma \int_{\xi_1}^{\xi_2} u_q(t)dt, \quad u_q := \frac{\sin(\pi q t)}{\pi q}; \tag{12.4}$$

For NBC (12.3) we introduce two entire functions

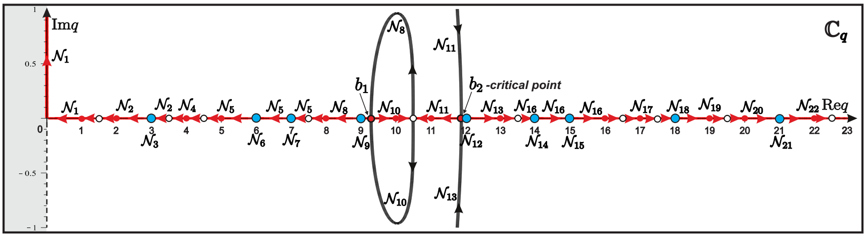
$$Z(z) := \frac{\sin(\pi z)}{\pi z}; \quad P_{\boldsymbol{\xi}}(z) := 2\frac{\sin(\pi z(\xi_1 + \xi_2)/2)}{\pi z} \cdot \frac{\sin(\pi z(\xi_2 - \xi_1)/2)}{\pi z}.$$



(a) ● – zero; ○ – pole; ● – CE Point; ● – Critical Point.



(b) Real CF



(c) projection CF onto $\mathbb{C}_q$.

Fig. 20. Zeroes, poles, CE points for SLP (12.1)–(12.3), $\boldsymbol{\xi} = (8/21, 20/21)$ [155].

Zeroes of these functions are important for description of the spectrum. Zeroes of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with EPs in the classical case $\gamma = 0$. We can rewrite equality (12.4) in the form:

$$Z(q) = \gamma P_{\boldsymbol{\xi}}(q), \quad q \in \mathbb{C}_q. \tag{12.5}$$

In Figure 19, one can see the roots (not all) of this equation for $\gamma = -17, 0, +17$ in the case $\boldsymbol{\xi} = (0.32, 0.61)$. There exist complex roots for $\gamma = -17, +17$.

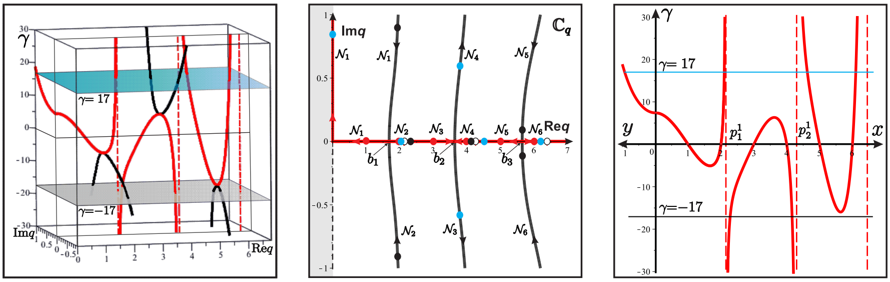
For NBC (12.3) we can find CEPs as solutions of the following system

$$Z(q) = 0, \quad P_{\boldsymbol{\xi}}(q) = 0,$$

i.e. CEP $c \in \mathbb{N}$ and $P_{\boldsymbol{\xi}}(c) = 0$. The notation $\mathcal{C}_\xi$ is used for the set of all CEPs.

If $q \notin \mathbb{N}$, i.e. $Z(q) \neq 0$, and q satisfies equation $P_{\boldsymbol{\xi}}(q) = 0$, then the equality (12.5) is not valid for all γ and such point q is PP. Notation of PP is connected with meromorphic function

$$\gamma_c(z) = \frac{Z(z)}{P_{\boldsymbol{\xi}}(z)}, \quad z \in \mathbb{C}. \tag{12.6}$$



(a) Complex-Real CF (b) Spectrum Domain (c) Real CF

Fig. 21. CF for $\boldsymbol{\xi} = (0.32, 0.61)$ and its projections [155].

This function is obtained by expressing γ from equation (12.5). If the denominator has a zero at $z = p$ and the numerator does not, then the value of the function will be infinite and we have a pole. If both parts have a zero at $z = p$, then one must compare the multiplicities of these zeroes. For our problem all zeroes $z_k = k \in \mathbb{N}$ of function $Z(z)$ are simple and positive if $z \in \mathbb{C}_q$. It follows that function $P_{\boldsymbol{\xi}}(z) = 2P^1_{\boldsymbol{\xi}}(z)P^2_{\boldsymbol{\xi}}(z)$, where

$$P^1_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_1 + \xi_2)/2)}{\pi z}, \quad P^2_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_2 - \xi_1)/2)}{\pi z}. \tag{12.7}$$

Zeroes of the functions $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ in the domain $\mathbb{C}_q$ are simple and positive, too. So, zeroes of function $P_{\boldsymbol{\xi}}$ can be simple or of the second order.

In Figure 21(a) one can see CF graph in the case $\boldsymbol{\xi} = (0.32, 0.61)$. CF $\gamma(q)$ is the restriction of function $\gamma_c(q)$ on a set $\mathcal{D} := \{q \in \mathbb{C}_q \colon \mathrm{Im}\gamma_c(q) = 0\}$. *Real CF* $\gamma(q)$ is defined on the domain $\{q \in \mathbb{C}_q \colon \lambda = (\pi q)^2 \in \mathbb{R}\}$ and describes only real eigenvalues. One can see the Real CF graph in Figure 20(b) for $\boldsymbol{\xi} = (8/21, 20/21)$ and in Figure 21(c) for $\boldsymbol{\xi} = (0.32, 0.61)$. In the case $\boldsymbol{\xi} = (8/21, 20/21)$ the vertical lines are added at the CEPs.

Spectrum Domain is the set $\mathcal{N} = \mathcal{D} \cup \mathcal{C}$. $\mathcal{D} = \bigcup_{\gamma \in \mathbb{R}} \mathcal{E}(\gamma)$, where $\mathcal{E}(\gamma_0) = \gamma^{-1}(\gamma_0)$ is the set of all nonconstant Eigenvalues Points for $\gamma_0 \in \mathbb{R}$, i.e. $\gamma|_{\mathcal{E}(\gamma_0)} \equiv \gamma_0$. Example of the Spectrum Domain one can see in Figure 21(b). We also add the EPs ($\gamma = -17, 0, +17$) from Figure 19 and PPs ($\gamma = \infty$). Spectrum Domain is symmetric with respect the real axis for $\mathrm{Re}\, q > 0$. If $\gamma = 0$, then the EPs are $q = z_k = k \in \mathbb{N}$. If $q \in \mathcal{E}(\gamma)$ is not CP then the set $\mathcal{E}(\gamma)$ is smooth curve $\mathcal{N} = \mathcal{N}(\gamma)$ in the neighbourhood of this point. We can numerate the part of $\mathcal{N}(\gamma)$ for this point by the classical case $\mathcal{N}_k(0) = z_k$, $k \in \mathbb{N}$. For every CEP $c_j = j$ we define $\mathcal{N}_j = \{c_j\}$, i.e. every such $\mathcal{N}_j$ has one point only (see Figure 20(c), Figure 21(b)). We call every $\mathcal{N}_k$, $k \in \mathbb{N}$, as a *Spectrum Curve*.

Remark 5. In the article [155, A. Skučaitė and Štikonas 2015] the term *Spectrum Curve* was used the first time.

13 Conclusions

Investigation of the Spectrum Curves gives useful information about the spectrum for problems with NBC. Part of the spectrum is defined by CEPs (nonregular Spectrum Curves) and do not depend on parameter γ in NBC. Another part of the spectrum depends on parameter γ and can be described by regular Spectrum Curves which are defined on special domain on part of complex plane. On this domain we can investigate CF. Zeroes, poles and CPs of this CF are important for investigation of complex eigenvalues and Spectrum Curves in the complex plane. Spectrum Curves in complex domain can be found only numerically.

Chapter 1

Sturm–Liouville Problem with one Dirichlet boundary condition and two-points Nonlocal Boundary Condition

Introduction

In this chapter we use *Characteristic Function* method [166, Štikonas and Štikonienė 2009] for investigation of the spectrum for differential SLP with two-point NBC. We describe zeroes, poles, critical points of Characteristic Function, *Constant Eigenvalue Points.* We investigate how Spectrum Curves depend on parameter ξ in NBC. The main results of this Chapter 1 are published in [9, Bingelė, Bankauskienė and Štikonas, 2020].

1 Sturm–Liouville problem with NBC

Let us analyze SLP with one classical BC and another two-point NBC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

$$u(1) = \gamma u'(\xi), \tag{1.3}$$

where parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. The eigenvalue $\lambda \in \mathbb{C}_\lambda := \mathbb{C}$ and eigenfunction $u(t)$ can be complex function.

If $\gamma = 0$, then we have the SLP with classical BCs. In this case eigenvalues and eigenfunction are known:

$$\lambda_k = (k\pi)^2, \qquad u_k(t) = \sin(k\pi t),\ k \in \mathbb{N} \tag{1.4}$$

We also use notation $\mathbb{N}_o := \{2k-1,\, k \in \mathbb{N}\}$, $\mathbb{N}_e := \{2k,\, k \in \mathbb{N}\}$. In the case $\xi = 1$, we obtain the third type (classical) BC. The case $\gamma = \infty$ corresponds to (1.1) with clasical BCs $u(0) = 0$ and $u'(\xi) = 0$, $\xi \in [0, 1]$, instead of condition (1.3) and eigenvalues and eigenfunction are:

$$\lambda_k = \big((k-1/2)\pi/\xi\big)^2, \qquad u_k(t) = \sin\big((k-1/2)\pi t/\xi\big),\ k \in \mathbb{N}. \tag{1.5}$$

If $\lambda = 0$, then a function $u(t) = Cu_0(t)$, where $u_0(t) := t$, satisfy (1.1) equation and BC (1.2). Substituting this function into the second NBC (1.3), we obtain that eigenvalue $\lambda = 0$ ($C \neq 0$) exists if and only if $\gamma = 1$.

If $\lambda \neq 0$ function $u(t) = Cu_q(t)$, $u_q(t) = \sin(\pi qt)/(\pi q)$, satisfies equation (1.1) and BC (1.2), where $\lambda = (\pi q)^2$. If we consider a map $\lambda : \mathbb{C}_\lambda \to \mathbb{C}$, $\lambda(q) = (\pi q)^2$, the inverse map is multivalued and point $\lambda = 0$ is the first order *Branch Point* (BP) of the second order. This point is important for our investigation and we call $q = 0$ *Ramification Point* (RP).

In this chapter $q \in \mathbb{C}_q := \mathbb{R}_q \cup \mathbb{C}_q^+ \cup \mathbb{C}_q^-$, where $\mathbb{R}_q := \mathbb{R}_q^- \cup \mathbb{R}_q^+ \cup \mathbb{R}_q^0$, $\mathbb{R}_q^- := \{q = x + \iota y \in \mathbb{C} : x = 0, y > 0\}$, $\mathbb{R}_q^+ := \{q = x + \iota y \in \mathbb{C} : x > 0, y = 0\}$, $\mathbb{R}_q^0 := \{q = 0\}$, $\mathbb{C}_q^+ := \{q = x + \iota y \in \mathbb{C} : x > 0, y > 0\}$ and $\mathbb{C}_q^- := \{q = x + \iota y \in \mathbb{C} :\ x > 0,\ y < 0\}$. Then a map $\lambda = (\pi q)^2$ is the bijection between $\mathbb{C}_q$ and $\mathbb{C}_\lambda$ [166, 2009]. Here $q = 0$ corresponds to $\lambda = 0$. This bijection is a conformal map, except the point $q = 0$. For each eigenvalue λ for SLP corresponds *Eigenvalue Point* (EP) $q \in \mathbb{C}_q$. Real eigenvalues are described by EP $q \in \mathbb{R}_q = \{q \in \mathbb{C}_q \colon \lambda = (\pi q)^2 \in \mathbb{R}\}$. If $\lambda = 0$ is eigenvalue then $q = 0$ we call as *Branch Eigenvalue Point* (BEP).

2 Constant Eigenvalues and Characteristic Function

We substitute $u_q(t)$ to BC (1.3) and get $u_q(1) = \gamma u_q'(\xi)$. So, a nontrivial solution of the problem (1.1)–(1.3) exists if $q \in \mathbb{C}_q$ is the root of a equation

$$\frac{\sin(\pi q)}{\pi q} = \gamma \cos(\xi \pi q). \tag{2.1}$$

For NBC (1.3) we introduce two entire functions:

$$Z(z) := \frac{\sin(\pi z)}{\pi z}; \quad P_\xi(z) := \cos(\xi \pi z), \quad z \in \mathbb{C}. \tag{2.2}$$

In this section we investigate relations between spectrum of SLP (1.1)–(1.3) and parameter γ for fixed ξ.

Zeroes points. Zeroes of these functions are important in analyzing and describing the spectrum. Moreover, zeroes z_k of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with EPs in the classical case $\gamma = 0$ (the graphs in Figure 1.1(a), (d)):

$$z_l = l \in \mathbb{N}. \tag{2.3}$$

We denote the corresponding set of points $\hat{\mathcal{Z}}$. All zeroes are simple, real (positive integer numbers).

In the case $\xi \neq 0$ all zeroes of the function $P_\xi(q)$ (see (2.2)) in $\mathbb{C}_q$ are simple, real and positive:

$$\overline{\mathcal{Z}}_\xi := \{p_k = (k - 1/2)/\xi, \ \ k \in \mathbb{N}\}. \tag{2.4}$$

In the case $\xi = 0$ the set $\overline{\mathcal{Z}}_\xi = \varnothing$.

We rewrite the equation (2.1) in the form:

$$Z(q) = \gamma P_\xi(q), \quad q \in \mathbb{C}_q. \tag{2.5}$$

Constant Eigenvalues. We will define a *Constant Eigenvalue* (CE) as the eigenvalue which does not depend on the parameter γ [166, 2009]. Then for any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \qquad P_\xi(q) = 0, \tag{2.6}$$

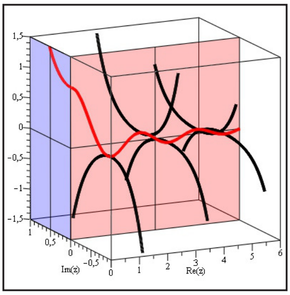
i.e., CEP $c \in \mathbb{N}$ and $P_\xi(c) = 0$. We denote the set of all CEP as $\mathcal{C}_\xi = \hat{\mathcal{Z}} \cap \overline{\mathcal{Z}}_\xi$. If $c \in \mathcal{C}_\xi$ and

$$Z(z) = (z-c)^\alpha \widetilde{Z}(z), \ \widetilde{Z}(c) \neq 0, \qquad P_\xi(z) = (z-c)^\beta \widetilde{P}(z), \ \widetilde{P}(c) \neq 0, \tag{2.7}$$

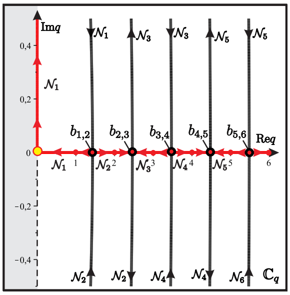
where $\alpha, \beta \in \mathbb{N}$, then number $\min\{\alpha, \beta\}$ is the order of CEP. For SLP (1.1)–(1.3) all CEPs are of the first order.

Remark 1.1. If the parameter $\xi = 0$, then from the formula (2.2) we obtain that $P_\xi \equiv 1$. So, $\overline{\mathcal{Z}}_\xi = \varnothing$ and CEPs do not exist. If $\xi = 1$ then there are no CEPs, because the functions $\sin(\pi q)$ and $\cos(\pi q)$ have no common zeroes (we have the third type BC).

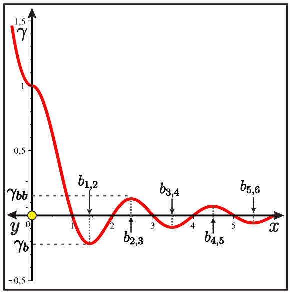
Remark 1.2. If the parameter $\xi \notin \mathbb{Q}$, then CEPs do not exist, because the equation $\xi l = k - \frac{1}{2}$ has not roots for $l, k \in \mathbb{N}$.



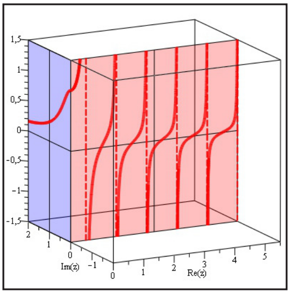
(a) CF ($\xi = 0$)



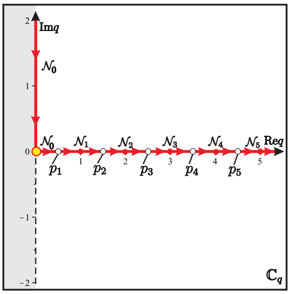
(b) Spectrum Curves ($\xi = 0$)



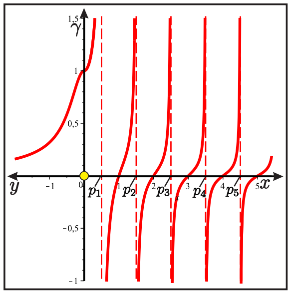
(c) Real CF ($\xi = 0$)



(d) CF ($\xi = 1$)



(e) Spectrum Curves ($\xi = 1$)



(f) Real CF ($\xi = 1$)

Fig. 1.1. CF, Spectrum Domain, Real CF for $\xi = 0$, $\xi = 1$. ● – Zero Point, ○ – Pole Point, ● – Ramification Point, ○ – Branch Eigenvalue Point, ● – Critical Point.

Let consider that $\xi \in \mathbb{Q}$, $\xi = m/n$, $m, n \in \mathbb{N}$, $0 < m \leqslant n$, and $\gcd(m, n) = 1$, where $\gcd(m, n)$ is the greatest common divisor of two positive integers m and n. In this case the system (2.6) is equivalent to equation $lm/n = k - 1/2$, where k, $l \in \mathbb{N}$ are unknowns. We can rewrite this equation in the following form

$$nk - lm = \tfrac{n}{2}. \tag{2.8}$$

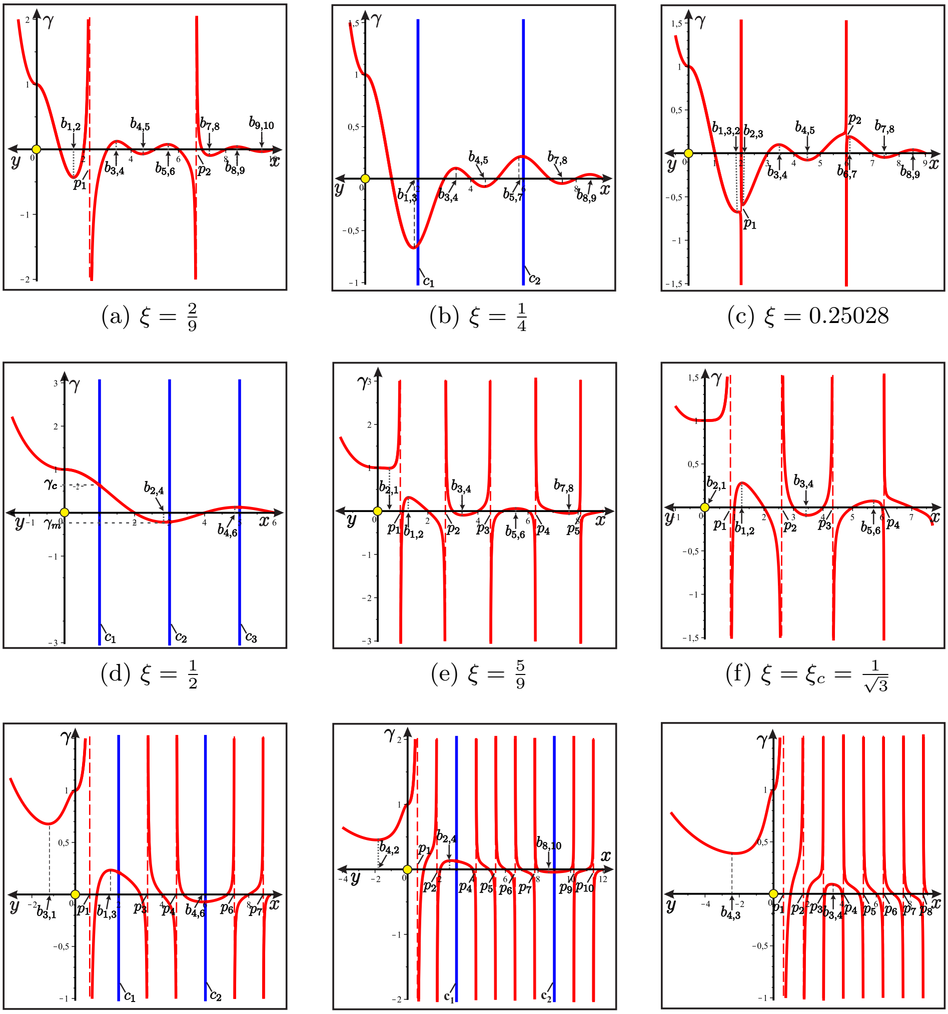
Remark 1.3. If $n \in \mathbb{N}_e$ then the right hand side of this equation is integer number. If $n \in \mathbb{N}_o$ then equation (2.8) has no roots.

Theorem 1.4. ([32], Gelfond 1978) *If* $\gcd(a, b) = 1$ *and* (α, β) *is any solution of equation:*

$$ax + by + c = 0, \tag{2.9}$$

then all solutions of this equation have a form;

$$x = \alpha - bt, \quad y = \beta + at, \quad t \in \mathbb{Z}.$$



(a) $\xi = \frac{2}{9}$ (b) $\xi = \frac{1}{4}$ (c) $\xi = 0.25028$

(d) $\xi = \frac{1}{2}$ (e) $\xi = \frac{5}{9}$ (f) $\xi = \xi_c = \frac{1}{\sqrt{3}}$

(g) $\xi = \frac{3}{4}$ (h) $\xi = \frac{5}{6}$ (i) $\xi = \frac{6}{7}$

Fig. 1.2. Real CF $\gamma_r(q;\xi)$ for various parameter ξ values.

Remark 1.5. Any solution (α, β) of (2.9), $\gcd(a, b) = 1$ can be found using Euclidean algorithm [32] and solving the classical equation

$$ax + by = \gcd(a, b).$$

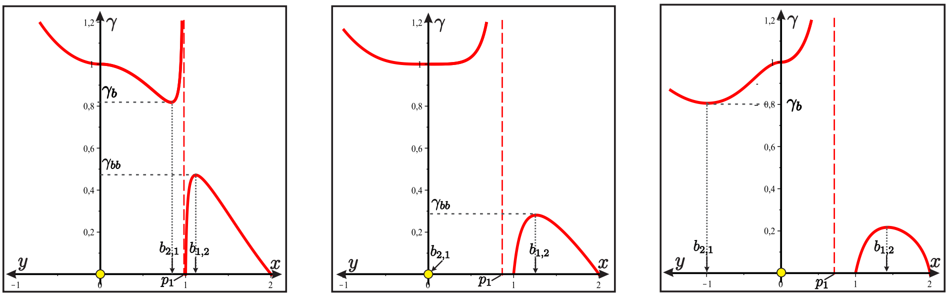
Lemma 1.6. *For SLP* (1.1)–(1.3) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0, 1)$*,* $m \in \mathbb{N}_o$*,* $n \in \mathbb{N}_e$*, values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$*,* $c_s := (s - 1/2)n$*,* $s \in \mathbb{N}$*.*

Proof. The case $\xi \notin \mathbb{Q}$ was discussed in Remark 1.2. Let $\xi \in \mathbb{Q}$, $\xi = m/n$, and from Remark 1.3 we have that $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$. From equation (2.8) we see that $l = \tau n/2$, $\tau \in \mathbb{N}$. We rewrite equation (2.8) as $m\tau = 2k - 1$. Right

Table 1. The first two real eigenvalues $\lambda_0 = \lambda_r(x_0)$ and $\lambda_1 = \lambda_r(x_1)$ ($\gamma \neq 0$).

ξ	γ	x_0	x_1
$\xi = 0$	$\gamma_b \leqslant \gamma < 0$	$(1, x_b]$	$[x_b, 2)$
$\gamma_b = \min\limits_{1<x<2} \gamma_r(x) = \gamma_r(x_b)$	$0 < \gamma \leqslant \gamma_{bb}$	$(0, 1)$	$(2, x_{bb}]$
	$\gamma_{bb} < \gamma < 1$	$(0, 1)$	$-$
$\gamma_{bb} = \max\limits_{2<x<3} \gamma_r(x) = \gamma_r(x_{bb})$	$\gamma = 1$	$= 0$	$-$
	$\gamma > 1$	< 0	$-$
$0 < \xi < 1/2$	$\gamma < 0$	> 1	> 1
	$0 < \gamma < 1$	$(0, 1)$	> 1
	$\gamma = 1$	$= 0$	> 1
	$\gamma > 1$	< 0	> 1
$\xi = 1/2$	$\gamma < \gamma_m$	$= 1$	$= 2$
	$\gamma_m \leqslant \gamma < 0$	$= 1$	$(2, x_m]$
	$0 < \gamma < \gamma_c$	$= 1$	$(1, 2)$
$\gamma_c = 2/\pi$	$\gamma = \gamma_c$	$= 1$	$= 1$
$\gamma_m = \min\limits_{2<x<3} \gamma_r(x) = \gamma_r(x_m)$	$\gamma_c < \gamma < 1$	$(0, 1)$	$= 1$
	$\gamma = 1$	$= 0$	$= 1$
	$\gamma > 1$	< 0	$= 1$
$1/2 < \xi < 1/\sqrt{3}$	$\gamma \leqslant \gamma_{bb}$	$(p_1, x_{bb}]$	$[x_{bb}, 2)$
$\gamma_b = \min\limits_{0<x<p_1} \gamma_r(x) = \gamma_r(x_b)$	$\gamma_{bb} < \gamma < \gamma_b$	$(p_2, 3)$	$(4, p_3)$
	$\gamma_b \leqslant \gamma < 1$	$(0, x_b]$	$[x_b, p_1)$
$\gamma_{bb} = \max\limits_{1<x<2} \gamma_r(x) = \gamma_r(x_{bb})$	$\gamma = 1$	$= 0$	(x_b, p_1)
	$\gamma > 1$	< 0	(x_b, p_1)
$\xi = 1/\sqrt{3}$	$\gamma < 0$	$(p_1, 1)$	$(2, p_2)$
$p_1 = \sqrt{3}/2$, $p_2 = 3\sqrt{3}/2$,	$0 < \gamma \leqslant \gamma_{bb}$	$(1, x_{bb}]$	$[x_{bb}, 2)$
$p_3 = 5\sqrt{3}/2$	$\gamma_{bb} < \gamma < 1$	$(p_2, 3)$	$(4, p_3)$
$\gamma_{bb} = \max\limits_{1<x<2} \gamma_r(x) = \gamma_r(x_{bb})$	$\gamma = 1$	$= 0$	$= 0$
	$\gamma > 1$	< 0	$(0, p_1)$
$1/\sqrt{3} < \xi < 1$	$\gamma < \gamma_b$	(p_1, p_3)	> 1
	$\gamma_b \leqslant \gamma < 1$	$\leqslant x_b$	$[x_b, 0)$
$\gamma_b = \min\limits_{x<0} \gamma_r(x) = \gamma_r(x_b)$	$\gamma = 1$	$< x_b$	$= 0$
	$\gamma > 1$	$< x_b$	$(0, p_1)$
$\xi = 1$	$\gamma < 0$	$(p_1, 1)$	$(p_2, 2)$
$p_1 = 1/2$, $p_2 = 3/2$	$0 < \gamma < 1$	< 0	$(1, p_2)$
	$\gamma = 1$	$= 0$	$(1, p_2)$
	$\gamma > 1$	$(0, p_1)$	$(1, p_2)$

side of this equation is odd number. So, τ must be odd number too, i.e., $\tau = 2t-1$, $t \in \mathbb{N}$. For t and k we have equation $mt-k = (m-1)/2$. Because $t_0 = 0$ and $k_0 = -(m-1)/2$ satisfy this equation, then the solution of this equation is $t = s$ and $k = -(m-1)/2 + ms$, according to Theorem 1.4. If $s \in \mathbb{N}$ then t, $k \in \mathbb{N}$. Therefore, we obtain that $q = l = (s - 1/2)n$, $s \in \mathbb{N}$, is CEP. $\square$



(a) $\xi = 0.51$ (b) $\xi = \xi_c = \frac{1}{\sqrt{3}} \approx 0,5773$ (c) $\xi = 0.7$

Fig. 1.3. Real CF at RP $q = 0$.

Characteristic Functions. Let to consider meromorphic function:

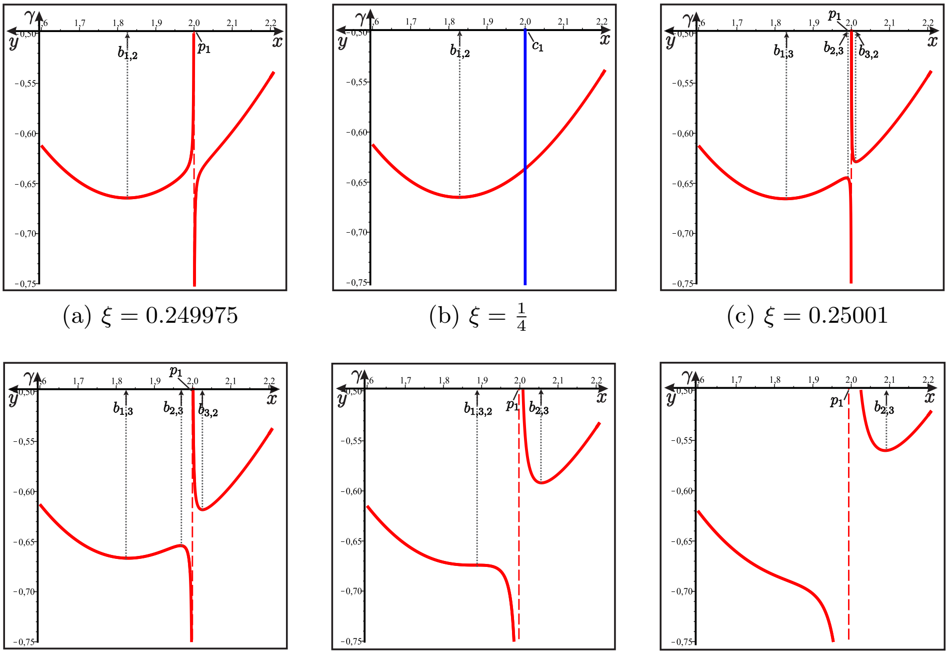
$$\gamma_c(z) = \gamma_c(z;\xi) := \frac{Z(z)}{P_\xi(z)}, \quad z \in \mathbb{C}, \tag{2.10}$$

where Z and P_ξ are entire functions (2.2). If $\lim_{z\to p} \gamma_c = \infty$ then we have a *Pole Point* (PP) at $z = p$. We have pole of the β order in the case $Z(p) \neq 0$ and $P_\xi(z) = (z-p)^\beta \widetilde{P}(z)$, $\widetilde{P}((p) \neq 0$, $\beta \in \mathbb{N}$. If $p \in \mathcal{C}_\xi$ and condition (2.7) is valid with $\beta > \alpha$, then pole $z = c$ is of the order $\beta - \alpha$. In the case $0 \leqslant \beta < \alpha$ function γ_c has zero of order $\alpha - \beta$. If $\beta = \alpha$ then point $z = c$ is removable singularity isolated point and $0 < |\gamma_c(c)| < \infty$.

We call the expression of the meromorphic function γ_c on $\mathbb{C}_q$ *Complex Characteristic Function* (Complex CF) [166, Štikonas and Štikonienė 2009], [150, A. Skučaitė 2016]. γ-points of Complex CF define EPs (and Eigenvalues, too) which depend on parameter γ. We call such EPs *Nonconstant Eigenvalue's Points* (NEP) and corresponding Eigenvalues as *Nonconstant Eigenvalues.*

As we noted, functions Z and P_ξ for SLP (1.1)–(1.3) have simple zeroes only. If $Z(z) = 0$ and $P_\xi(z) \neq 0$ then we have zero point of CF at the point z; if $Z(p) \neq 0$ and $P_\xi(p) = 0$ then we have PP of CF at the point p. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. So, $p_k \in \overline{\mathcal{Z}}_\xi$ is PP if and only if $p_k \notin \mathbb{N}$. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. If $c \in \mathcal{C}_\xi$, i.e. $Z(c) = 0$ and $P_\xi(c) = 0$, then we have removable singularity isolated point and we have (for $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$) sequence of such points

$$\begin{aligned} &c_s = p_{k_s} = z_{l_s} = n(s - 1/2), \;\; s \in \mathbb{N}, \\ &k_s = m(s - 1/2) + 1/2, \;\; l_s = n(s - 1/2). \end{aligned} \tag{2.11}$$



(a) $\xi = 0.249975$ (b) $\xi = \frac{1}{4}$ (c) $\xi = 0.25001$

(d) $\xi = 0.25005$ (e) $\xi = \xi_{2b}$ (f) $\xi = 0.2508$

Fig. 1.4. Real CF in the neighborhood CEP $x = 2$ for $\xi = 1/4$ and in the neighborhood of the second order critical point for $\xi = \xi_{2b} \approx 0.25028$.

Remark 1.7. In the case $\xi = 0$ function $P_\xi \equiv 1$ and PPs do not exist. If $\xi > 0$ a set of poles $\mathcal{P}_\xi = \varnothing$ or countable. It follows from (2.11) that $1 = k_1 = (m+1)/2$ if $m = 1$. So, PPs exist if $\xi \neq 1/n$.

Remark 1.8. Case $\xi = 1/n$, $n \in \mathbb{N}$. From de Moivre formula:

$$\begin{aligned} \sin(2kz) &:= 2k\cos^{2k-1} z \sin z - \binom{2k}{3} \cos^{2k-3} z \sin^3 z + ... \\ &\quad +(-1)^{k-1} 2k \cos z \sin^{2k-1} z, \quad k \in \mathbb{N} \cup \{0\}. \qquad (2.12) \\ \sin((2k+1)z) &:= (2k+1)\cos^{2k} z \sin z - \binom{2k+1}{3} \cos^{2k-2} z \sin^3 z + ... \\ &\quad +(-1)^k \sin^{2k} z, \quad k \in \mathbb{N} \cup \{0\}, \qquad (2.13) \end{aligned}$$

we have $\mathcal{P}_\xi = \overline{\mathcal{Z}}_\xi$, $\mathcal{C}_\xi = \varnothing$ for $\xi = 1/(2k+1)$, $k \in \mathbb{N}$; $\mathcal{P}_\xi = \varnothing$, $\mathcal{C}_\xi = \overline{\mathcal{Z}}_\xi$ for $\xi = 1/(2k)$, $k \in \mathbb{N}$. So, for $n \in \mathbb{N}_o$ there are PPs, but CEPs do not exist, for $n \in \mathbb{N}_e$ there are no PPs and we have CEPs.

The same result it follows from formulae $p_k = (k - 1/2)n$, $k \in \mathbb{N}$, and $c_s = (s - 1/2)n$, $s \in \mathbb{N}$, $n \in \mathbb{N}_e$.

Remark 1.9. A point $q = \infty \notin \mathbb{C}_q$. This point is singular (isolated essential point if $\mathcal{P}_\xi = \varnothing$, otherwise we have cluster of poles) point.

Complex-Real Characteristic Function (CF) [166, 2009] $\gamma = \gamma(q)$ is the restriction of Complex CF γ_c on a domain $\mathcal{D}_\xi := \{q \in \mathbb{C}_q : \mathrm{Im}\gamma_c(q) = 0\}$, i.e., $\gamma\colon \mathcal{D}_\xi \to \mathbb{R}$. CF $\gamma(q)$ describes the value of the parameter γ at the point $q \in \mathcal{D}_\xi$, such that there exist the Nonconstant Eigenvalue $\lambda = (\pi q)^2$. A set $\mathcal{E}_\xi(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all NEPs for $\gamma_0 \in \mathbb{R}$. So, $\mathcal{D}_\xi = \cup_{\gamma\in\mathbb{R}}\mathcal{E}_\xi(\gamma)$. The *Spectrum Domain* $\mathcal{N}_\xi$ is the set $\mathcal{D}_\xi \cup \mathcal{C}_\xi$ for fixed ξ [155, A. Skučaitė and Štikonas 2015]. We denote sets $\overline{\mathcal{D}_\xi} := \mathcal{D}_\xi \cup \mathcal{P}_\xi \cup \{\infty\}$, $\partial\mathcal{D}_\xi = \mathcal{P}_\xi \cup \{\infty\}$. We can see the Spectrum Domain in Figure 1.5 for various ξ.

Remark 1.10. Examples of CF graphs are presented in Figure 1.1(a),(d) for $\xi = 0$ and $\xi = 1$. We see Spectrum Domains for these cases in Figure 1.1(b),(e). In the case $\xi = 1$ Spectrum Domain $\mathcal{D}_1 \subset \mathbb{R}_q$. So, all eigenvalues are real. In the case $\xi = 0$ part of $\mathcal{D}_0$ belongs to $\mathbb{C}_q^+ \cup \mathbb{C}_q^-$ and complex eigenvalues exist for some values of γ. If the parameter $\xi = 0$ then from the equation (2.1) we obtain that CF have no PPs (see Remark 1.1 and Figure 1.1(a)–(c)). If $\xi = 1$ then there are no CEPs, but we have family of poles p_k, $k \in \mathbb{N}$, defined by formula (2.4) (see Figure 1.1(d)–(f)).

Real Eigenvalues. *Real Characteristic Function* (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$. We can use the argument $x \in \mathbb{R}$ for Real CF:

$$\gamma_r(x) = \gamma_r(x;\xi) := \begin{cases} \gamma(-\imath x;\xi) = \frac{\sinh(\pi x)}{\pi x\cosh(\xi\pi x)}, & x \leqslant 0; \\ \gamma(x;\xi) = \frac{\sin(\pi x)}{\pi x\cos(\xi\pi x)}, & x \geqslant 0. \end{cases} \tag{2.14}$$

This function is useful for investigation of real negative, zero and positive eigenvalues

$$\lambda = \lambda_r(x) = \lambda_r(x;\xi) := \begin{cases} -(\pi x)^2, & x \leqslant 0; \\ (\pi x)^2, & x \geqslant 0. \end{cases} \tag{2.15}$$

For positive and zero eigenvalues EPs for CF γ and Real CF γ_r are the same. For negative eigenvalues EPs for CF γ and Real CF γ_r are related by formula $q = -\imath x$, $x < 0$. The graphs of these Real CFs for some parameter ξ values are presented in Figure 1.1(c),(f), Figure 1.2. In Figure 1.2 the vertical (blue) solid lines correspond to the CEP, vertical (red) dashed lines cross the x-axis at the PPs. Real CF for SLP (1.1)–(1.3) was investigated in [102, Pečiulytė and Štikonas]. In this paper values of Real CF and it's derivatives at CEP $c_s := (s - 1/2)n$, $s \in \mathbb{N}$, $\xi = m/n \in (0, 1)$, $m \in \mathbb{N}_o$,

$n \in \mathbb{N}_e$, were found:

$$\gamma_s(\xi) := \gamma(c_s;\xi) = (-1)^{s+(n-m+1)/2} c_s^{-1}\xi^{-1}\pi^{-1}, \qquad (2.16)$$
$$\gamma_s'(\xi) := \gamma'(c_s;\xi) = -c_s^{-1}\gamma_s, \qquad (2.17)$$
$$\gamma_s''(\xi) := \gamma''(c_s;\xi) = \big(2c_s^{-2} - \pi^2(1-\xi^2)/3\big)\gamma_s. \qquad (2.18)$$

We see, that $\gamma_s \neq 0$, $\gamma_s' \neq 0$, $\gamma_s'' \neq 0$ for all ξ and s. Graphs of Real CF in the neighborhood CEP are presented in Figure 1.4(b) (see Figure 1.4(a),(c), too).

Some results about the first two real eigenvalues are presented in Table 1. We note, that in the case $\xi = 0$ and $\gamma_r < \gamma_b$ real eigenvalues do not exist. The location of these two eigenvalues can be more accurate if we take smaller intervals of parameter ξ. In [107, Pečiulytė *et al.* 2008] statements about negative eigenvalues were formulated. For example, if $\xi = 1/\sqrt{3}$ then double negative eigenvalue exists. Some results about Real CF one can find in [102, 103, Pečiulytė and Štikonas 2006, 2007].

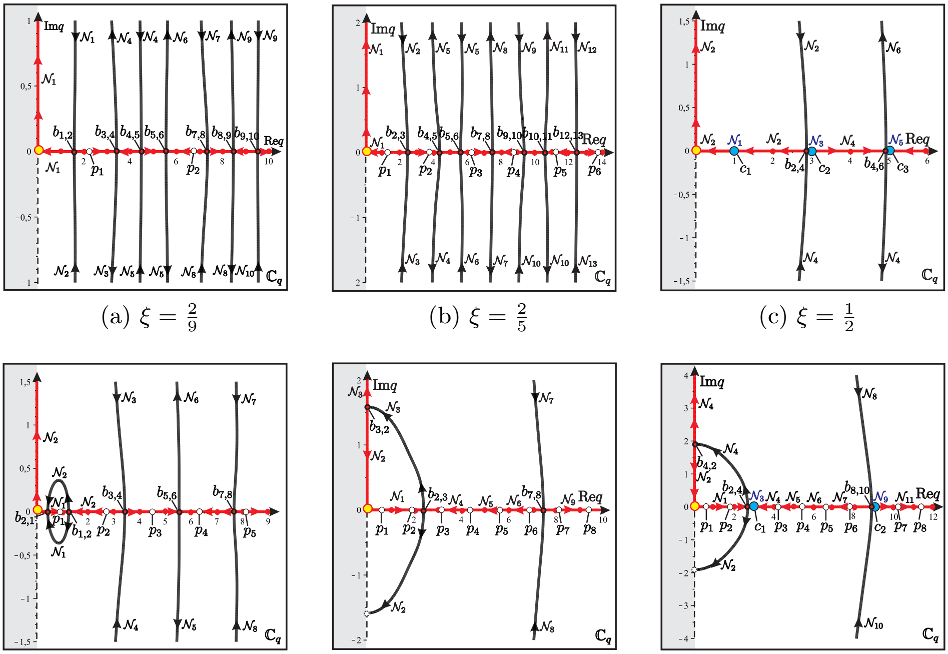
Ramification Point. The Taylor series for CF $\gamma(q)$ at RP $q = 0$ is:

$$\begin{aligned}\gamma(q;\xi) \ := \ & 1 + \big(-\tfrac{1}{6} + \tfrac{1}{2}\xi^2\big)\pi^2 q^2 \\ & + \big(\tfrac{1}{120} - \tfrac{1}{24}\xi^4 - \big(\tfrac{1}{2}\big(\tfrac{1}{6} - \tfrac{1}{2}\xi^2\big)\big)\xi^2\big)\pi^4 q^4 + \mathcal{O}(q^6), \quad q \in \mathbb{C}_q. \qquad (2.19)\end{aligned}$$

Multiplier of q^2 is positive if, $\xi > \xi_c = 1/\sqrt{3}$, and negative if $\xi < \xi_c$. When $\xi = \xi_c$ the second term vanishes in (2.19), and the coefficient in the third term of this series is positive and equal $\pi^4/270 > 0$. Graphs of Real CF in the neighborhood RP $q = 0$ are presented in Figure 1.3.

Critical Points. For the SLP (1.1)–(1.3) with two-points NBCs we obtain three types of critical points: the first, the second and the third order. Let to consider function γ_c (2.10). If $\gamma_c'(b) = 0$, $b \in \mathbb{C}$, then b we call *Critical Point* (CP) of the function γ_c, and value $\gamma_c(b)$ is a *critical value* of the function γ_c [160, 2007]. CPs are saddle points of Complex CF. For Real CF it can be a half-saddle points, maximum or minimum points and also can be inflection points. CPs of the CF are important for investigation of multiple eigenvalues. The critical point b depend on the parameter ξ continuously. If the function γ_c at CP $b \in \mathbb{C}_q$ satisfies $\gamma_c'(b) = 0, \ldots, \gamma_c^{(k)}(b) = 0$, $\gamma_c^{(k+1)}(b) \neq 0$, then b is called the *k-order Critical Point* (kCP). The set of CPs we denote $\mathcal{K}_\xi$.

In paper [107, Pečiulytė *et al.* 2008] the properties of negative CPs are investigated. For the SLP (1.1)–(1.3) we have one negative CP ($b \in \mathbb{R}_q^-$) if $\sqrt{3}/3 < \xi < 1$.



(a) $\xi = \frac{2}{9}$ (b) $\xi = \frac{2}{5}$ (c) $\xi = \frac{1}{2}$

(d) $\xi = \frac{5}{9}$ (e) $\xi = \frac{4}{5}$ (f) $\xi = \frac{5}{6}$

Fig. 1.5. (a)–(f) Spectrum Domain (Spectrum Curves) for various parameter ξ values. ● – Constant Eigenvalue Point.

Remark 1.11. In the case $\xi = \xi_c = \frac{1}{\sqrt{3}}$ the point $q = 0$ is 3CP in the domain $\mathbb{C}_q$, but for $\lambda = 0$ it is only 1CP, because $q = 0$ is RP for map $\lambda = (\pi q)^2$. In the complex plane $\mathbb{C}_\lambda$ the Taylor series (2.19) have a form

$$\gamma(\lambda, \xi) := 1 + \left(-\tfrac{1}{6} + \tfrac{1}{2}\xi^2\right)\lambda + \left(\tfrac{1}{120} - \tfrac{1}{24}\xi^4 - \left(\tfrac{1}{2}\left(\tfrac{1}{6} - \tfrac{1}{2}\xi^2\right)\right)\xi^2\right)\lambda^2 + \mathcal{O}(\lambda^3). \quad (2.20)$$

If $\xi \neq \xi_c$, then point $q = 0$ and $\lambda = 0$ are not CPs.

The *first order real critical point* (1CP) $b \in \mathring{\mathbb{R}}_q = \mathbb{R}_q^- \cup \mathbb{R}_q^+$ can be found as root of an equation

$$\gamma'(b; \xi) = 0 \quad (2.21)$$

for fixed ξ. For example, when $\xi = 0$ we can see 1CP ($b_{1,2} \approx 1.43$, $b_{2,3} \approx 2.46$) in Figure 1.1. If 1CP is between two zeroes of Real CF, then these zeroes we use to numerate CP in most cases. More precisely, the index of CP we explain in the next section.

The *second order critical points* (2CP) arise when two 1CP coincide in the same point b. 2CP $b \in \mathring{\mathbb{R}}_q$ and ξ value we can find by solving system:

$$\gamma'(b; \xi) = 0, \quad \gamma''(b; \xi) = 0. \quad (2.22)$$

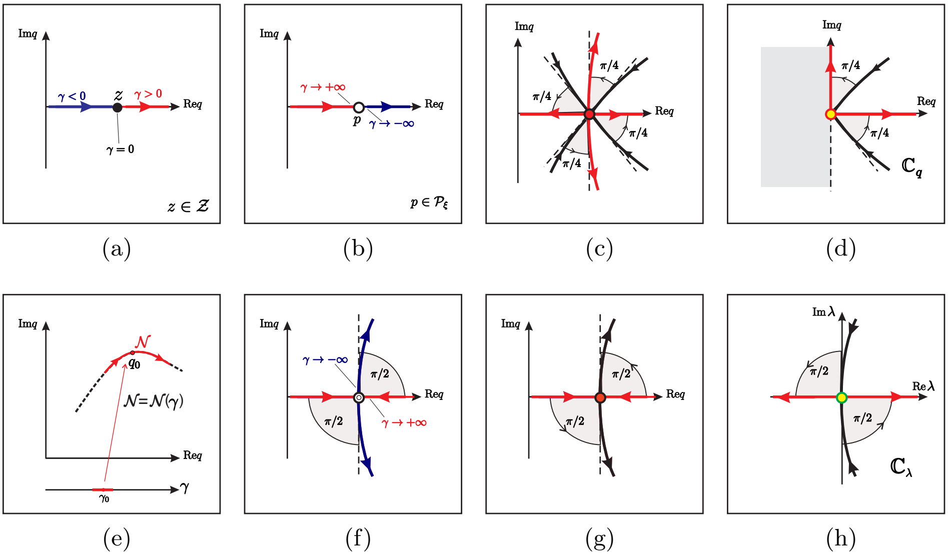


Fig. 1.6. Spectrum Curves. ⊚ – Pole Point of the second order, ○ – Branch Point

For example, 2CP we obtain in the inflection point (see Figure 1.2(c), when $\xi = \xi_{2b} \approx 0.25028$ at the point $b_{1,3,2} \approx 1.883$). Graphs of Real CF in the neighborhood 2CP are presented in Figure 1.4(e) (see Figure 1.4(d),(f), too).

The *third order real critical point* (3CP) $b \in \mathring{\mathbb{R}}_q$ satisfies the following system:

$$\gamma'(b;\xi) = 0, \quad \gamma''(b;\xi) = 0, \quad \gamma'''(b;\xi) = 0. \tag{2.23}$$

For problem (1.1)–(1.3) we have only one 3CP, $b_{2,1} = 0$, $\xi = \xi_c$. But as we note in Remark 1.8, this point is 1CP in the domain $\mathbb{C}_\lambda$ (see in Figure 1.2(f) and Figure 1.3(b) at the point $b_{2,1}$, $\xi = \xi_c$).

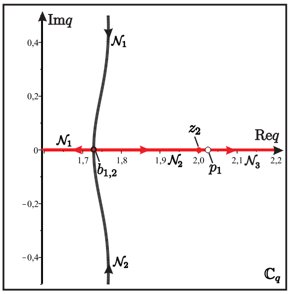
Lemma 1.12. *Zero Point of CF can not be CP.*

Proof. First of all $q = 0$ is not azero. For CF (2.10) (see (2.2), too) we have

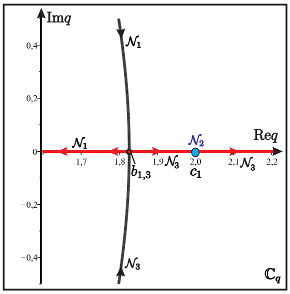
$$\gamma' = \left(\frac{\sin(\pi q)}{\pi q \cos(\xi\pi q)}\right)' = \frac{-1}{q^2\pi} \cdot \frac{\sin(\pi q)}{\cos^2(\xi\pi q)} + \frac{\xi}{q} \cdot \frac{\sin(\pi q)}{\cos(\xi\pi q)} \sin(\xi\pi q) + \frac{1}{q} \cdot \frac{\cos(\pi q)}{\cos(\xi\pi q)}$$
$$= -\gamma(q)\left(\frac{1}{q} - \xi\pi\frac{\sin(\xi\pi q)}{\cos(\xi\pi q)}\right) + \frac{\cos(\pi q)}{q\cos(\xi\pi q)}.$$

If $\gamma(q_z) = 0$ then $q_z \notin \mathcal{C}_\xi$ (see (2.16)), $\sin(\pi q_z) = 0$, $\cos(\pi q_z) \neq 0$, $\cos(\xi\pi q_z) \neq 0$. So, $\gamma'(q_z) \neq 0$. □

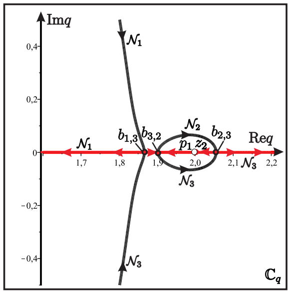
Remark 1.13. Pole Point of CF is not CP. Function γ^{-1} has CP at this point only if order of the pole is greater than the first.



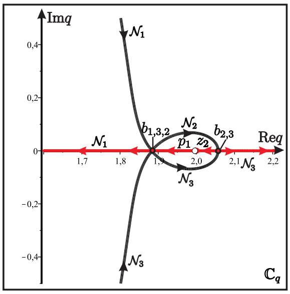
(a) $\xi = 0.247$



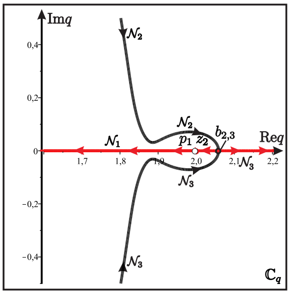
(b) $\xi = \frac{1}{4}$



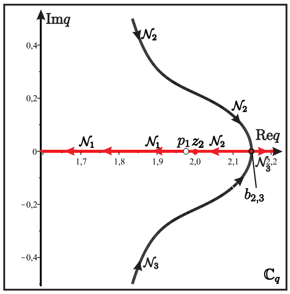
(c) $\xi = 0.25026$



(d) $\xi = \xi_{2b} \approx 0.25028$



(e) $\xi = 0.2503$



(f) $\xi = 0.253$

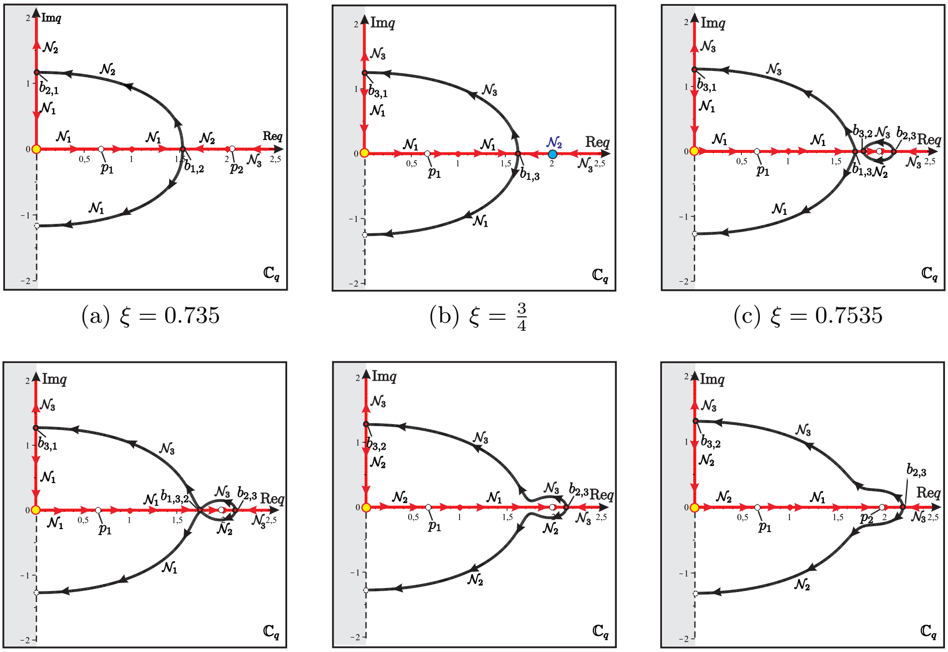
Fig. 1.7. Spectrum Curves for various parameter ξ values, bifurcations.

3 Spectrum Curves

Spectrum Domain is a union $\mathcal{N}_\xi = \mathcal{D}_\xi \cup \mathcal{C}_\xi$ in the complex domain $\mathbb{C}_q$. In the classical case $\gamma = 0$ Spectrum is $\mathbb{N} = \mathcal{Z}_\xi + \mathcal{C}_\xi$, $\mathcal{Z}_\xi = \mathcal{E}_\xi(0)$.

First of all, we consider $q_0 \in \mathcal{D}_\xi = \cup_{\gamma\in\mathbb{R}}\mathcal{E}_\xi(\gamma)$ and ${\gamma_c}'(q_0) \neq 0$, i.e., q_0 is not CP. Then $\mathcal{E}_\xi(\gamma)$ is a smooth parametric curve $\mathcal{N} : (\gamma_0 - \delta_1, \gamma_0 + \delta_2) \subset \mathbb{R} \to \mathbb{C}_q$ in the neighborhood of the point q_0 and $\mathcal{N}(\gamma_0) = q_0$ (see Figure 1.6(a),(e)). We can add arrows to this curve. Arrows show the direction in which γ is increasing. So, EP from $\mathcal{E}_\xi(\gamma)$ is moving along this curve. We call this curve the *Spectrum Curve*. Zero Point is not CP (see Lemma 1.12) and in the neighborhood of Zero Point the Spectrum Curve belongs to $\mathbb{R}_q^+$ (see Figure 1.6(a)). When $\gamma \to \pm\infty$ the Spectrum Curve $\mathcal{N}(\gamma)$ is approaching to $\partial\mathcal{D}_\xi = \mathcal{P}_\xi \cup \{\infty\}$ (see Figure 1.5). PP is not CP, too, and all Poles are of the first order. So, in the neighborhood of PP, Spectrum Curves belong to $\mathbb{R}_q^+$ (see Figure 1.6(b)), and γ values in the limit correspond to $-\infty$ and $+\infty$. For other problems the structure of Spectrum Curves may be more complex (see Figure 1.6(f) for a pole of the second order [155, A. Skučaitė and Štikonas 2015], [150, A. Skučaitė 2016]).

If $q_0 \in \mathcal{K}_\xi$ and $q_0 \neq 0$, i.e. we have CP, then $0 < |\gamma(q_0)| < \infty$ (see



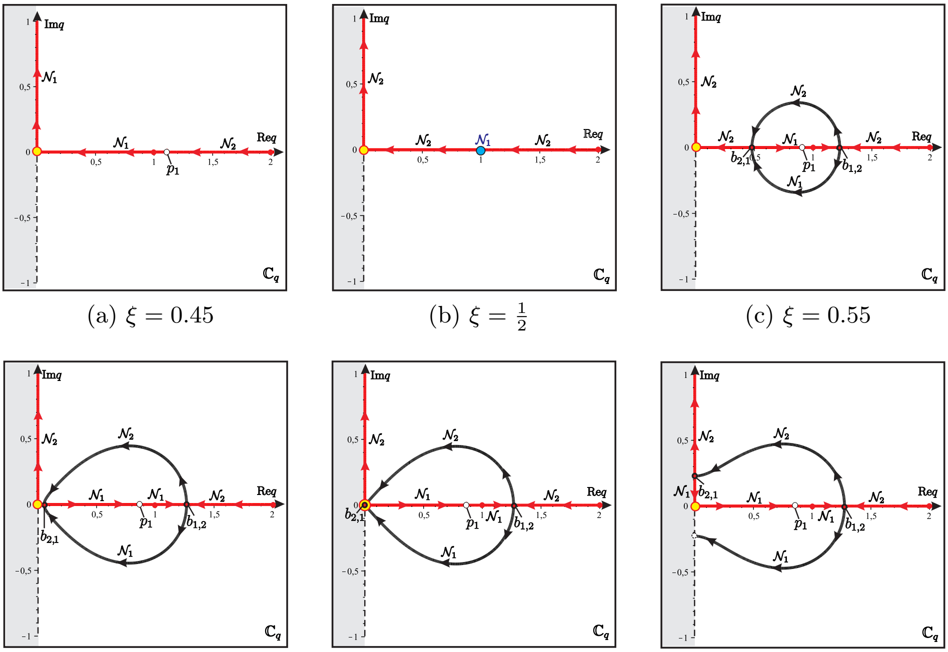
(a) $\xi = 0.735$ (b) $\xi = \frac{3}{4}$ (c) $\xi = 0.7535$

(d) $\xi = \xi_{2b} \approx 0.7539...$ (e) $\xi = 0.7545$ (f) $\xi = 0.76$

Fig. 1.8. Spectrum Curves for various parameter ξ values.

Lemma 1.12). At this point eigenvalue is not simple. The view of Spectrum Curves near to CPs is shown in Lemma 1.6(c) (3CP) and Figure 1.6(g) (1CP). For SLP (1.1)–(1.3) CPs of the first order and the second order exist (see Figure 1.5 and Figure 1.7). Since $0 < |\gamma(q_0)| < \infty$ at CP q_0, we can assume that a few Spectrum Curves are intersecting at this CP when parameter $\gamma = \gamma(q_0)$. At kCP Spectrum Curves change direction and the angle between the old and new direction is $\pi/(k+1)$ (see Figure 1.6(c) and Figure 1.6(g)). We use "right hand rule". So, the Spectrum Curve turns to the right. Then the parameter $\gamma \in \mathbb{R}$ for all Spectrum Curves, but we have exception in the case $\xi = 1$ when Spectrum Curve belongs to imaginary axis for $\gamma \in (0; 1]$ and real axis for $\gamma \in [1; +\infty)$ (see Figure 1.1(d)–(f)). We enumerate Spectral Curves by classical case: if for $\gamma = 0$ the point $q = l \in \mathcal{Z}_\xi$ belongs to the Spectrum Curve then we denote this *regular Spectrum Curve* $\mathcal{N}_l$. So, $\mathcal{N}_l = \{\mathcal{N}(\gamma), \gamma \in \mathbb{R}, \mathcal{N}(0) = l\}$, $l \in \mathcal{Z}_\xi$. In the case $\xi = 1$ we have additional *semi-regular Spectrum Curve* $\mathcal{N}_0 := \{\mathcal{N}(\gamma), \gamma \in (0; +\infty), \mathcal{N}(1) = 0\} = \mathbb{R}_q^- \cup [0; 1/2)$. Then we have $\mathcal{D}_\xi = \cup_{l \in \mathcal{Z}_\xi} \mathcal{N}_l$ for $\xi \neq 1$ and $\mathcal{D}_\xi = \cup_{l \in \{0\} \cup \mathcal{Z}_\xi} \mathcal{N}_l$ for $\xi = 1$.

Remark 1.14. In the case $\xi = 1$ and $\gamma \neq 0$ we can consider boundary condition $u'(1) = \tilde{\gamma} u(1)$, $\tilde{\gamma} \in \mathbb{R}$, where $\tilde{\gamma} = \gamma^{-1}$. Now CF is $\tilde{\gamma} = \pi q \cos(\pi q)/\sin(\pi q)$

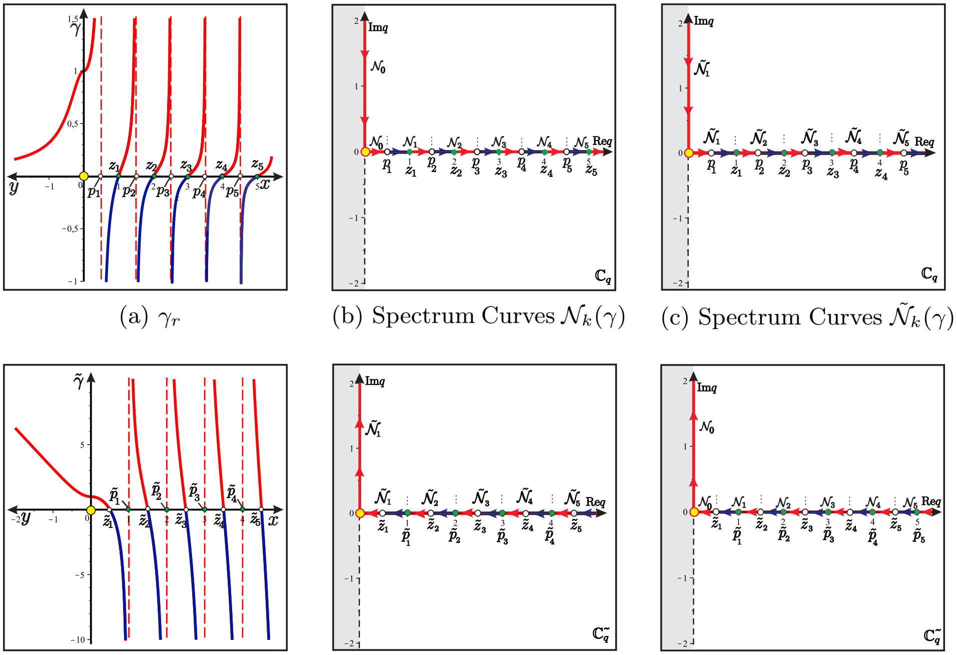


(a) $\xi = 0.45$ (b) $\xi = \frac{1}{2}$ (c) $\xi = 0.55$

(d) $\xi = 0.577$ (e) $\xi = \xi_c = \frac{1}{\sqrt{3}}$ (f) $\xi = 0.584$

Fig. 1.9. Spectrum Curves for various parameter ξ values.
◉ – Critical Point at Branch Eigenvalue Point.

and its zeroes are $\tilde{z}_k = p_k$, $k \in \mathbb{N}$, poles are $\tilde{p}_k = z_k$, $k \in \mathbb{N}$ (for $\xi = 1$ CEPs do not exist, but in the general case $\tilde{c}_k = c_k$ for all k). For parameter $\tilde{\gamma} \in \mathbb{R}$ all Spectrum Curves will be regular. The graphs of Real CFs γ_r and $\tilde{\gamma}_r$ are presented in Figure 1.10(a) and Figure 1.10(d). In Figure 1.10(b) and Figure 1.10(e) we can see the corresponding Spectrum Curves. In the first case the Spectrum Curves are parameterised by parameter γ, in the second case parameter is $\tilde{\gamma}$. Sometimes is useful to show points $\tilde{p}_k$, $\tilde{z}_l$ and Spectrum Curves $\mathcal{N}_s(\tilde{\gamma})$ (see Figure 1.10(f), and otherwise (see Figure 1.10(c)). The part of a semi-regular Spectrum Curve $\mathcal{N}_0$, $\gamma \in (+0, +\infty)$, can be described as $\mathcal{N}_0(\tilde{\gamma})$, $\tilde{\gamma} \in (+0, +\infty)$, but the directions of these two Spectral Curves are opposite. Spectral Curve $\mathcal{N}_0(\tilde{\gamma})$ belongs to Spectrum Curve $\tilde{\mathcal{N}}_1(\tilde{\gamma})$. Regular Spectrum Curve $\mathcal{N}_1(\gamma)$, $\gamma \in (-\infty, -0) \cup \{0\} \cup (+0, +\infty)$, we describe as $\mathcal{N}_1(\tilde{\gamma})$, $\tilde{\gamma} \in (+0, +\infty) \cup \{\infty\} \cup (-\infty, -0)$, and $\mathcal{N}_1(\tilde{\gamma})$, $\tilde{\gamma} \in (+0, +\infty)$ belongs to $\tilde{\mathcal{N}}_2(\tilde{\gamma})$, $\mathcal{N}_1(\tilde{\gamma})$, $\tilde{\gamma} \in (-\infty, -0)$ belongs to $\tilde{\mathcal{N}}_1(\tilde{\gamma})$. More generally, we can investigate SLP with parameter $\gamma \in \mathbb{R}P^1$ (projective line). In the case $\xi = 1$ we can consider one "super" Spectrum Curve (a union $\cup_{s=0}^{\infty} \mathcal{N}_s \cup (\cup_{k=1}^{\infty} p_k)$ of all Spectrum Curves and PPs, with beginning and end in infinity) for CF $\gamma\colon \mathbb{R}_q \to \mathbb{R}P^1$ (see [161, Štikonas 2011], too).



(a) γ_r (b) Spectrum Curves $\mathcal{N}_k(\gamma)$ (c) Spectrum Curves $\tilde{\mathcal{N}}_k(\gamma)$

(d) $\tilde{\gamma}_r$ (e) Spectrum Curves $\tilde{\mathcal{N}}_k(\tilde{\gamma})$ (f) Spectrum Curves $\mathcal{N}_k(\tilde{\gamma})$

Fig. 1.10. Real CFs γ_r and $\tilde{\gamma}_r$ and Spectrum Curves for $\xi = 1$.
• – Zero Point z_l $(\gamma = 0)$, Pole Point $\tilde{p}_k$ $(\tilde{\gamma} = \infty)$
∘ – Pole Point p_l $(\gamma = \infty)$, Zero Point $\tilde{z}_k$ $(\tilde{\gamma} = 0)$
— – $\gamma, \tilde{\gamma} \in (-\infty, -0)$, — – $\gamma, \tilde{\gamma} \in (+0, +\infty)$

The point $q = 0$ is RP. This point belongs to the regular Spectrum Curve in the case $\xi \neq 1$, $\xi \neq 1/\sqrt{3}$ and semi-regular Spectrum Curve in the case $\xi = 1$. RP $q = 0$ is CP for $\xi = \xi_c = 1/\sqrt{3}$ (see Remark 1.10). The Spectrum Curve near to this RP $q = 0$ and BP $\lambda = 0$ has different properties. For example, the Spectrum Curve turns perpendicular to the right at this BEP for $\xi \in [0; 1/\sqrt{3})$ and the Spectrum Curve turns perpendicular to the left for $\xi \in (1/\sqrt{3}; 1]$. In $\mathbb{C}_\lambda$ the image of the Spectrum Curve lies in the real axis. For $\xi = \xi_c = 1/\sqrt{3}$ we have different angles (see Figure 1.6(d) and Figure 1.6(h)) for $\mathbb{C}_q$ and $\mathbb{C}_\lambda$.

The index of CP is formed by the indices of the Spectrum Curves, intersecting in this CP. If the CP is real, then the left index coincides with the index of Spectral Curve, which is defined by the smaller real λ values, and the right index is defined by greater λ values (see Figure 1.5, Figure 1.7, Figure 1.9). We put the indices of other Spectrum Curves in the ascending order between left and right indices (see Figure 1.7(d)).

For every CEP $c_j = j$ we define *non-regular Spectrum Curve* (consisting of one point only) $\mathcal{N}_j = \{c_j\}$. We note, that non-regular Spectrum Curves can overlap with a point of a regular Spectrum Curve. In this point eigenvalue is not simple and generalized eigenvectors exist. Finally, we have that $\mathcal{N}_\xi$ is a countable union of Spectrum Curves $\mathcal{N}_l$, where $l \in \mathbb{N}$ for $\xi \neq 1$ and $l \in \{0\} \cup \mathbb{N}$ for $\xi = 1$.

3.1 Dynamics of Spectrum Curves

There are many papers, in which real eigenvalues of SLP are analyzed. However, a complex spectrum of this problem is investigated that much and it is more complicated. By changing the value of the parameter ξ we get various type projection of Spectrum Curves in the complex domain $\mathbb{C}_q$. In Figure 1.1(b),(e) and Figure 1.5 we can see qualitative view of Spectrum Curves dependence on the parameter ξ.

If $\xi \in [0, 1]$ is increasing from 0 to 1, then zeroes $p_k = (k - 1/2)/\xi$, $k \in \mathbb{N}$, of the function $P_\xi(q)$ are moving to the left and zeroes $z_l = l$, $l \in \mathbb{N}$, of the function $Z(q)$ remain unchanged. In the limit case $\xi = 1$ PPs are $p_k = k - 1/2$, $k \in \mathbb{N}$, and we do not have PPs for $\xi = 0$. So, every p_k coincides with $z_l = l$, $l = k, k+1, \ldots$, for $\xi = (2k-1)/(2l)$ and we have CEP $c_s = p_{k_s} = z_{l_s} = n(s - 1/2)$, $k_s = m(2s-1)/2 + 1/2$, $l_s = n(s - 1/2)$, $s \in \mathbb{N}$ (see (2.11)). Formula (2.17) shows that we have the same situation for all CEPs. In Figure 1.4(a)–(c) we can see how the Real CFs depend on the value of the parameter ξ in the neighbourhood of $\xi = \frac{1}{4}$. The Spectrum Curves near to $\xi = \frac{1}{4}$ are presented in Figure 1.7(a)–(c). When $\xi \lesssim \frac{1}{4}$ the PP p_1 is moving from right side to the zero point $z_2 = 2$. For $\xi = 1/4$ we have CEP at $c_1 = 2$. Next, if the value of $\xi \gtrsim \frac{1}{4}$ is increasing, the PP p_1 moves to the left from the zero point. A loop type curve appears, which consists of Spectrum Curves $\mathcal{N}_2$, $\mathcal{N}_3$ and two critical points $b_{3,2}$ and $b_{2,3}$ of the first order. While the value of ξ is increasing, the PP p_1 is moving to the left and the loop grows. We denote such *Zero and Pole bifurcation* type by $\beta_{ZP}\colon (z_{l_s}, p_{k_s}) \to c_s \to (b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$. The CEPs exist for all rational $\xi = m/n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$. We have such type bifurcation for $\xi = 1/2$ and $\xi = 3/4$ in the neighborhood of CEPs $c_1 = 1$ and $c_1 = 2$, respectively (see Figure 1.8(a)–(c), Figure 1.9(a)–(c)). Every such ξ is a point of bifurcation near to all CEPs for this value of ξ.

Remark 1.15. For $\xi = 1/4$ we have $c_1 = 2$, $c_2 = 6$. β_{ZP} bifurcations at these two CEPs give similar view Spectrum Curves but the directions are opposite. The direction of Spectrum Curves depends on sign $\gamma(c_s;\xi) = (-1)^{ms+(n-m+1)/2} c_s^{-1} \xi^{-1} \pi^{-1}$ (see (2.16)).

CEP $c_1 = 1$ exists only for $\xi = 1/2$. If $\xi < 1/2$ then $\mathbb{R}_q^-$ and interval

$[0,1)$ belong to regular Spectrum Curve $\mathcal{N}_1$. For $\xi = 1/2$ we have β_{ZP} type bifurcation and there are no CPs on $\mathcal{N}_2$ for $\gamma > 0$. If $\xi \gtrsim \frac{1}{2}$ then the smallest real eigenvalue is described by Spectrum Curve $\mathcal{N}_2$. We note that after bifurcation the CP $b_{2,1}$ exists (see Figure 1.9(c)) and this CP is moving to the left (Figure 1.9(d)). For $\xi = \xi_c = 1/\sqrt{3}$ this CP is at BEP $q = 0$. For $\xi > \xi_c$ this CP is moving along $\mathbb{R}_q^-$ (the imaginary axis) to ∞. So, for $\xi \colon \xi_c < \xi < 1$, we have loop type curve with one CP in $\mathbb{R}_q^-$ and another CP in $\mathbb{R}_q^+$ (see Figure 1.9(e)–(f), Figure 1.8).

Remark 1.16. During β_{ZP} bifurcation we get CEP c_s (see Lemma 1.6) and have a new configuration of Spectrum Curves:

1) regular Spectrum Curve $\mathcal{N}_{l_s}$ becomes non-regular;

2) on the left and on the rigth of CEP we have the same regular Spectrum Curve $\mathcal{N}_{l_s+1}$;

3) the CP b_{l_s-1,l_s} (if exists) becomes b_{l_s-1,l_s+1}, and the CP $b_{l_s,l_s-1} \in R_q^-$ (if exists) becomes b_{l_s+1,l_s-1}; at these CPs we have Spectrum Curves $\mathcal{N}_{l_s-1}$ and $\mathcal{N}_{l_s+1}$ instead $\mathcal{N}_{l_s-1}$ and $\mathcal{N}_{l_s}$.

The β_{ZP} bifurcation create configuration of the points b_{l_s-1,l_s+1}, b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1} (see Figure 1.4(c),(d), Figure 1.7(c), Figure 1.8(c)) and loop type curve (formed by parts of the Spectrum Curves $\mathcal{N}_{l_s}$ and $\mathcal{N}_{l_s+1}$) in complex part of $\mathbb{C}_q$. While value of the parameter ξ is increasing, this loop type curve grows, the PP p_{k_s} is moving to the left and is pushing the CP b_{l_s+1,l_s} towards the CP b_{l_s-1,l_s+1}. These two 1CPs are between zero point z_{l_s-1} and PP p_{k_s}. For $\xi = \xi_{2b}$ these CPs merge into one 2CP b_{l_s-1,l_s+1,l_s} (see Figure 1.4(e), Figure 1.7(d), Figure 1.8(d)) and we have *the second order CP* bifurcation β_{2B}. We note, that ξ_{2b} depends on s and we can calculate the location of b_{l_s-1,l_s+1,l_s} only numerically ($z_{l_s-1} < b_{l_s-1,l_s+1,l_s} < p_{k_s}$). When $\xi \gtrsim \xi_{2b}$ loop type curve (around p_{k_s} and z_{l_s}) disappears and we have two Spectral Curves $\mathcal{N}_{l_s}$ and $\mathcal{N}_{l_s+1}$ which intersect in CP b_{l_s,l_s+1} (see Figure 1.4(f), Figure 1.7(e),(f), Figure 1.8(e),(f)). So, $\beta_{2B} \colon (b_{l_s-1,l_s+1}, b_{l_s+1,l_s}) \to b_{l_s-1,l_s+1,l_s} \to \varnothing$.

Remark 1.17. During β_{2B} bifurcation we have a new configuration of Spectrum Curves:

1) the part of Spectrum Curve $\mathcal{N}_{l_s-1}$ becomes as part of Spectrum Curve $\mathcal{N}_{l_s}$;

2) after bifurcation Spectrum Curve $\mathcal{N}_{l_s-1}$ starts in PP p_{k_s} and not at infinity.

Finally, these two bifurcations β_{ZP} and β_{2B} interchange sequence of the points $(b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$ for $z_{l_s} \neq 1$. If $\xi = 1/2$ then the loop type curve around $q = 1$ exists for $1/2 < \xi < 1$. When value of ξ is increasing then more and more zeroes and poles get inside the loop. This loop disappears only for $\xi = 1$. In Figure 1.1(b), Figure 1.5, Figure 1.1(e) we see how global view of Spectrum Curves depends on parameter $\xi \in [0, 1]$.

4 Conclusions

Below we present principal conclusions of this chapter:

1. For SLP (1.1)–(1.3) CEs do not exist for irrational parameter ξ and exist only for $\xi = \frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$.

2. SLP (1.1)–(1.3) has two types CPs: the first, the second order. For this problem we have only one 3CP, $b_{2,1} = 0$, $\xi = \xi_c = 1/\sqrt{3}$. But this point is 1CP in the domain $\mathbb{C}_\lambda$ (see in Figure 1.2(f) and Figure 1.3(b) at the point $b_{2,1}$, $\xi = \xi_c$). The negative CP exists only for $\xi > \xi_c$. CPs of the second order are positive and they exist only for some values of $\xi \in (0, 1)$. CPs of the first order exist (the infinite number) for all $\xi \in [0, 1)$ (then we always have complex eigenvalues).

3. For SLP (1.1)–(1.3) we obtain two types' bifurcations:

   - $\beta_{ZP}\colon (z_{l_s}, p_{k_s}) \to c_s \to (b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$ when zero and pole of CF merge into CEP and we get a loop type curve.
   - $\beta_{2B}\colon (b_{l_s-1,l_s+1}, b_{l_s+1,l_s}) \to b_{l_s-1,l_s+1,l_s} \to \varnothing$ when two 1CPs merge into one 2CP. At this bifurcation the loop type curve vanish.

Chapter 2

Spectrum Curves for another types of Nonlocal Boundary Conditions

Introduction

The SLP with classical and two-point NBCs is considered in this chapter. These problems with NBC are not self-adjoint, so the spectrum has complex points. The purpose is to analyze a complex eigenvalues problem for a stationary differential operator with one classical and one two-point NBC. We investigate how the real and complex eigenvalues of these problems depend on the two point NBCs parameters γ and ξ.

In Section 1 we analyze the SLP with the first Dirichlet BC $u(0) = 0$ and another nonlocal two-point BC of Samarskii–Bitsadze type:

$$u'(1) = \gamma u(\xi), \qquad (1_1)$$

$$u'(1) = \gamma u'(\xi), \qquad (1_2)$$

Note, if we write the index in the formula number, as in (1_1) or (1_2), then the first formula is related to Case 1, the second formula is related to Case 2.

Real eigenvalues of this problems were analyzed by others authors [102, 104, Pečiulytė and Štikonas 2006, 2007], [107, Pečiulytė *et al.* 2008]. The main results of Section 1 are published in [157, K. Skučaitė *et al.* 2009], [158, K. Skučaitė-Bingelė and Štikonas 2011].

In Section 2 for SLP we changed the first classical BC to Neumann type BC $u'(0) = 0$. We investigate four cases of SLP with two-point NBCs

$(0 \leqslant \xi \leqslant 1)$:

$$u'(1) = \gamma u(\xi), \tag{2_1}$$

$$u'(1) = \gamma u'(\xi), \tag{2_2}$$

$$u(1) = \gamma u'(\xi), \tag{2_3}$$

$$u(1) = \gamma u(\xi), \tag{2_4}$$

and analyze these problems real and complex eigenvalues and dynamic of Spectrum Curves. The main results of this Section 2 are published in [158].

In Section 3 we analyze SLP with one classical Dirichlet BC $u(0) = 0$ and the second symmetrical type $u(\xi) = \gamma u(1-\xi)$ NC. We investigated the dependence of this problem real and complex part of spectrum on NBCs parameters. The main results of this Section 3 are published in [159, K. Skučaitė-Bingelė and Štikonas 2013].

As the theoretical investigation of the complex spectrum is a very difficult problem. So, we present in these Sections 1–3 modelling results and illustrate in these problems existing situations in graphs.

1 The Sturm–Liouville Problems with one classical Dirichlet condition and another Two-Point NBC

Let us analyze SLP with one classical BC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

and another two-point NBC of Samarskii–Bitsadze type Case 1 and Case 2:

$$u'(1) = \gamma u(\xi), \tag{1.3_1}$$

$$u'(1) = \gamma u'(\xi), \tag{1.3_2}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. Real eigenvalues of this problem were analyzed in [100, 102–104, Pečiulytė and Štikonas 2005–2007], [107, Pečiulytė *et al.* 2008].

If $\gamma = 0$, we have problems with classical BCs. In this case, all the eigenvalues are positive and eigenfunctions do not depend on the parameter ξ:

$$\lambda_k = \pi^2(k-1/2)^2, \quad u_k = \sin(\pi(k-1/2)t), \quad k \in \mathbb{N}. \tag{$1.4_{1,2}$}$$

When $\gamma \neq 0$ then we have the classical eigenvalues and eigenfunctions (1.4) if $\xi = 0$ in Case 1: $u'(1) = \gamma u(0) = 0$. For the $\xi = 1$ in Case 2 we have the classical problem only for $\gamma \neq 1$, and the second BC is trivial for $\gamma = 1$. For $\xi = 1$ in Case 1 we have third type classical BC.

If $\lambda = 0$, then all the function $u(t) = Ct$ satisfy the problem (1.1)-(1.2). Substituting this solution into the second BCs (1.3) we get $C = C\gamma\xi$ (Case 1) and $C = C\gamma$ (Case 2). So, the eigenvalue $\lambda = 0$ exists only if $\gamma = 1/\xi$ in (1.3_1)) and $\gamma = 1$ in (1.3_2)).

For $\lambda \neq 0$, eigenfunctions are $u = c\sin(\pi q t)$ and eigenvalues $\lambda = (\pi q)^2, q \in \mathbb{C}_q \backslash \{0\}$.

In both cases ($q = 0$ and $q \neq 0$), we can write one formula for the solution:

$$u = C\frac{\sin(\pi q t)}{\pi q}, \quad q \in \mathbb{C}_q. \tag{1.5}$$

1.1 Constant Eigenvalues and Characteristic Function

Let us investigate a problem (1.1)–(1.3) and consider $\xi \in (0,1]$ in Case 1 and $\xi \in [0,1)$ in Case 2. If we substitute function (1.5) into the second BC (1.3) then we get the equality:

$$C\big(\gamma\frac{\sin(\xi\pi q)}{\pi q} - \cos(\pi q)\big) = 0, \tag{1.6_1}$$

$$C\big(\gamma\cos(\xi\pi q) - \cos(\pi q)\big) = 0. \tag{1.6_2}$$

There exists a nontrivial solution (eigenfunction) if q is the root of the function:

$$\cos(\pi q) = \gamma\frac{\sin(\xi\pi q)}{\pi q}, \tag{1.7_1}$$

$$\cos(\pi q) = \gamma\cos(\xi\pi q). \tag{1.7_2}$$

We introduce two entire functions:

$$Z(z) := \cos(\pi q); \quad P_\xi(z) := \frac{\sin(\xi\pi q)}{\pi q}, \quad z \in \mathbb{C}, \tag{1.8_1}$$

$$Z(z) := \cos(\pi q); \quad P_\xi(z) := \cos(\xi\pi q), \quad z \in \mathbb{C}. \tag{1.8_2}$$

Zeroes points. Zeroes set

$$\hat{\mathcal{Z}} := \{z_l = l - 1/2, \quad l \in \mathbb{N}\}. \tag{1.9}$$

of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with EPs in the classical case $\gamma = 0$. All zeroes are simple, real and positive.

In the case $\xi \neq 0$ all zeroes of the function $P_\xi(q)$ (see (2.2)) in $\mathbb{C}_q$ are simple, real and positive:

$$\overline{\mathcal{Z}}_\xi := \{p_k = k/\xi, \quad k \in \mathbb{N}\}, \tag{1.10_1}$$

$$\overline{\mathcal{Z}}_\xi := \{p_k = (k - 1/2)/\xi, \ \ k \in \mathbb{N}\}. \tag{1.10_2}$$

In Case 2 and $\xi = 0$ the set $\overline{\mathcal{Z}}_\xi = \varnothing$.

We rewrite the equation (1.7) in the form:

$$Z(q) = \gamma P_\xi(q), \quad q \in \mathbb{C}_q. \tag{1.11}$$

Constant Eigenvalues. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \qquad P_\xi(q) = 0. \tag{1.12}$$

Remark 2.1. In the Case 1, if the parameter $\xi = 1$, then CEPs do not exist, because the functions $\sin(\pi q)$ and $\cos(\pi q)$ have no common zeroes. In the Case 2, if the parameter $\xi = 0$, then $P_\xi \equiv 1$. So, $\mathcal{P}_\xi = \varnothing$ and CEPs do not exist, too.

Remark 2.2. If the parameter $\xi \notin \mathbb{Q}$, then CEPs does not exist, because the equations

$$\xi(l - 1/2) = k, \tag{1.13_1}$$

$$\xi(l - 1/2) = k - \tfrac{1}{2} \tag{1.13_2}$$

have not roots for $l, k \in \mathbb{N}$.

Lemma 2.3. *For SLP* (1.1)–(1.3_1) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0,\, 1)$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s := (s - 1/2)n$, $s \in \mathbb{N}$.

Proof. The case $\xi \notin \mathbb{Q}$ was discussed in Remark 2.2. Let $\xi \in \mathbb{Q}$, $\xi = m/n$, $0 < m < n$, $\gcd(n, m) = 1$. We rewrite equation (1.13_1) as $ml - nk = m/2$. So, m must be even, i.e. $m \in \mathbb{N}_e$ and $n \in \mathbb{N}_o$. In this case we have solution $l = ns - (n-1)/2$, $k = ms - m/2$, $s \in \mathbb{N}$. Then $c_s = l - 1/2 = n(s - 1/2)$, $s \in \mathbb{N}$. □

Lemma 2.4. *For SLP* (1.1)–(1.3_2) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0,\, 1)$, $m, n \in \mathbb{N}_o$, *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.

Proof. We rewrite equation (1.13_2) as $ml - nk = (m - n)/2$. So, m and n must be odd, i.e. $m, n \in \mathbb{N}_o$. In this case we have solution $l = ns - (n-1)/2$, $k = ms - (m-1)/2$, $s \in \mathbb{N}$. Then $c_s = l - 1/2 = n(s - 1/2)$, $s \in \mathbb{N}$. □

Complex Characteristic Function. For SLP (1.1)–(1.3) we have meromorphic Complex Characteristic Functions (Complex CF)

$$\gamma_c(q) = \frac{Z(q)}{P_\xi(q)} = \frac{\pi q\cos(\pi q)}{\sin(\xi\pi q)}, \quad q\in\mathbb{C}_q, \tag{1.14$_1$}$$

$$\gamma_c(q) = \frac{Z(q)}{P_\xi(q)} = \frac{\cos(\pi q)}{\cos(\xi\pi q)}, \quad q\in\mathbb{C}_q. \tag{1.14$_2$}$$

All zeroes and poles of meromorphic function $\gamma_c(q)$ lie on the positive part of real axis. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. So, $p_k \in \overline{\mathcal{Z}}_\xi$ is PP if and only if $p_k + 1/2 \notin \mathbb{N}$. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. If $c\in\mathcal{C}_\xi$, i.e. $Z(c)=0$ and $P_\xi(c)=0$, then we have removable singularity isolated point and we have sequence of such points $c_s = p_{k_s} = z_{l_s} = n(s-1/2)$, $s\in\mathbb{N}$,

$$k_s = m(s-1/2), \qquad l_s = n(s-1/2)+1/2, \quad m\in\mathbb{N}_e, n\in\mathbb{N}_o, \tag{1.15$_1$}$$

$$k_s = m(s-1/2)+1/2, \quad l_s = n(s-1/2)+1/2, \quad m\in\mathbb{N}_o, n\in\mathbb{N}_o. \tag{1.15$_2$}$$

Remark 2.5. In Case 2 and $\xi=0$ function $P_\xi \equiv 1$. So, PPs do not exist. If $\xi>0$ then it follows from $1 = k_1 = m/2+1/2$ that $c_1 = p_1$ if $m=1$. Then $p_k = (k-1/2)n$, $c_k = (k-1/2)n$, $n\in\mathbb{N}_o$ and we have CEPs and PPs do not exist for $\xi = 1/(2k-1)$, $k\in\mathbb{N}$. In Case 1 it follows from $1 = k_1 = m/2$ that $c_1 = p_1$ if $m=2$. Then $p_2 = n < c_2 = 3n/2$ and PPs always exist.

Real Characteristic Function. *Real Characteristic Function* (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$:

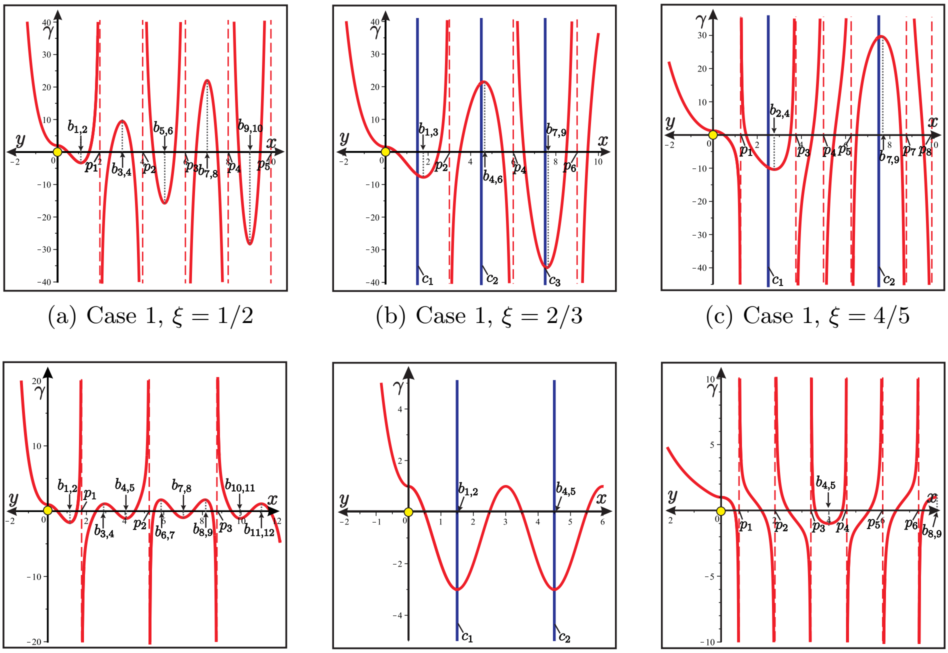
$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} \frac{\pi x\cosh(\pi x)}{\sinh(\xi\pi x)}, & x\leqslant 0; \\ \frac{\pi x\cos(\pi x)}{\sin(\xi\pi x)}, & x\geqslant 0. \end{cases} \tag{1.16$_1$}$$

$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} \frac{\cosh(\pi x)}{\cosh(\xi\pi x)}, & x\leqslant 0; \\ \frac{\cos(\pi x)}{\cos(\xi\pi x)}, & x\geqslant 0. \end{cases} \tag{1.16$_2$}$$

Real CF for SLP (1.1)–(1.3) was investigated in [102, Pečiulytė and Štikonas]. In this paper values of Real CF and it's derivatives at CEP $c_s := (s-1/2)n$, $s\in\mathbb{N}$, $\xi = m/n \in (0,\,1)$, were found:

$$\gamma_s(\xi) := \gamma(c_s;\xi) = (-1)^{s+(n-m-1)/2} c_s\pi\xi^{-1}, \qquad m\in\mathbb{N}_e, n\in\mathbb{N}_o, \tag{1.17$_1$}$$

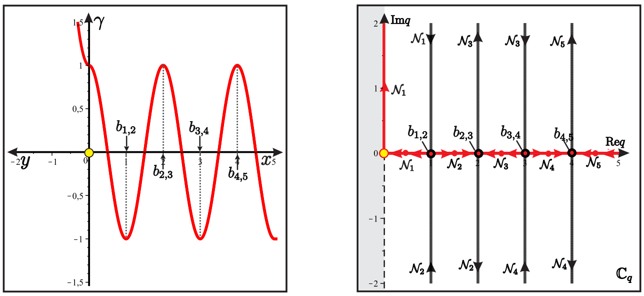
$$\gamma_s(\xi) := \gamma(c_s;\xi) = (-1)^{(n-m)/2}\xi^{-1}, \qquad m\in\mathbb{N}_o, n\in\mathbb{N}_o, \tag{1.17$_2$}$$



(a) Case 1, $\xi = 1/2$ (b) Case 1, $\xi = 2/3$ (c) Case 1, $\xi = 4/5$

(d) Case 2, $\xi = 2/7$ (e) Case 2, $\xi = 1/3$ (f) Case 2, $\xi = 3/4$

Fig. 2.1. Real CF for various ξ.



(a) Real CF (b) Spectrum Curves

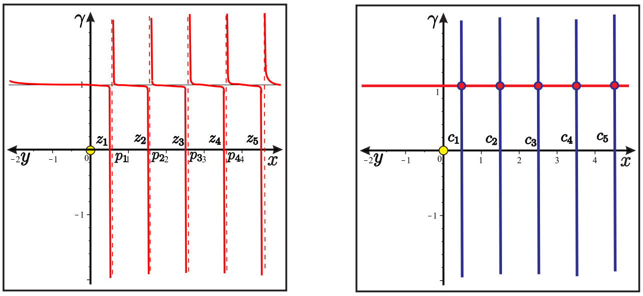
Fig. 2.2. Real CF and Spectrum Curves for $\xi = 0$ in Case 2.

$$\gamma_s'(\xi) := \gamma'(c_s; \xi) = c_s^{-1}\gamma_s, \qquad (1.18_1)$$

$$\gamma_s'(\xi) := \gamma'(c_s; \xi) = 0, \qquad (1.18_2)$$

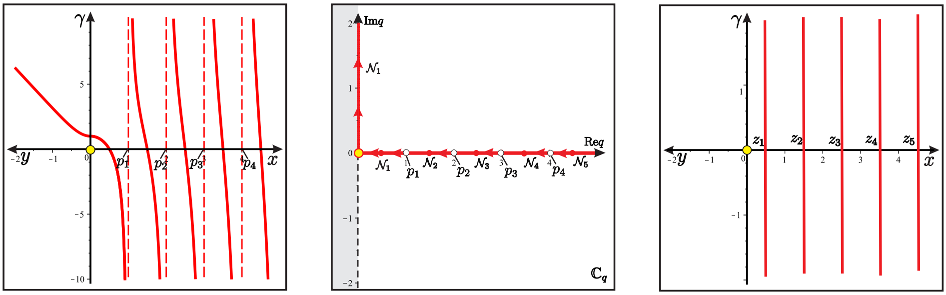
$$\gamma_s''(\xi) := \gamma''(c_s; \xi) = -\pi^2(1 - \xi^2)/3 \cdot \gamma_s. \qquad (1.19_{1,2})$$

We see, that $\gamma_s \neq 0$, $\gamma_s'' \neq 0$ for all ξ and s. Graphs of the function $\gamma_r(x)$ for various ξ are shown in Figure 2.1 and Figure 2.2(a) for $\xi = 0$ in Case 2.



(a) $\xi = 499/500$ (b) $\xi = 1$ (the limit case)

Fig. 2.3. Real CF for $\xi \lesssim 1$ in Case 2.



(a) Real CF (b) Spectrum Curves (c) $\xi = 0$ (the limit case)

Fig. 2.4. Real CF and Spectrum Curves for $\xi = 1$ and Real CF for $\xi = 0$ in Case 1.

Ramification Point and Critical Points. The Taylor series for CF $\gamma(q)$ at RP $q = 0$ is:

$$\gamma(q;\xi) = \frac{1}{\xi} - \frac{1}{6\xi}(3-\xi^2)\pi^2 q^2 + \mathcal{O}(q^4), \qquad (1.20_1)$$

$$\gamma(q;\xi) = 1 - \tfrac{1}{2}(1-\xi^2)\pi^2 q^2 + \mathcal{O}(q^4). \qquad (1.20_2)$$

Multiplier of q^2 is nonzero. So, $q = 0$ is not CP in $\mathbb{C}_\lambda$.

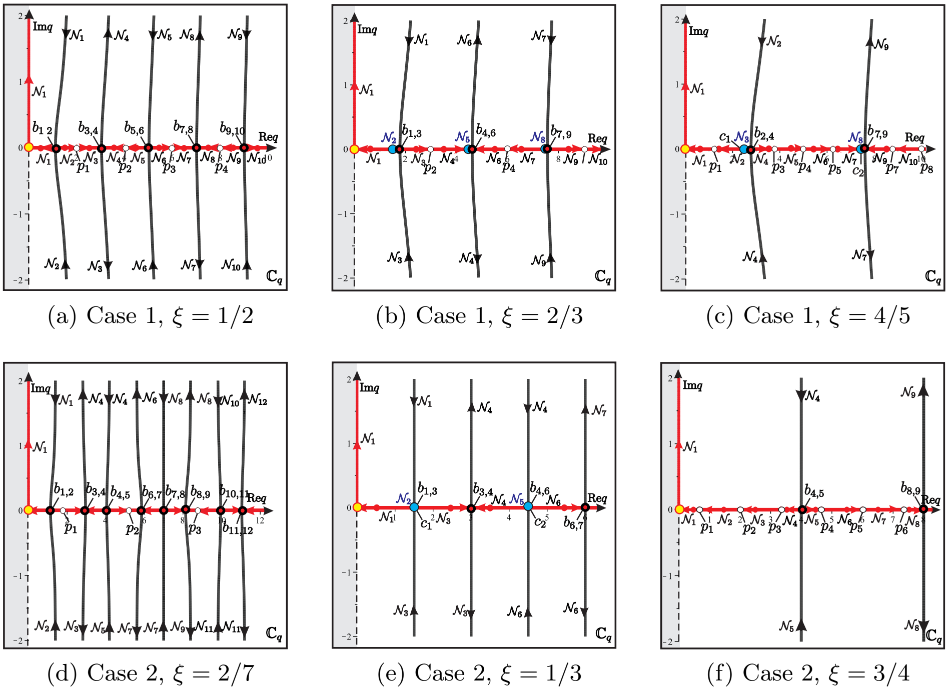
For the SLPs (1.1)–(1.3) negative CP ($b \in \mathbb{R}_q^-$) do not exists [107, Pečiulytė *et al.* 2008].

Lemma 2.6. *Zero Point of CF can not be CP.*

Proof. For CF (1.14) we have

$$\gamma' = \gamma(q)\Big(\frac{1}{q} - \xi\pi\frac{\cos(\xi\pi q)}{\sin(\xi\pi q)}\Big) - \pi^2 q\frac{\sin(\pi q)}{\sin(\xi\pi q)}, \qquad (1.21_1)$$

$$\gamma' = \gamma(q)\xi\pi\frac{\sin(\xi\pi q)}{\cos(\xi\pi q)} - \pi\frac{\sin(\pi q)}{\cos(\xi\pi q)}. \qquad (1.21_2)$$



(a) Case 1, $\xi = 1/2$ (b) Case 1, $\xi = 2/3$ (c) Case 1, $\xi = 4/5$

(d) Case 2, $\xi = 2/7$ (e) Case 2, $\xi = 1/3$ (f) Case 2, $\xi = 3/4$

Fig. 2.5. Domain $\mathcal{N}_\xi$ for CF $\gamma(q)$.

If $\gamma(q_z) = 0$ then $q_z \notin \mathcal{C}_\xi$. So, $\cos(\pi q_z) = 0$, $\sin(\pi q_z) \neq 0$, $\sin(\xi\pi q_z) \neq 0$ in Case 1; $\cos(\pi q_z) = 0$, $\sin(\pi q_z) \neq 0$, $\cos(\xi\pi q_z) \neq 0$ in Case 2. It follows from (1.21) for both cases that $\gamma'(q_z) \neq 0$. □

1.2 Spectrum Curves

Domain $\mathcal{N}_\xi$ for various ξ for CF $\gamma(q)$ is shown in Figure 2.5, Figure 2.4(b) for $\xi = 1$ in Case 1, Figure 2.2(b) for $\xi = 0$ in Case 2. All Spectrum Curves are regular.

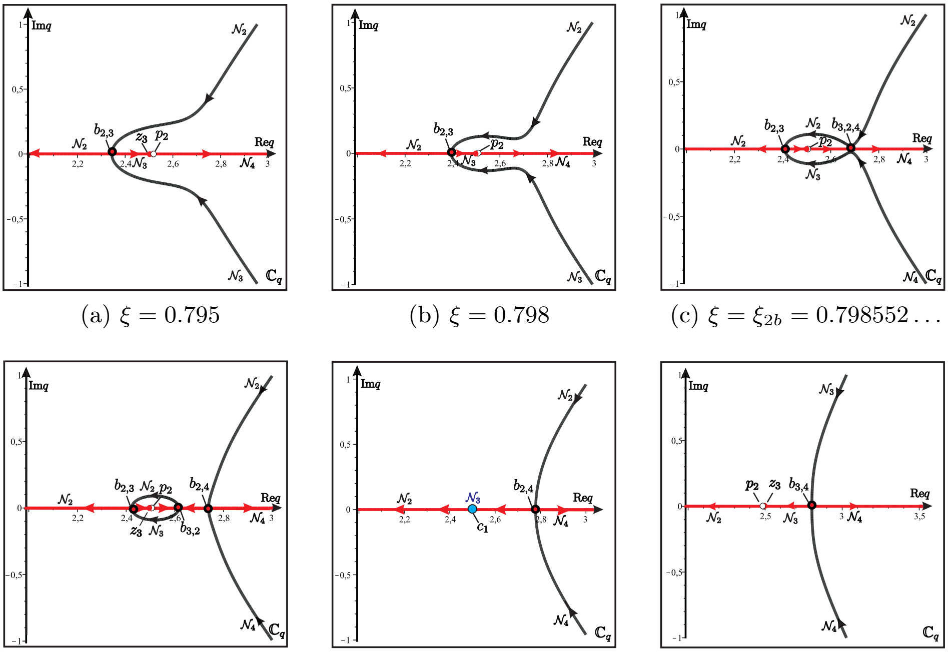
Remark 2.7. In Case 1 and $\xi = 1$ CF $\gamma = \pi q \cos(\pi q)/\sin(\pi q)$ is the same as function $\tilde{\gamma} = \pi q \cos(\pi q)/\sin(\pi q)$ for SLP in Chapter 1 for $\xi = 1$. Real CF and Spectrum Curves are the same as in Figure 1.10(d) and Figure 1.10(e) (see Figure 2.4).

The most interesting thing is how complex eigenvalues is changing near to CEP.

Bifurcations in Case 1. In this case we obtain the similar situation as we have in Chapter 1. We note, that now $\gamma'(c_s;\xi) = c_s^{-1}\gamma_s$ instead of $\gamma'(c_s;\xi) = -c_s^{-1}\gamma_s$ (see (2.17) in Chapter 1). So, we investigate bifurcations when parameter $\xi \in (0, 1]$ is decreasing. If ξ is decreasing, then zeroes $p_k = k/\xi$, $k \in \mathbb{N}$, of the function $P_\xi(q)$ are moving to the right and zeroes $z_l = l - 1/2$, $l \in \mathbb{N}$, of the function $Z(q)$ remain unchanged. So, every p_k coincides with $q = l + 1/2$, $l = k, k+1, \ldots$, for $\xi = 2k/(2l+1)$ and we have CEP $c_s = p_{k_s} = n(s - 1/2)$, $k_s = m(s - 1/2)$ (see (1.15_1)). Formula (1.18_1) shows that we have the same situation for all CEPs. As example, let to consider $\xi = 4/5$ (see Figure 2.1(c) and Figure 2.5(c)). The first CEP $c_1 = 2.5$ and Spectrum Curves for the parameter ξ in the neighbourhood of $\xi = 4/5$ are presented in Figure 2.6. If $\xi \gtrsim 4/5$ and ξ is decreasing then PP p_2 is moving from left side to the zero point $z_3 = 2.5$. For $\xi = 4/5$ we have CEP at $c_1 = 2.5$. Next, if the value of $\xi \lesssim \frac{4}{5}$ is decreasing, then PP p_2 moves to the right from the zero point. A loop type curve appears, which consists of Spectrum Curves $\mathcal{N}_2$, $\mathcal{N}_3$ and two critical points $b_{2,3}$ and $b_{3,2}$ of the first order. While the value of ξ is decreasing, the PP p_2 is moving to the right and the loop grows. Such bifurcation we have in Chapter 1 when parameter ξ was increasing. So, we denote such *inverse Zero and Pole bifurcation* type by $\beta_{ZP}^{-1}\colon (b_{l_s-1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1}) \to c_s \to (z_{l_s}, p_{k_s})$. The CEPs exist for all rational $\xi = m/n$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$. So, every such ξ is a point of bifurcation near all CEPs for this value of ξ. For example, we have such type bifurcation for $\xi = 2/3$ and $\xi = 4/5$ in the neighborhood of CEPs $c_1 = 1.5$, $c_2 = 4.5$, $c_3 = 7.5$ and $c_1 = 2.5$, $c_2 = 7.5$, respectively (see Figure 2.1(b)–(c), Figure 2.5(b)–(c)). During β_{ZP}^{-1} bifurcation we have a new configuration of Spectrum Curves.

Before the β_{ZP}^{-1} bifurcation we have configuration of the points b_{l_s-1,l_s}, z_{l_s}, p_{k_s}, b_{l_s,l_s-1}, b_{l_s-1,l_s+1} (see Figure 2.6(d), Figure 2.7(c)) and loop type curve (formed by parts of the Spectrum Curves $\mathcal{N}_{l_s-1}$ and $\mathcal{N}_{l_s}$) in complex part of $\mathbb{C}_q$. While value of the parameter ξ is decreasing, this loop type curve grows, the PP p_{k_s} is moving to the right and is pushing the CP b_{l_s,l_s-1} towards the CP b_{l_s-1,l_s+1}. These two 1CPs are between PP p_{k_s} and zero point z_{l_s+1}. For $\xi = \xi_{2b}$ these CPs merge into one 2CP b_{l_s,l_s-1,l_s+1} (see Figure 2.6(c), Figure 2.7(b)) and we have *the inverse the second order CP* bifurcation β_{2B}^{-1}. When $\xi \lesssim \xi_{2b}$ loop type curve (around p_{k_s} and z_{l_s}) disappears and we have two Spectral Curves $\mathcal{N}_{l_s-1}$ and $\mathcal{N}_{l_s}$ which intersect in CP b_{l_s-1,l_s} (see Figure 2.6(a),(b), Figure 2.7(a)). So, $\beta_{2B}^{-1}\colon \varnothing \to b_{l_s,l_s-1,l_s+1} \to (b_{l_s,l_s-1}, b_{l_s-1,l_s+1})$.

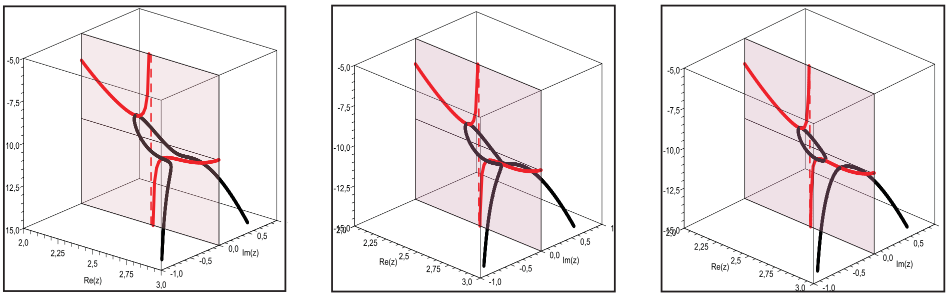
Finally, these two bifurcations β_{2B}^{-1} and β_{ZP}^{-1} interchange sequence of the points $(b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$ for $z_{l_s} \neq 1$. If $\xi = 1$ then we have situation without CP and between two zeroes exist PP (see Figure 2.4). If $\xi \to 0$ then $p_k \to +\infty$.



(a) $\xi = 0.795$ (b) $\xi = 0.798$ (c) $\xi = \xi_{2b} = 0.798552\ldots$

(d) $\xi = 0.799$ (e) $\xi = \xi_c = 4/5$ (f) $\xi = 0.801$

Fig. 2.6. Bifurcations in Case 1.

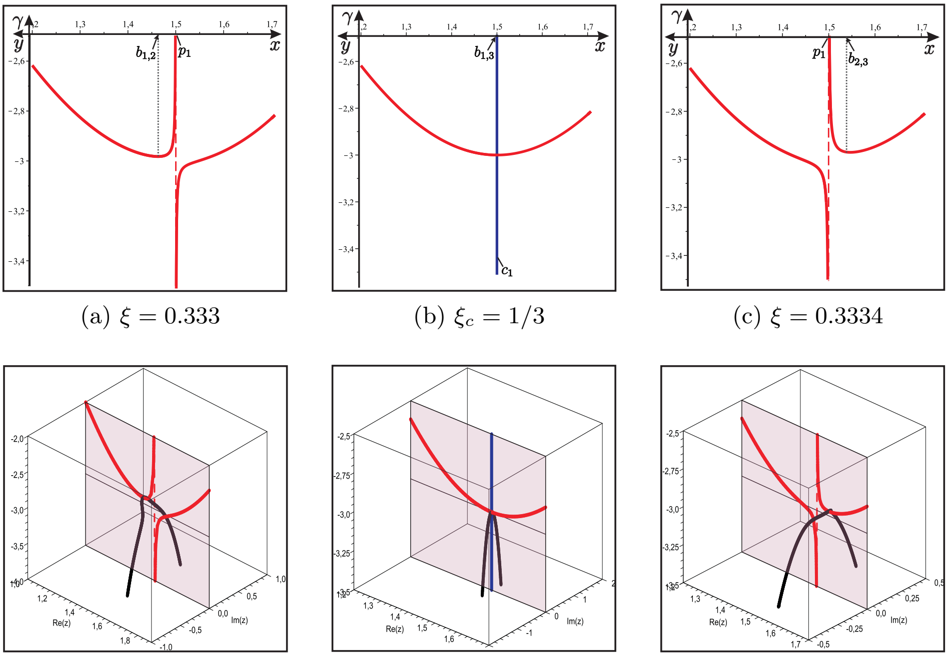


(a) $\xi = 0.798$ (b) $\xi = \xi_{2b} = 0.798552\ldots$ (c) $\xi = 0.799$

Fig. 2.7. CF in Case 1.

Bifurcations in Case 2. In this case the spectrum of complex eigenvalues is not so complicated like in the first case.

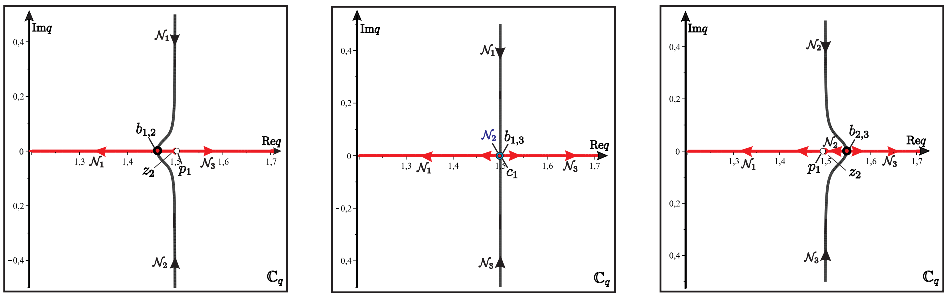
If $\xi \in [0, 1)$ is increasing from 0 to 1, then zeroes $p_k = (k-1/2)/\xi$, $k \in \mathbb{N}$, of the function $P_\xi(q)$ are moving to the left and zeroes $z_l = l - 1/2$, $l \in \mathbb{N}$, of the function $Z(q)$ remain unchanged. If $\xi = 0$ then PPs do not exist. In the limit case $\xi \to 1$ we have $p_k \to z_k$ (see Figure 2.3). Every p_k coincides with $z_l = l$, $l = k+1, \ldots$, for $\xi = (2k-1)/(2l-1)$ and we have CEP $c_s = p_{k_s} = z_{l_s} = n(s-1/2)$, $k_s = m(2s-1)/2 + 1/2$, $l_s = n(2s-1)/2 + 1/2$,



(a) $\xi = 0.333$ (b) $\xi_c = 1/3$ (c) $\xi = 0.3334$

(d) $\xi = 0.333$ (e) $\xi_c = 1/3$ (f) $\xi = 0.3334$

Fig. 2.8. CF in Case 2.



(a) $\xi = 0.333$ (b) $\xi_c = 1/3$ (c) $\xi = 0.3334$

Fig. 2.9. Bifurcations in Case 2.

$s \in \mathbb{N}$ (see (1.15_1)). Formula (1.18_1) shows that we have the same situation for all CEPs and every CEP is CP.

In Figure 2.8(a)–(c) and Figure 2.8(d)–(f) we can see how the CFs depend on the value of the parameter ξ in the neighbourhood of $\xi_c = \frac{1}{3}$. The first CEP $c_1 = 1.5$ and Spectrum Curves for the parameter ξ in the neighbourhood of $\xi_c = 1/3$ are presented in Figure 2.9. When $\xi \lesssim \frac{1}{3}$ the PP p_1 is moving from right side to the zero point $z_2 = 1.5$. For $\xi = 1/3$ we have CEP at $c_1 = 1.5$. Next, if the value of $\xi \gtrsim \frac{1}{3}$ is increasing, the PP p_1 moves to the left from the zero point. Because every CEP is CP we have symmetrical situation and the complex part of Spectrum Curves is simple (the CP is conected with infinity). We denote such *symmetric Zero and Pole bifurcation* type by β^0_{ZP}: $(b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to c_s = b_{l_s-1,l_s+1} \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$. The CEPs exist for all rational $\xi = m/n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_o$. So, every such ξ is a point of bifurcation near all CEPs for this value of ξ.

Remark 2.8. If $\xi = 1$ in Case 2 (see Figure 2.3(b)) we have CEPs $c_l = z_l = p_l = l - 1/2$, $l \in \mathbb{N}$, and CF $\gamma \equiv 1$. If $\xi = 0$ in Case 1 (see Figure 2.4(c)) the CF looks like CEPs $z_l = l - 1/2$, $l \in \mathbb{N}$.

2 The Sturm–Liouville Problems with one classical Neumann condition and another Two-Point NBC

Let us investigate SLP

$$-u'' = \lambda u, \quad t \in (0,1), \tag{2.1}$$

with one classical (Neumann type) BC:

$$u'(0) = 0, \tag{2.2}$$

and another two-point NBC ($0 \leqslant \xi \leqslant 1$):

$$u'(1) = \gamma u(\xi), \tag{2.3$_1$}$$

$$u'(1) = \gamma u'(\xi), \tag{2.3$_2$}$$

$$u(1) = \gamma u'(\xi), \tag{2.3$_3$}$$

$$u(1) = \gamma u(\xi), \tag{2.3$_4$}$$

where parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. The eigenvalue $\lambda \in \mathbb{C}_\lambda := \mathbb{C}$ and eigenfunction $u(t)$ can be complex function.

If $\gamma = \infty$, instead BCs (2.3) we obtain the SLP with BCs ($\xi > 0$): $u(\xi) = 0$ (Cases 1 and 4) and $u'(\xi) = 0$ (Cases 2 and 3). If $\gamma = 0$, we obtain classical BVP. In this case, all the eigenvalues are positive and eigenfunctions do not depend on the parameter ξ:

$$\lambda_k = (\pi k)^2, \qquad u_k = \cos(\pi k t), \qquad k \in \mathbb{N}, \tag{2.4$_{1,2}$}$$

$$\lambda_k = \pi^2 (k - 1/2)^2, \qquad u_k = \cos(\pi (k - 1/2) t), \qquad k \in \mathbb{N}. \tag{2.4$_{3,4}$}$$

If $\gamma \neq 0$, classical eigenvalues and eigenfunctions (2.4) we also obtain, when parameter $\xi = 0$ (Cases 2 and 3). For $\xi = 1$ (Cases 1 and 3) we have third type classical BC. In Cases 2 and 4 we have the classical problem only for $\xi = 1$ and $\gamma \neq 1$. The second BC is trivial for $\gamma = 1$ (Cases 2 and 4).

2.1 Constant Eigenvalues and Characteristic Function

Let us investigate a problem (2.1)–(2.3) and consider $\xi \in [0, 1]$ in Case 1, $\xi \in (0, 1)$ in Case 2, $\xi \in (0, 1]$ in Case 3 and $\xi \in [0, 1)$ in Case 4.

If $\lambda = 0$, then a function $u(t) = Cu_0(t)$, where $u_0(t) := 1$, satisfy (2.1) equation and BC (2.2). Substituting this function into the second NBC (2.3), we obtain that eigenvalue $\lambda = 0$ ($C \neq 0$) exists if and only if: $\gamma = 0$ in Case 1, γ is any number in Case 2, $\gamma = 1$ in Case 4. In Case 3 eigenvalue $\lambda = 0$ does not exist. So, in Case 2 $q = 0$ is CEP.

If $\lambda \neq 0$ function $u(t) = Cu_q(t)$, $u_q(t) = \cos(\pi qt)$, satisfies equation (2.1) and BC (2.2), where $\lambda = (\pi q)^2$. Then a map $\lambda = (\pi q)^2$ is the bijection between $\mathbb{C}_q$ and $\mathbb{C}_\lambda$ [166, Štikonas and Štikonienė 2009]. Here $q = 0$ corresponds to $\lambda = 0$. This bijection is a conformal map, except the point $q = 0$.

If we substitute function $u_q(t)$ into NBC (2.3) then we get the equality:

$$-C\pi q \sin(\pi q) = C\gamma \cos(\pi q\xi), \qquad (2.5_1)$$

$$-C\pi q \sin(\pi q) = -C\gamma\pi q \sin(\pi q\xi), \qquad (2.5_2)$$

$$C \cos(\pi q) = -C\gamma\pi q \sin(\pi q\xi), \qquad (2.5_3)$$

$$C \cos(\pi q) = C\gamma \cos(\pi q\xi). \qquad (2.5_4)$$

There exists a nontrivial solution (eigenfunction) if q is the root of the function:

$$-\pi q \sin(\pi q) = \gamma \cos(\pi q\xi), \qquad (2.6_1)$$

$$q \sin(\pi q) = \gamma q \sin(\pi q\xi), \qquad (2.6_2)$$

$$-\cos(\pi q) = \gamma\pi q \sin(\pi q\xi), \qquad (2.6_3)$$

$$\cos(\pi q) = \gamma \cos(\pi q\xi). \qquad (2.6_4)$$

We see that (2.6) in Case 4 is the same as (1.7) in Case 2.

Theorem 2.9. *Spectra for SLPs* (1.1)–(1.3$_2$) *and* (2.1)–(2.3$_2$) *overlap for all* γ *and* ξ.

Corollary 2.10. *Spectrum Curves and Spectrum Domain* $\mathcal{N}_\xi$ *for SLPs* (1.1)–(1.3$_2$) *and* (2.1)–(2.3$_2$) *are the same.*

Remark 2.11. Eigenfuntions for SLP (1.1)–(1.3$_2$) are $u = \sin(\pi qt)/(\pi q)$, $q \in \mathcal{N}_\xi$. Eigenfuntions for SLP (2.1)–(2.3$_4$) are $u = \cos(\pi qt)$, $q \in \mathcal{N}_\xi$.

We introduce two entire functions:

$$Z(z) := \pi z \sin(\pi z); \qquad P_\xi(z) := -\cos(\xi \pi z), \qquad z \in \mathbb{C}, \tag{2.7$_1$}$$
$$Z(z) := \pi z \sin(\pi z); \qquad P_\xi(z) := \pi z \sin(\pi z \xi), \qquad z \in \mathbb{C}, \tag{2.7$_2$}$$
$$Z(z) := \cos(\pi z); \qquad P_\xi(z) := -\pi z \sin(\pi z \xi), \qquad z \in \mathbb{C}. \tag{2.7$_3$}$$

Zeroes points. Zeroes set

$$\hat{\mathcal{Z}} := \{z_l = l, \quad l \in \mathbb{N}_0\}, \tag{2.8$_{1,2}$}$$
$$\hat{\mathcal{Z}} := \{z_l = l - 1/2, \quad l \in \mathbb{N}\}. \tag{2.8$_3$}$$

of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with EPs in the classical case $\gamma = 0$. All zeroes for $l \in \mathbb{N}$ are simple and positive. In Cases 1 and 2 $z_0 = 0$ is zero of the second order.

All zeroes of the function $P_\xi(q)$ in $\mathbb{C}_q$ are simple, real and positive:

$$\overline{\mathcal{Z}}_\xi := \{p_k = (k - 1/2)/\xi, \ \ k \in \mathbb{N}\}, \tag{2.9$_1$}$$
$$\overline{\mathcal{Z}}_\xi := \{p_k = k/\xi, \quad k \in \mathbb{N}_0\}. \tag{2.9$_{2,3}$}$$

All zeroes for $k \in \mathbb{N}$ are simple, nonnegative numbers. In Cases 2 and 3 $p_0 = 0$ is zero of the second order.

We rewrite the equation (2.6) in the form:

$$Z(q) = \gamma P_\xi(q), \quad q \in \mathbb{C}_q. \tag{2.10}$$

Constant Eigenvalues. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z(q) = 0, \qquad P_\xi(q) = 0. \tag{2.11}$$

Remark 2.12. In the Case 1, if the parameter $\xi = 0$ or $\xi = 1$, then CEPs do not exist. In the Case 3, if the parameter $\xi = 1$, then CEPs do not exist.

Remark 2.13. In Case 2 RP $q = 0$ is CEP (of the second order). For the other cases CEPs are positive.

Remark 2.14. If the parameter $\xi \notin \mathbb{Q}$, then CEPs does not exist, because the equations

$$\xi l = k - \tfrac{1}{2}, \tag{2.12$_1$}$$
$$\xi l = k, \tag{2.12$_2$}$$
$$\xi (l - 1/2) = k \tag{2.12$_3$}$$

have not roots for integer l, k.

Lemma 2.15. *For SLP* (2.1)–(2.3$_1$) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0,\,1)$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, $\gcd(m,n) = 1$, *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.

Proof. We rewrite equation (2.12$_1$) as $ml - nk = -n/2$. So, n must be even, i.e. $m \in \mathbb{N}_o$ and $n \in \mathbb{N}_e$. In this case we have solution $l = ns - n/2$, $k = ms - (m-1)/2$, $s \in \mathbb{N}$. Then $c_s = n(s - 1/2)$, $s \in \mathbb{N}$. □

Lemma 2.16. *For SLP* (2.1)–(2.3$_2$) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0,\,1)$, $m, n \in \mathbb{N}$, $\gcd(m,n) = 1$, *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s := ns$, $s \in \mathbb{N}_0$.

Proof. We rewrite equation (2.12$_2$) as $ml - nk = 0$. In this case we have solution $l = ns$, $k = ms$, $s \in \mathbb{N}_0$. Then $c_s = ns$, $s \in \mathbb{N}_0$. □

Lemma 2.17. *For SLP* (2.1)–(2.3$_2$) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0,\,1)$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, $\gcd(m,n) = 1$, *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s := n(s - 1/2)$, $s \in \mathbb{N}$.

Proof. We rewrite equation (2.12$_3$) as $ml - nk = m/2$. So, m must be even, i.e. $m \in \mathbb{N}_e$ and $n \in \mathbb{N}_o$. In this case we have solution $l = ns - (n-1)/2$, $k = ms - m/2$, $s \in \mathbb{N}$. Then $c_s = n(s - 1/2)$, $s \in \mathbb{N}$. □

Remark 2.18. CEP at Ramification Point $c_0 = 0$ in Case 2 is double in $\mathbb{C}_q$ but corresponding CE $\lambda = 0$ is simple.

Complex Characteristic Function. For SLP (2.1)–(2.3) we have meromorphic Complex Characteristic Functions (Complex CF)

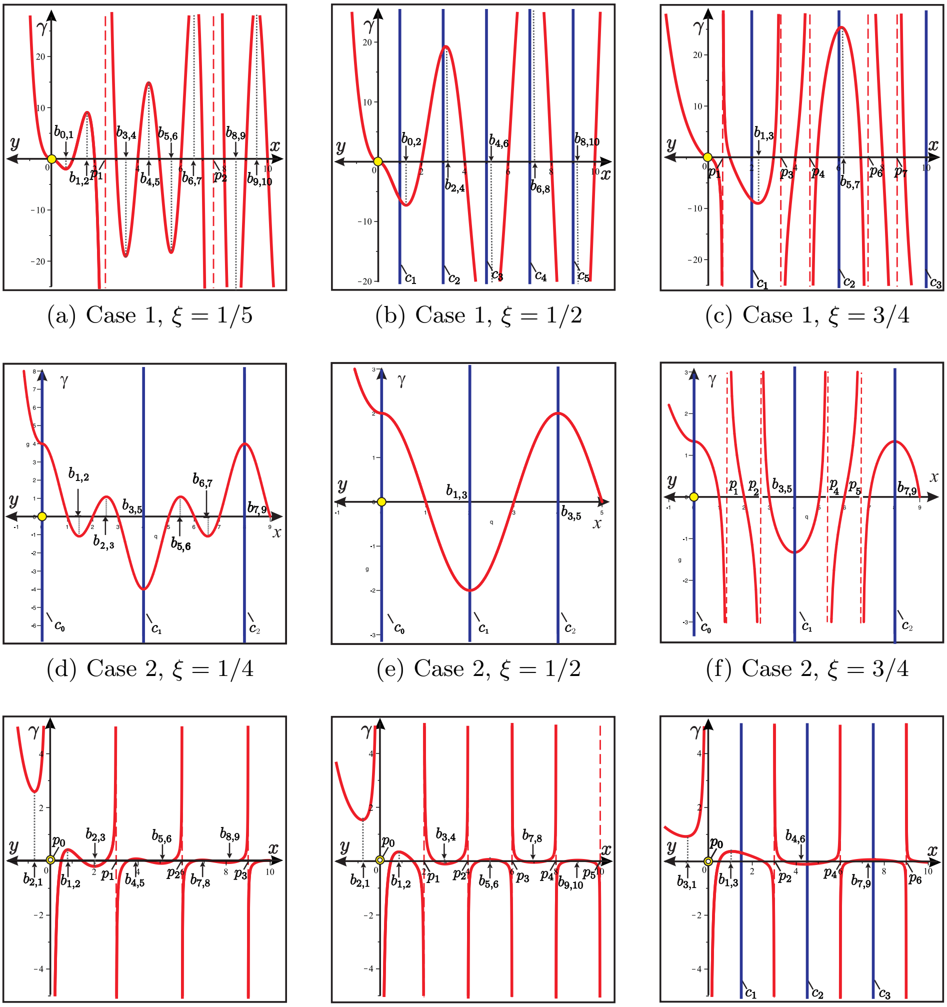
$$\gamma_c(q) : -\frac{\pi q \sin(\pi q)}{\cos(\xi \pi q)}, \tag{2.13$_1$}$$

$$\gamma_c(q) := \frac{\sin(\pi q)}{\sin(\xi \pi q)}, \tag{2.13$_2$}$$

$$\gamma_c(q) := -\frac{\cos(\pi q)}{\pi q \sin(\xi \pi q)}. \tag{2.13$_3$}$$

Remark 2.19. In the article [166, 2009] were analyzed CF (2.13$_2$) in Case 2.

All zeroes and poles of meromorphic function $\gamma_c(q)$ lie on the nonnegative part of real axis. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. So, $p_k \in \overline{\mathcal{Z}}_\xi$ is PP if and only if $p_k \notin \mathbb{N}$ in Case 1 and $p_k + 1/2 \notin \mathbb{N}$ in Case 3. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. If



(a) Case 1, $\xi = 1/5$ (b) Case 1, $\xi = 1/2$ (c) Case 1, $\xi = 3/4$

(d) Case 2, $\xi = 1/4$ (e) Case 2, $\xi = 1/2$ (f) Case 2, $\xi = 3/4$

(g) Case 3, $\xi = 1/3$ (h) Case 3, $\xi = 1/2$ (i) Case 3, $\xi = 2/3$

Fig. 2.10. Real CF (Neumann BC) for various ξ.

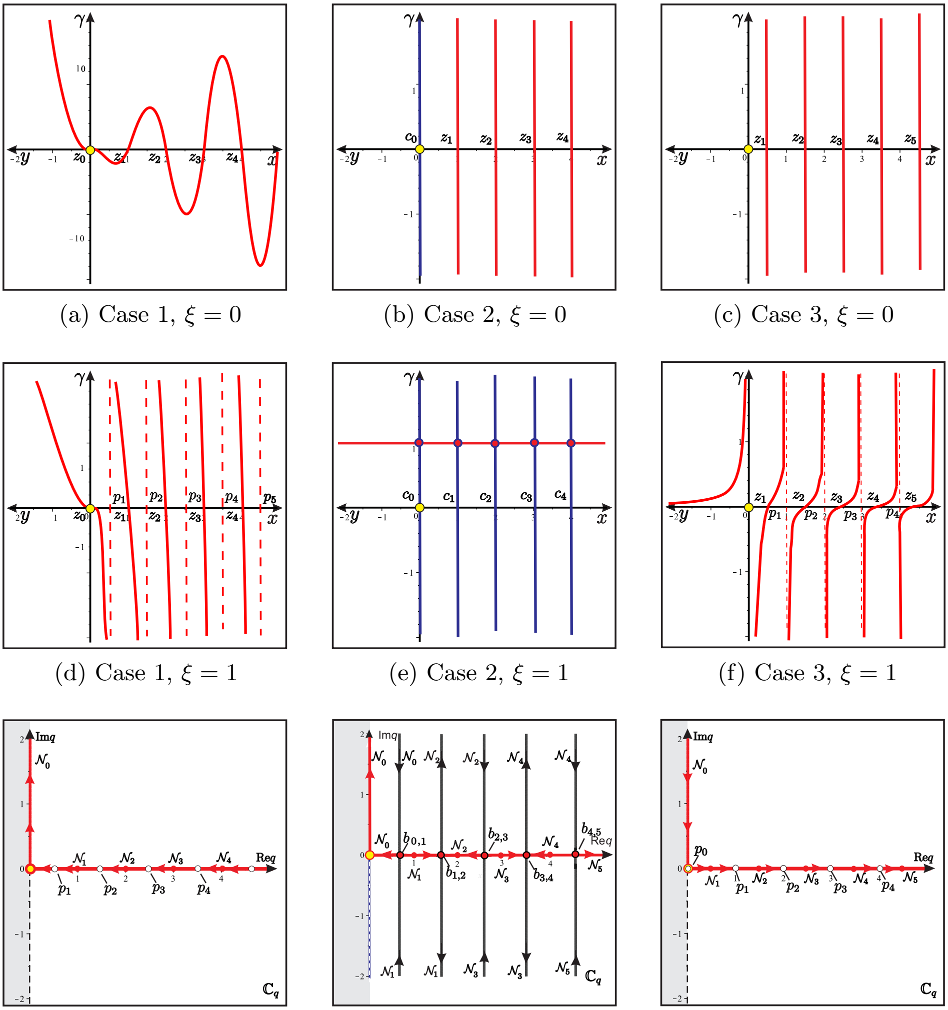
$c \in \mathcal{C}_\xi$, i.e. $Z(c) = 0$ and $P_\xi(c) = 0$, then we have removable singularity isolated point and we have sequence of such points $c_s = p_{k_s} = z_{l_s}$, $s \in \mathbb{N}$,

$$k_s = m(s - 1/2) + 1/2, \quad l_s = n(s - 1/2), \qquad m \in \mathbb{N}_o, n \in \mathbb{N}_e, \quad (2.14_1)$$

$$k_s = ms, \quad l_s = ns, \qquad m, n \in \mathbb{N}, \quad (2.14_2)$$

$$k_s = m(s - 1/2), \quad l_s = n(s - 1/2) + 1/2, \quad m \in \mathbb{N}_e, n \in \mathbb{N}_o. \quad (2.14_3)$$

Remark 2.20. In Case 1 and $\xi = 0$ function $P_\xi \equiv 1$. So, PPs do not exist. If $\xi > 0$ then it follows from (2.14) that $1 = k_1$ if $m = 1$ in Cases 1,2 and $m = 2$ in Case 3. Then $p_k = n(k - 1/2)$ in Case 1, $p_k = nk$ in Cases 2,3 and $c_k = n(k - 1/2)$ in Cases 1,3 $c_k = nk$ in Case 2 ($k \in \mathbb{N}$). In Case 3 $p_1 = c_1$



(a) Case 1, $\xi = 0$ (b) Case 2, $\xi = 0$ (c) Case 3, $\xi = 0$

(d) Case 1, $\xi = 1$ (e) Case 2, $\xi = 1$ (f) Case 3, $\xi = 1$

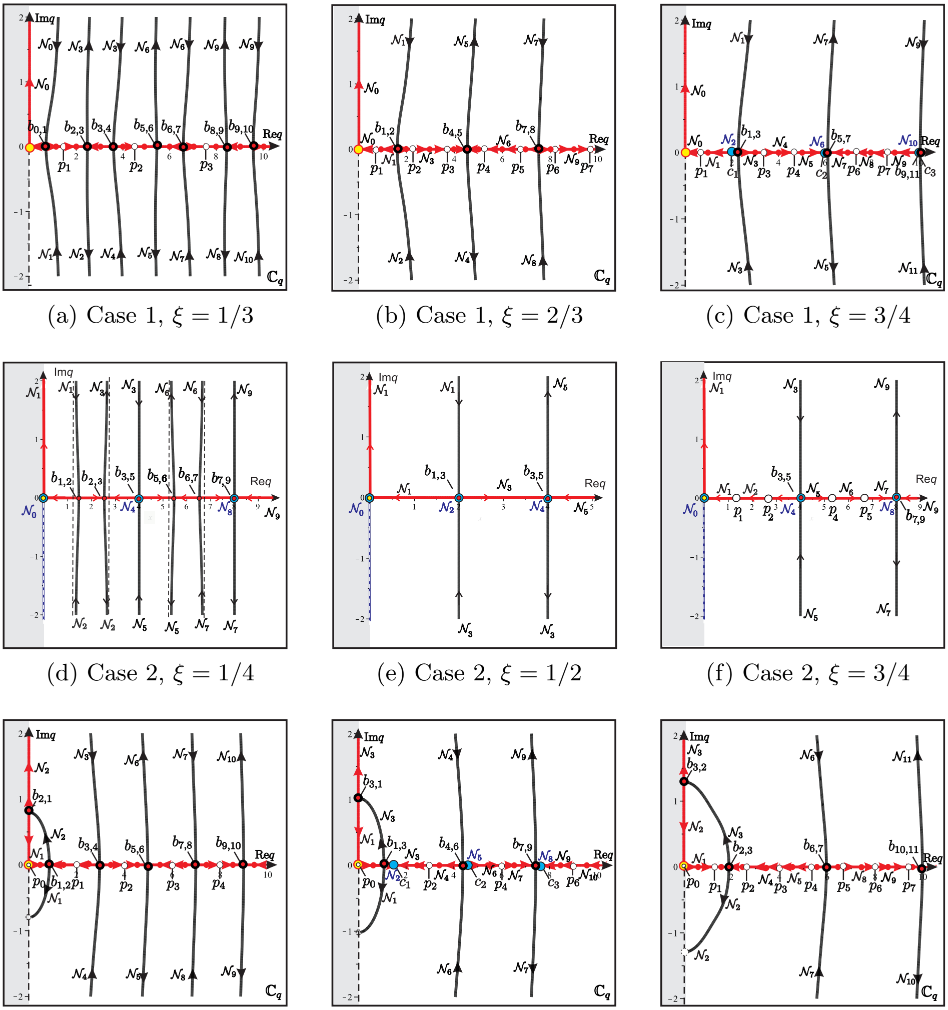
(g) Case 1, $\xi = 1$ (h) Case 1, $\xi = 0$ (i) Case 3, $\xi = 1$

Fig. 2.11. (a)–(f) Real CF (Neumann BC) for $\xi = 0$ and $\xi = 1$. (g)–(i) Spectrum Curves in Case 1 for $\xi = 0$, $\xi = 1$ and in Case 3 $\xi = 1$.

but $p_2 < c_2$ and therefore PPs exist. If $\xi = 1/n$ then PPs do not exist for $n \in \mathbb{N}_e$ in Case 1 and $n \in \mathbb{N}$ in Case 2.

Remark 2.21. In Case 3 RP is PP of the second order in $\mathbb{C}_q$ (of the first order in $\mathbb{C}_\lambda$) for all $\xi \in (0, 1]$.

Real Characteristic Function. *Real Characteristic Function* (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Com-



(a) Case 1, $\xi = 1/3$ (b) Case 1, $\xi = 2/3$ (c) Case 1, $\xi = 3/4$

(d) Case 2, $\xi = 1/4$ (e) Case 2, $\xi = 1/2$ (f) Case 2, $\xi = 3/4$

(g) Case 3, $\xi = 1/2$ (h) Case 3, $\xi = 2/3$ (i) Case 3, $\xi = 3/4$

Fig. 2.12. Domain $\mathcal{N}_\xi$ for CF $\gamma(q)$ (Neumann BC).
⊙ – Constant Eigenvalue Point at Ramification Point

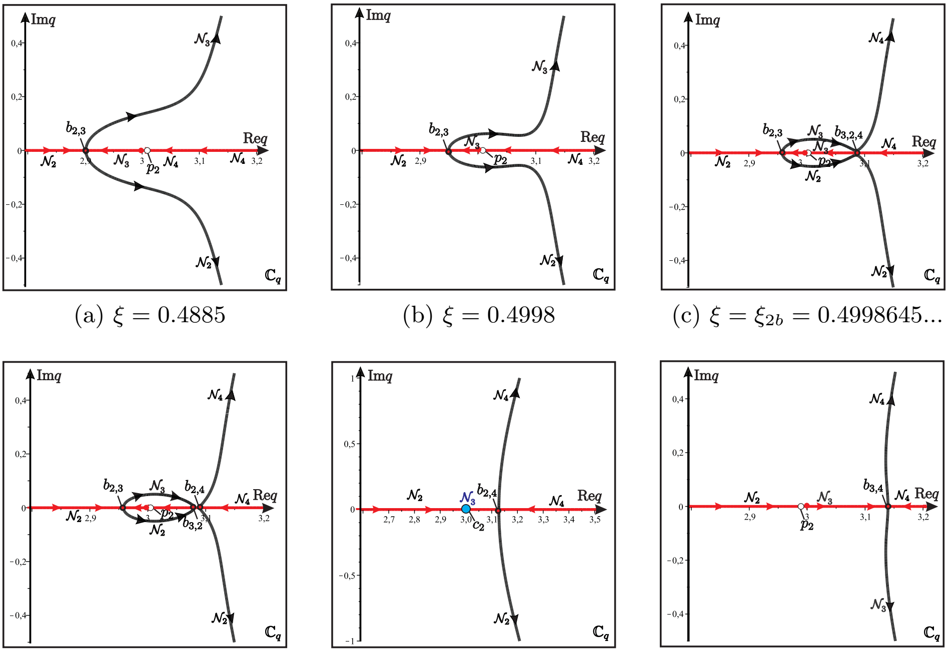
plex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$:

$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} -\dfrac{\pi x \sinh(\pi x)}{\cosh(\xi \pi x)}, & x \leqslant 0; \\ -\dfrac{\pi x \sin(\pi x)}{\cos(\xi \pi x)}, & x \geqslant 0. \end{cases} \tag{2.15$_1$}$$

$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} \dfrac{\sinh(\pi x)}{\sinh(\xi \pi x)}, & x \leqslant 0; \\ \dfrac{\sin(\pi x)}{\sin(\xi \pi x)}, & x \geqslant 0. \end{cases} \tag{2.15$_2$}$$

$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} -\dfrac{\cosh(\pi x)}{\pi x \sinh(\xi \pi x)}, & x > 0; \\ -\dfrac{\cos(\pi x)}{\pi x \sin(\xi \pi x)}, & x < 0. \end{cases} \tag{2.15$_3$}$$

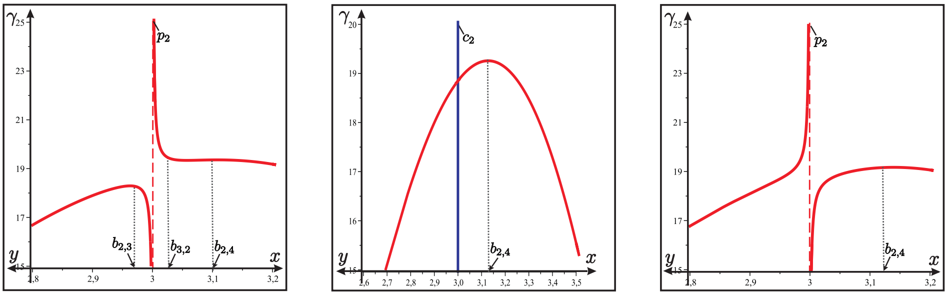
We calculate values of Real CF and it's derivatives at CEP ($c_s := n(s -$



(a) $\xi = 0.4885$ (b) $\xi = 0.4998$ (c) $\xi = \xi_{2b} = 0.4998645\ldots$

(d) $\xi = 0.499867$ (e) $\xi = \xi_c = 1/2$ (f) $\xi = 0.5001$

Fig. 2.13. Bifurcations in Case 1 (Neumann BC).



(a) $\xi = 0.4999$ (b) $\xi = \xi_c = 1/2$ (c) $\xi = 0.5001$

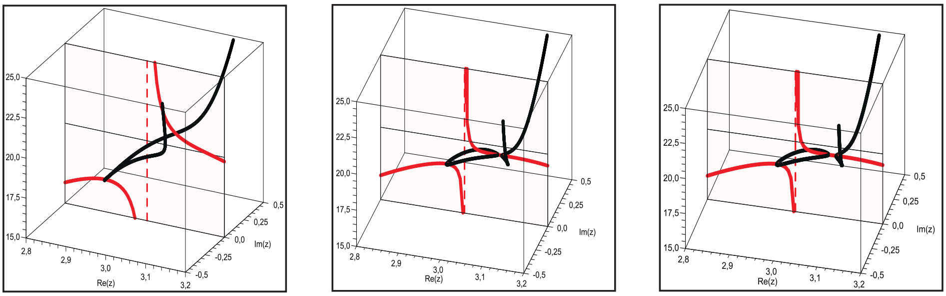
Fig. 2.14. Real CF (Neumann BC) for $\xi \in (0.4999; 0.5001)$ in Case 1 in the neighborhood $x = 3$.

$1/2)$, $s \in \mathbb{N}$, $\xi = m/n \in (0, 1)$ in Cases 1 and 3; $c_s := ns$, $s \in \mathbb{N}_0$, $\xi = m/n \in (0, 1)$ in Case 2):

$$\gamma_s(\xi) := \gamma(c_s; \xi) = (-1)^{s+(n-m-1)/2} c_s \pi \xi^{-1}, \qquad m \in \mathbb{N}_o, n \in \mathbb{N}_e, \quad (2.16_1)$$

$$\gamma_s(\xi) := \gamma(c_s; \xi) = (-1)^{(n-m)s} \xi^{-1} = (-1)^{(1-\xi)c_s} \xi^{-1}, \quad (2.16_2)$$

$$\gamma_s(\xi) := \gamma(c_s; \xi) = (-1)^{s+(n-m+1)/2} c_s^{-1} \pi^{-1} \xi^{-1}, \quad m \in \mathbb{N}_e, n \in \mathbb{N}_o, \quad (2.16_3)$$



(a) $\xi = 0.499$ (b) $\xi = \xi_{2b} = 0.499867...$ (c) $\xi = 0.49987$

Fig. 2.15. CF for $\xi \in (0.499; 0.49987)$ in Case 1 (Neumann BC) in the neighborhood $q = 3$.

$$\gamma'_s(\xi) := \gamma'(c_s;\xi) = c_s^{-1}\gamma_s, \tag{2.17$_1$}$$

$$\gamma'_s(\xi) := \gamma'(c_s;\xi) = 0, \tag{2.17$_2$}$$

$$\gamma'_s(\xi) := \gamma'(c_s;\xi) = -c_s^{-1}\gamma_s, \tag{2.17$_3$}$$

$$\gamma''_s(\xi) := \gamma''(c_s;\xi) = -\pi^2(1-\xi^2)/3\cdot\gamma_s, \tag{2.18$_1$}$$

$$\gamma''_s(\xi) := \gamma''(c_s;\xi) = -\pi^2(1-\xi^2)/3\cdot\gamma_s, \tag{2.18$_2$}$$

$$\gamma''_s(\xi) := \gamma''(c_s;\xi) = \big(2c_s^{-2} - \pi^2(1-\xi^2)/3\big)\gamma_s. \tag{2.18$_3$}$$

We see, that $\gamma_s \neq 0$, $\gamma''_s \neq 0$ for all ξ and s. In the article [166, 2009] were analyzed CF (2.13$_2$) for Case 2. Graphs of the function $\gamma_r(x)$ for various ξ are shown in Figure 2.10.

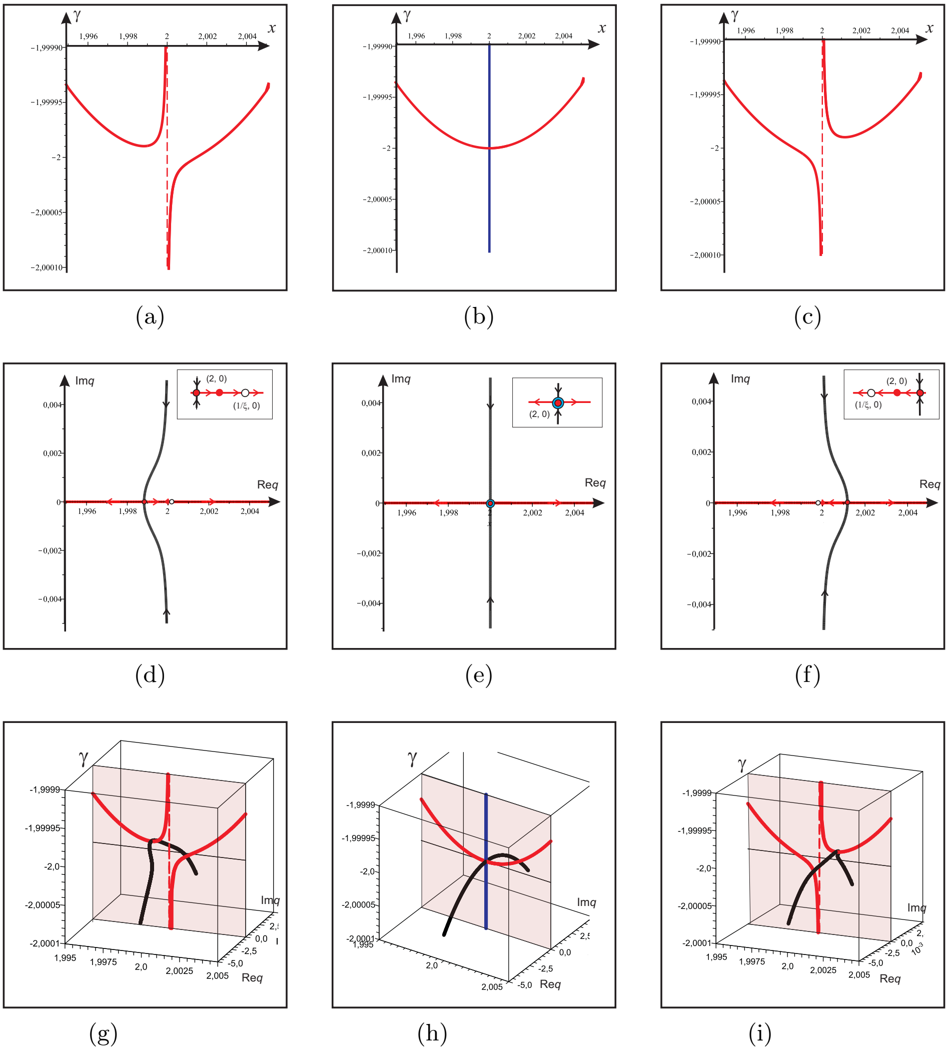
Remark 2.22. In Case 1 and $\xi = 0$ we have entire CF (PPs do not exist, see Figure 2.11(a)). In Case 3 and $\xi = 1$ we have one semi-regular Spectral Curve which describes negative eigenvalues for $\gamma \in (0, +\infty)$. We note that in this case for $\tilde{\gamma}$ parametrization all Spectrum Curves are regular and CF coincide with CF in Case 1 for $\xi = 1$ (see Figure 2.11(d) and Figure 2.11(f)). Degenerate cases ($\xi = 0$ and $\xi = 1$ in Case 2, $\xi = 0$ in Case 3) are presented in Figure 2.11(b),(c),(e). Complex eigenvalues exist only in Case 1 for $\xi = 0$.

Ramification Point and Critical Points. The Taylor (Laurent) series for CF $\gamma(q)$ at RP $q = 0$ is:

$$\gamma(q;\xi) = -\pi^2 q^2 + \frac{1-3\xi^2}{6}\pi^4 q^4 + \mathcal{O}(q^6), \tag{2.19$_1$}$$

$$\gamma(q;\xi) = \frac{1}{\xi} - \frac{1-\xi^2}{6\xi}\pi^2 q^2 + \mathcal{O}(q^4), \tag{2.19$_2$}$$

$$\gamma(q;\xi) = -\frac{1}{\pi^2\xi}q^{-2} - \frac{3-\xi^2}{6\xi} + \mathcal{O}(q^2). \tag{2.19$_3$}$$

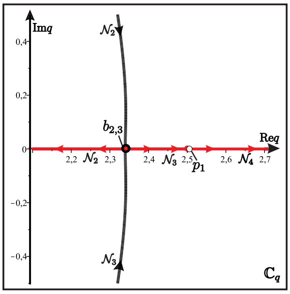


$\xi = 0.499999999$ $\qquad$ $\xi = \xi_c = 1/2$ $\qquad$ $\xi = 0.500000001$

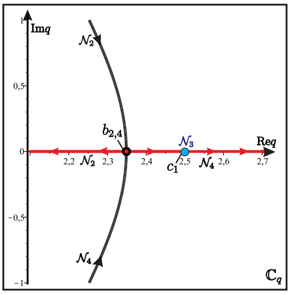
Fig. 2.16. Complex-real characteristic functions (real, domain $\mathcal{N}$ and complex-real). Their dependence on the parameter ξ in the neighborhood of the constant eigenvalue point $\xi = 0.5$ and $q = 2$. [166, Štikonas and Štikonienė 2009]

Remark 2.23. In Cases 1 and 2 multiplier of q^2 is nonzero. So, $q = 0$ is CP of the first order in $\mathbb{C}_q$, but corresponding eigenvalue $\lambda = 0$ is simple. In Case 3 $q = 0$ is pole of the second order in $\mathbb{C}_q$, but corresponding point in $\mathbb{C}_\lambda$ has properties as pole of the first order.

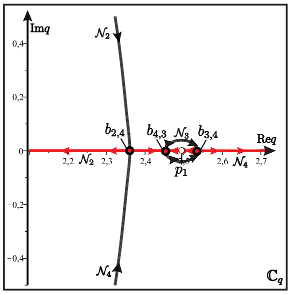
For the SLPs (2.1)–(2.3_1) and (2.1)–(2.3_2) negative CP ($b \in \mathbb{R}_q^-$) do not exist [107, Pečiulytė *et al.* 2008]. For the SLP (2.1)–(2.3_3) we have one negative CP for $0 < \xi < 1$ [107].



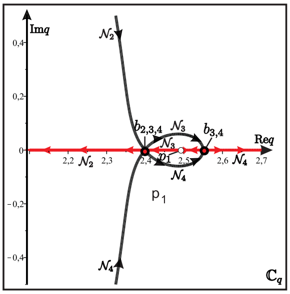
(a) $\xi = 0.3999$



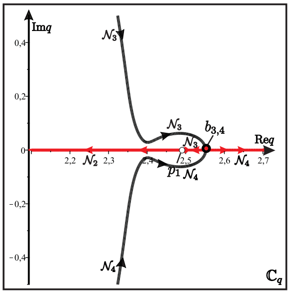
(b) $\xi = \xi_c = 2/5$



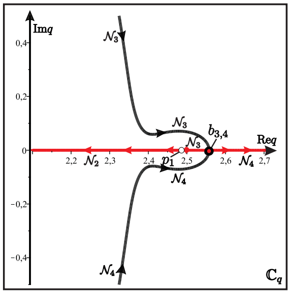
(c) $\xi = 0.4001$



(d) $\xi = \xi_{2b} = 0.400218...$

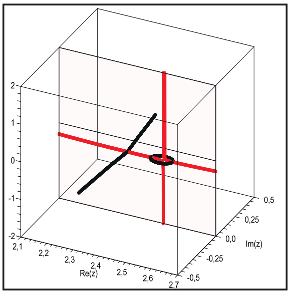


(e) $\xi = 0.400235$

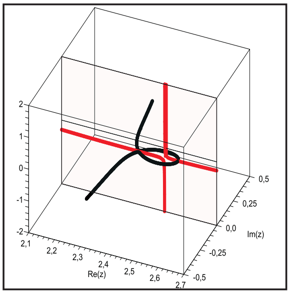


(f) $\xi = 0.40036$

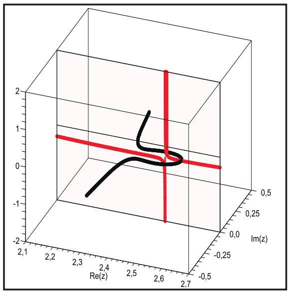
Fig. 2.17. Bifurcations in Case 3 (Neumann BC).



(a) $\xi = 0.4001$



(b) $\xi = \xi_{2b} = 0.400218...$



(c) $\xi = 0.4003$

Fig. 2.18. CF for $\xi \in (0.4001; 0.4003)$ in Case 3.

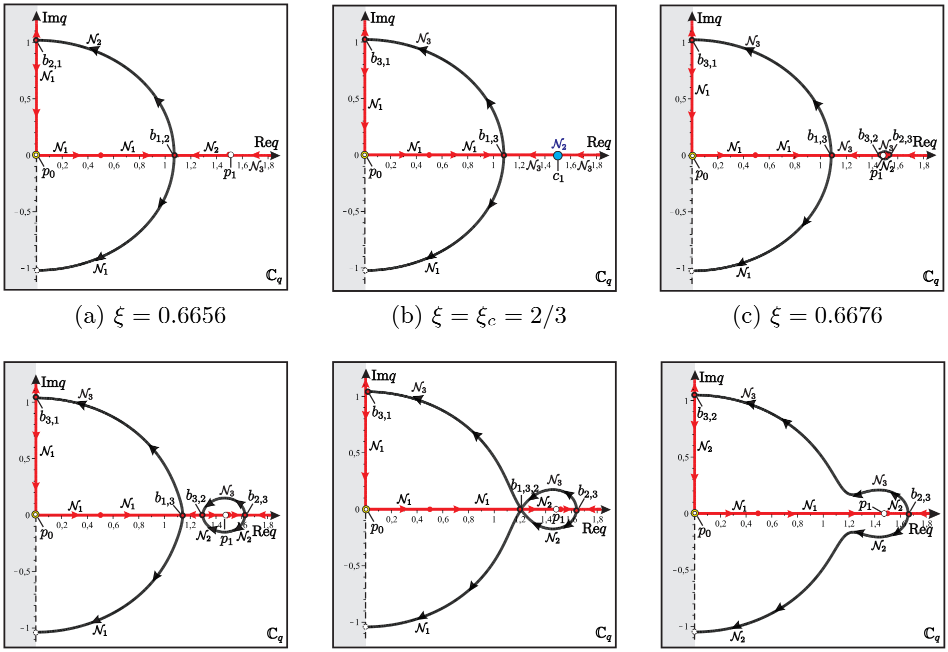
Lemma 2.24. *Zero Point $q \neq 0$ of CF can not be CP.*

Proof. For CF (2.13) we have

$$\gamma' = \gamma(q)\Big(\frac{1}{q} + \xi\pi\frac{\sin(\xi\pi q)}{\cos(\xi\pi q)}\Big) + \pi^2 q\frac{\cos(\pi q)}{\cos(\xi\pi q)}, \tag{2.20_1}$$

$$\gamma' = -\gamma(q)\xi\pi\frac{\cos(\xi\pi q)}{\sin(\xi\pi q)} + \pi\frac{\cos(\pi q)}{\sin(\xi\pi q)}. \tag{2.20_2}$$

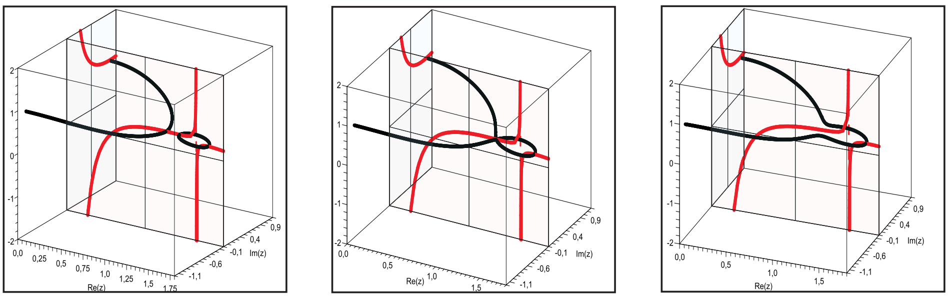
$$\gamma' = -\gamma(q)\Big(\frac{1}{q} + \xi\pi\frac{\cos(\xi\pi q)}{\sin(\xi\pi q)}\Big) + \frac{\sin(\pi q)}{q\sin(\xi\pi q)}, \tag{2.20_3}$$



(a) $\xi = 0.6656$ (b) $\xi = \xi_c = 2/3$ (c) $\xi = 0.6676$

(d) $\xi = 0.673$ (e) $\xi = \xi_{2b} = 0.67495...$ (f) $\xi = 0.678$

Fig. 2.19. Bifurcations in Case 3 (Neumann BC) near to point $q = 0$.



(a) $\xi = 0.673$ (b) $\xi = \xi_{2b} = 0.67495...$ (c) $\xi = 0.678$

Fig. 2.20. CF for $\xi \in (0.673; 0.678)$ in Case 3 (Neumann BC).

If $\gamma(q_z) = 0$ and $q_z \neq 0$ then $q_z \notin \mathcal{C}_\xi$. So, $\sin(\pi q_z) = 0$, $\cos(\pi q_z) \neq 0$, $\cos(\xi\pi q_z) \neq 0$ in Case 1; $\sin(\pi q_z) = 0$, $\cos(\pi q_z) \neq 0$, $\sin(\xi\pi q_z) \neq 0$ in Case 2; $\cos(\pi q_z) = 0$, $\sin(\pi q_z) \neq 0$, $\sin(\xi\pi q_z) \neq 0$ in Case 2. It follows from from (2.20) for all three cases that $\gamma'(q_z) \neq 0$. □

For the SLPs (2.1)–(2.3$_1$) and (2.1)–(2.3$_2$) we have CPs of the first and the second order. For the SLP (2.1)–(2.3$_2$) we have nonzero CPs of the first order and CP $b = 0$ has properties of the second order CP only in $\mathbb{C}_q$.

Finally we formulate theorem about Case 2.

Theorem 2.25. *Spectrum of SLP* (2.1)–(2.3$_2$) *has one additional simple eigenvalue* $\lambda = 0$ *in addition to eigenvalues of spectrum of SLP* (10.1)–(10.2) *in Introduction [166, Štikonas and Štikonienė 2009] for all* γ *and* $\xi \in (0, 1)$*.*

Corollary 2.26. *Additional eigenvalue* $\lambda = 0$ *for SLP* (2.1)–(2.3$_2$) *corresponds to nonregular Spectrum Curve (CEP* $q = 0$*)* $\mathcal{N}_0$*. The other Spectrum curves for both SLPs overlap (see Figure 2.12(d)–(f) and Figure 12(a)–(c)).*

2.2 Spectrum Curves

Spectrum Curves are presented in Figure 2.12(a)–(c), Figure 2.11(g),(h) (Case 1), Figure 2.12(d)–(f) (Case 2) and Figure 2.12(g)–(i), Figure 2.11(i) (Case 3). In Cases 1 and 2 one negative eigenvalue exists for $\gamma > 0$ (Case 1), $\gamma > 1/\xi$.

In Case 3 around the PP $q = 0$ we for all $\xi \in (0, 1)$ we have a loop type Spetrum Curve and one CP $b \in \mathbb{R}_q^-$. So, for some $\gamma_b > 0$ double negative eigenvalue exist and for $\gamma > \gamma_b$ we have two simple negative eigenvalues. In case $\xi = 0$ CP disappears (∞) and we have only positive CEP type eigenvalues. In case $\xi = 1$ one negative eigenvalue exists for $\gamma > 0$ (semi-regular Spectrum Curve $\mathcal{N}_0$). PP $p_0 = 0$ and $z_1 = 1/2$ are inside the loop for all $\xi \in (0, 1)$. If $\xi \to 1$ then more and more PPs are inside the loop around the RP $q = 0$.

Bifurcations in Case 1. In this case we have two types of bifurcations $\beta_{ZP}^{-1}\colon (b_{l_s-1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1}) \to c_s \to (z_{l_s}, p_{k_s})$ and $\beta_{2B}^{-1}\colon \varnothing \to b_{l_s,l_s-1,l_s+1} \to (b_{l_s,l_s-1}, b_{l_s-1,l_s+1})$ (as in subsection 1.2 in Case 1). The same types of bifurcations are defined by expressions in formulas (1.18) and (2.17$_1$). The CEPs exist for all rational $\xi = m/n$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$ and every such ξ is a point of bifurcation near all CEPs for this value of ξ. For example, we have such type bifurcation for $\xi = 1/2, 1/4, 3/4$. Finally, these two bifurcations β_{2B}^{-1} and β_{ZP}^{-1} interchange sequence of the points $(b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$. We can see these bifurcations in Figure 2.13(a)–(f), Figure 2.14, Figure 2.15 in the neighborhood $q = 3$ for $\xi = \xi_c = 1/2$.

Bifurcations in Case 2. In this case we have two types of bifurcations $\beta_{ZP}^{0}\colon (b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to c_s = b_{l_s-1,l_s+1} \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$ (as in [166, Štikonas and Štikonienė 2009] and subsection 1.2 in Case 2). We can see these bifurcations in Figure 2.16 in the neighborhood $q = 2$ for $\xi = \xi_c = 1/2$.

Bifurcations in Case 3. In this case we have two types of bifurcations $\beta_{ZP}\colon (b_{l_s-1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1}) \to c_s \to (z_{l_s}, p_{k_s})$ and $\beta_{2B}\colon \varnothing \to b_{l_s,l_s-1,l_s+1} \to (b_{l_s,l_s-1}, b_{l_s-1,l_s+1})$ (as in Section 3 in Chapter 1). The same types of bifurcations are defined by expressions in formulas (2.17) in Chappter 1 and (2.17_3). The CEPs exist for all rational $\xi = m/n$, $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$ and every such ξ is a point of bifurcation near all CEPs for this value of ξ. For example, we have such type bifurcation for $\xi = 2/3, 2/5$. Finally, these two bifurcations β_{ZP} and β_{2B} interchange sequence of the points $(b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1})$. We can see these bifurcations in Figure 2.17(a)–(f), Figure 2.18 in the neighborhood $q = 2.5$ for $\xi = \xi_c = 2/3$.

PP $p_1 = 1/\xi$ and $z_2 = 1.5$ are inside the loop around the RP only after bifurcation β_{2B} for $\xi = \xi_{2b} \approx 0.67495 > 2/3$ (see Figure 2.19(a)–(f), Figure 2.20).

3 Sturm–Liouville Problem with one symmetrical type NC

Let us analyze the SLP with one classical BC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{3.1}$$

$$u(0) = 0, \tag{3.2}$$

and another two-point NC

$$u(\xi) = \gamma u(1-\xi), \tag{3.3}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$.

Remark 2.27. Case $\gamma = 0$. If $\xi = 0$, then have problem (3.1),(3.2) with one BC $u(0) = 0$ only. If $0 < \xi \leqslant 1$, then we have the classical BVP in the interval $[0,\xi]$ with BCs $u(0) = 0$, $u(\xi) = 0$, and its eigenvalues and eigenfunctions are

$$\lambda_k = \left(\frac{\pi k}{\xi}\right)^2, \quad u_k(t) = \sin\left(\frac{\pi k t}{\xi}\right), \quad k \in \mathbb{N}. \tag{3.4}$$

In the case $0 < \xi < 1$ we also have the initial value problem in the interval $[\xi, 1]$ with the initial conditions $u(\xi) = 0$ and $u'(\xi)$ (this value is known from SLP in $[0,\xi]$).

Case $\gamma = \infty$. If $\xi = 1$, we have problem (3.1)–(3.2) with one BC $u(0) = 0$. If $0 \leqslant \xi < 1$ then we have the same situation as in Case $\gamma = 0$ with the BVP in the interval $[0, 1-\xi]$ and the initial value problem in the interval $[1-\xi, 1]$. In the case ξ=0 we have BVP only.

Case $\xi = \frac{1}{2}$. If $\gamma = 1$, then we have problem (3.1)–(3.2) with one BC $u(0) = 0$. If $\gamma \neq 1$, then we have BVP in the interval $[0, \frac{1}{2}]$ and the initial value problem in the interval $\left[\frac{1}{2}, 1\right]$ (see case $\gamma = 0$).

Case $\xi = 0$ or $\xi = 1$ ($\gamma \neq 0$, $\gamma \neq \infty$). We have the same case as in case $\gamma = 0$ ($\xi = 1$).

Let us return to the problem (3.1)–(3.3) and consider that $0 < \xi < 1$, $\xi \neq \frac{1}{2}$. If $\lambda = 0$, then the function $u(t) = ct$ satisfies problem (3.1)–(3.2). By substituting this solution into NC, we derive that there exists a nontrivial solution ($c \neq 0$) if $c\xi = \gamma c(1 - \xi)$. So, the following lemma is valid.

Lemma 2.28. *There exists the eigenvalue $\lambda = 0$ if and only if $\gamma = \frac{\xi}{1-\xi}$.*

In general case the problem (3.1)–(3.2) has solution:

$$u = C\frac{\sin(\pi q t)}{\pi q}, \quad q \in \mathbb{C}_q. \tag{3.5}$$

3.1 Constant Eigenvalues and Characteristic Function

The NC gives equation:

$$C\frac{\sin(\xi \pi q)}{\pi q} = C\gamma \frac{\sin\big((1-\xi)\pi q\big)}{\pi q}.$$

There exists a nontrivial solution, if q is the root of the equation

$$\frac{\sin(\xi \pi q)}{\pi q} = \gamma \frac{\sin\big((1-\xi)\pi q\big)}{\pi q}. \tag{3.6}$$

We introduce two entire functions:

$$Z_\xi(z) := \frac{\sin(\xi \pi z)}{\pi z}; \quad P_\xi(z) := Z_{1-\xi}(z) = \frac{\sin\big((1-\xi)\pi z\big)}{\pi z}, \quad z \in \mathbb{C}. \tag{3.7}$$

Zeroes points. Zeroes sets of the functions $Z_\xi(q)$ and $P_\xi(q) = Z_{1-\xi}(q)$, $q \in \mathbb{C}_q$, are

$$\hat{\mathcal{Z}}_\xi := \{z_l = l/\xi, l \in \mathbb{N}\}, \tag{3.8}$$

$$\overline{\mathcal{Z}}_\xi := \{p_k = k/(1-\xi), k \in \mathbb{N}\}. \tag{3.9}$$

All zeroes of these functions are simple, real (positive integer numbers).

We rewrite the equation (3.6) in the form:

$$Z_\xi(q) = \gamma P_\xi(q) = \gamma Z_{1-\xi}(q), \quad q \in \mathbb{C}_q. \tag{3.10}$$

Constant Eigenvalues. So, if the next system is valid

$$\sin(\pi q\xi) = 0, \qquad \sin\big(\pi q(1-\xi)\big) = 0, \quad q \neq 0, \tag{3.11}$$

then equation (3.10) is valid for all γ. In this case, we get CE $\lambda = (\pi c_k)^2$, $k \in \mathbb{N}$ and CEPs c_k are the roots of system (3.6). We expand the second equation and use the first equation in expression

$$0 = \sin\big(\pi q(1-\xi)\big) = \sin(\pi q)\,\cos(\xi\pi q) - \cos(\pi q)\,\sin(\xi\pi q) = \cos(\xi\pi q)\sin(\pi q).$$

If $\sin(\pi q\xi) = 0$ then $\cos(\pi q\xi) \neq 0$. We obtain a new system

$$\sin(\pi q) = 0, \qquad \sin(\pi q\xi) = 0, \tag{3.12}$$

the roots of which are CEP of SLP (3.1)–(3.3). The system (3.12) was investigated in [166] (see Section 10 in Introduction, too).

Lemma 2.29. *For SLP* (3.1)–(3.3) *Constant Eigenvalues exist only for rational parameter* $\xi = m/n \in (0, 1)$, $m, n \in \mathbb{N}$, $\xi \neq 1/2$, *values and those eigenvalues are equal to* $\lambda_s = (\pi c_s)^2$, $c_s = ns$, $s \in \mathbb{N}$.

Complex Characteristic Function. For SLP (3.1)–(3.3) we have meromorphic Complex CF

$$\gamma_c(q) = \frac{Z(q)}{Z_{1-\xi}(q)} = \frac{\sin(\xi\pi q)}{\sin\big((1-\xi)\pi q\big)}, \quad q \in \mathbb{C}_q. \tag{3.13}$$

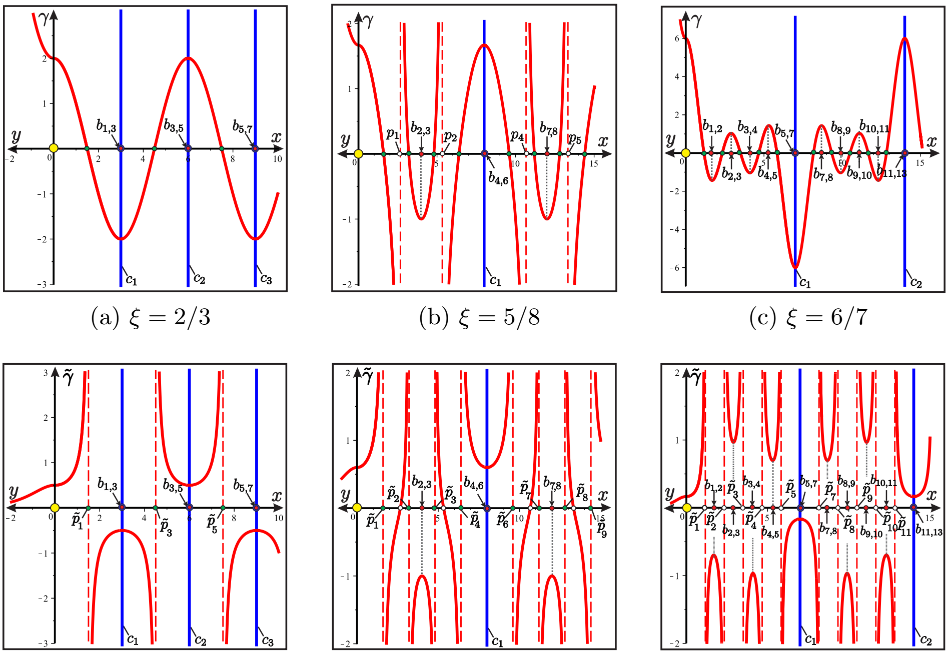
All zeroes and poles of meromorphic function $\gamma_c(q)$ lie on the positive part of real axis. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. If $c \in \mathcal{C}_\xi$, i.e. $Z(c) = 0$ and $Z_{1-\xi}(c) = 0$, then function γ_c has removable singularity isolated point and we have sequence of such points $c_s = z_{l_s} = p_{k_s}$, $s \in \mathbb{N}$,

$$l_s = ms, \quad k_s = (n-m)s. \tag{3.14}$$

Remark 2.30. If $1/2 < \xi < 1$ then it follows from (3.14) that $1 = k_1$ if $\xi = (n-1)/n$. Then $p_k = nk = c_k$, $k \in \mathbb{N}$, and PPs do not exist.

Real Characteristic Function. *Real Characteristic Function* (Real CF) describes only real Nonconstant Eigenvalues and it is restriction of the Complex CF $\gamma_c(q)$ on the set $\mathbb{R}_q$:

$$\gamma_r(x) = \gamma_r(x;\xi) = \begin{cases} \frac{\sinh(\xi\pi x)}{\sinh((1-\xi)\pi x)}, & x \leqslant 0; \\ \frac{\sin(\xi\pi x)}{\sin((1-\xi)\pi x)}, & x \geqslant 0. \end{cases} \tag{3.15}$$



(a) $\xi = 2/3$ (b) $\xi = 5/8$ (c) $\xi = 6/7$

(d) $\xi = 2/3$ (e) $\xi = 5/8$ (f) $\xi = 6/7$

Fig. 2.21. Real CF $\gamma_r(q)$ and $\tilde{\gamma}_r(q)$ for $1/2 < \xi < 1$.

Values of Real CF and it's derivatives at CEP c_s, $s \in \mathbb{N}$, $\xi = m/n \in (0,\,1)$, $\xi \neq 1/2$, are:

$$\gamma_s(\xi) := \gamma(c_s;\xi) = (-1)^{ns}\tfrac{\xi}{1-\xi} = (-1)^{c_s}\tfrac{\xi}{1-\xi}, \tag{3.16}$$

$$\gamma_s'(\xi) := \gamma'(c_s;\xi) = 0, \tag{3.17}$$

$$\gamma_s''(\xi) := \gamma''(c_s;\xi) = \pi^2(1-2\xi)/3\cdot\gamma_s. \tag{3.18}$$

We see, that $\gamma_s \neq 0$, $\gamma_s' = 0$, $\gamma_s'' \neq 0$ for all ξ and s. Graphs of the function $\gamma_r(x)$ for various $1/2 < \xi < 1$ are shown in Figure 2.21(a)–(c).

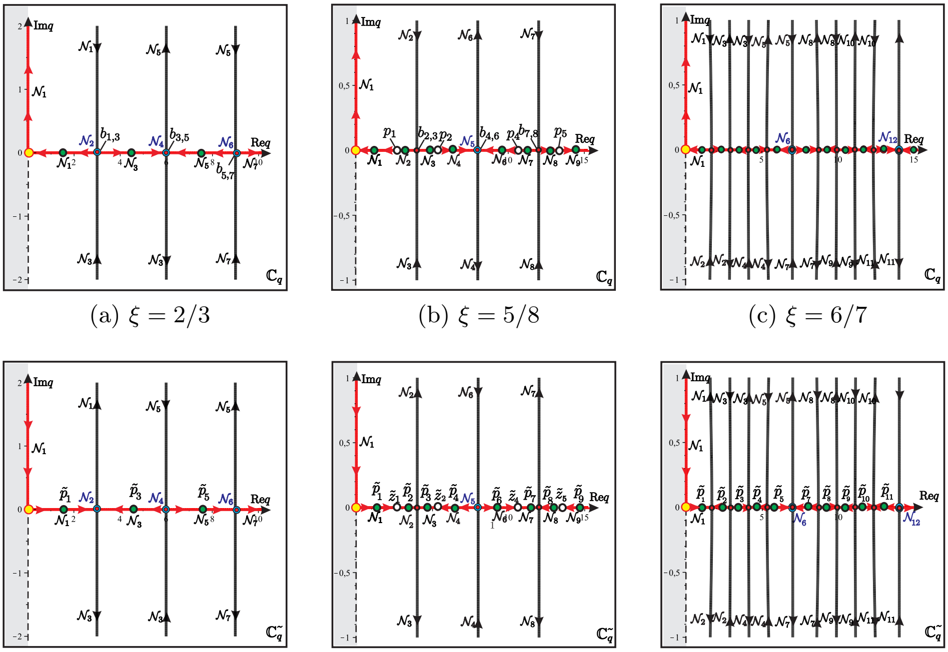
Remark 2.31. NC (3.3) we can rewrite as

$$u(1-\xi) = \tilde{\gamma}u(\xi), \qquad \tilde{\gamma} = 1/\gamma. \tag{3.19}$$

Graphs of the function $\tilde{\gamma}_r(x)$ for various $1/2 < \xi < 1$ are shown in Figure 2.21(d)–(f). NC (3.19) we can rewrite as

$$u(\tilde{\xi}) = \tilde{\gamma}u(1-\tilde{\xi}). \tag{3.20}$$

So, the spectrum for SPL (3.1)–(3.3) with parameters $0 < \xi < 1/2$ and γ is the same as the spectrum for SPL (3.1)–(3.2), (3.21) with parameters $1/2 < \tilde{\xi} < 1$ and $\tilde{\gamma} = 1/\gamma$. Thus, it is enough to investigate problem (3.1)–(3.3) with the parameter $1/2 < \xi < 1$.



(a) $\xi = 2/3$ (b) $\xi = 5/8$ (c) $\xi = 6/7$

(d) $\xi = 2/3$ (e) $\xi = 5/8$ (f) $\xi = 6/7$

Fig. 2.22. Spectrum Curves $N_k(\gamma)$ and $N_k(\tilde{\gamma})$ for $1/2 < \xi < 1$.

Ramification Point and Critical Points. The Taylor series for CF $\gamma(q)$ at RP $q = 0$ is:

$$\gamma(q;\xi) = \frac{\xi}{1-\xi} + \frac{\xi(1-2\xi)}{6(1-\xi)}\pi^2 q^2 + \mathcal{O}(q^4). \tag{3.21}$$

Multiplier of q^2 is nonzero for $0 < \xi < 1/2$ and $1/2 < \xi < 1$. So, $q = 0$ is not CP in $\mathbb{C}_\lambda$.

Lemma 2.32. *Zero Point of CF can not be CP.*

Proof. For CF (3.13) we have

$$\gamma' = -\gamma(q)(1-\xi)\frac{\cos((1-\xi)\pi q)}{\sin((1-\xi)\pi q)} + \xi\pi\frac{\cos(\xi\pi q)}{\sin((1-\xi)\pi q)}. \tag{3.22}$$

If $\gamma(q_z) = 0$ then $q_z \notin \mathcal{C}_\xi$. So, $\sin(\xi\pi q_z) = 0$, $\cos(\xi\pi q_z) \neq 0$, $\sin((1-\xi)\pi q_z) \neq 0$ $(\xi \neq 1/2)$. It follows from (3.22) that $\gamma'(q_z) \neq 0$. □

3.2 Spectrum Curves

Spectrum Curves for SLP (3.1)–(3.3) are presented in Figure 2.22 ((d)–(f) $\mathcal{N}(\tilde{\gamma})$ describe Spectrum Curves in case $0 < \xi < 1/2$). If we are increasing the value of parameter ξ, the poles are moving to the right, and the zeroes are moving to the left. When zeroes and poles coincide, we have CEPs.

Theorem 2.33. *Spectrum for SLP* (3.1)–(3.3) *for* $1/2 < \xi < 1$ *is equivalent to Spectrum for SLP* (10.1)–(10.2) *in Introduction.*

Proof. If we change argument $t = \xi\tau$ in SLP (3.1)–(3.3) then we get SLP

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} = \lambda_\chi u, \quad \tau \in (0, 1+\chi), \tag{3.23}$$

$$u(0) = 0, \tag{3.24}$$

$$u(1) = \gamma u(\chi), \tag{3.25}$$

where $\chi = (1-\xi)/\xi \in (0,1)$, $\lambda_\chi = \lambda\xi^2$. Relation between points in domain $\mathbb{C}_q$ is $q_\xi = q_\chi \xi$. □

4 Conclusions

Below we present principal conclusions of this chapter:

1. For SLP with Dirichlet type BC (two cases SLP_1^d, SLP_2^d) (1.1)–(1.3), SLP with Neumann type BC (three cases SLP_1^n, SLP_2^n, SLP_3^n and $\mathrm{SLP}_4^n \sim \mathrm{SLP}_2^d$) (2.1)–(2.3) and SLP with symmetrical type BC (SLP^s) (3.1)–(3.3) CEs do not exist for irrational parameter ξ and exist only for rational $\xi = \frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $\gcd(m,n) = 1$:

   (SLP_1^d) $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$;

   (SLP_2^d) $m, n \in \mathbb{N}_o$;

   (SLP_1^n) $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$;

   (SLP_2^n) $m, n \in \mathbb{N}$;

   (SLP_3^n) $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$;

   (SLP^s) $m, n \in \mathbb{N}$.

2. CPs of the first order (and complex eigenvalues) exist for all SLP in case $\xi \in (0,1)$, for SLP_2^d in case $\xi = 0$, SLP_1^n in case $\xi = 0$. We have infinite number of such CPs of the first order. Negative CP of the first order exists for SLP_1^n in case $\xi \in (0,1)$. CPs of the second order exist for SLP_2^d, SLP_1^n, SLP_3^n (only for some $\xi \in (0,1)$).

3. For SSLP_1^d, SLP_2^d, SLP_1^n, SLP_2^n, SLP_3^n, SLP^s we obtain five types bifurcations

$$\beta_{ZP}\colon (z_{l_s}, p_{k_s}) \to c_s \to (b_{l_s+1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1}),$$
$$\beta_{2B}\colon (b_{l_s-1,l_s+1}, b_{l_s+1,l_s}) \to b_{l_s-1,l_s+1,l_s} \to \varnothing,$$
$$\beta_{ZP}^{-1}\colon (b_{l_s-1,l_s}, p_{k_s}, z_{l_s}, b_{l_s,l_s+1}) \to c_s \to (z_{l_s}, p_{k_s}),$$
$$\beta_{2B}^{-1}\colon \varnothing \to b_{l_s,l_s-1,l_s+1} \to (b_{l_s,l_s-1}, b_{l_s-1,l_s+1}),$$
$$\beta_{ZP}^{0}\colon (b_{l_s-1,l_s}, z_{l_s}, p_{k_s}) \to c_s = b_{l_s-1,l_s+1} \to (p_{k_s}, z_{l_s}, b_{l_s,l_s+1}):$$

(SLP_1^d) β_{2B}^{-1} and β_{ZP}^{-1};

(SLP_2^d) β_{ZP}^{0};

(SLP_1^n) β_{2B}^{-1} and β_{ZP}^{-1};

(SLP_2^n) β_{ZP}^{0};

(SLP_3^n) β_{ZP} and β_{ZP};

(SLP^s) β_{ZP}^{0}.

Remark 2.34. Let us consider SLP with one classical BC

$$-u'' = \lambda u, \quad t \in (0,1), \tag{4.1}$$
$$u(1) = 0, \tag{4.2}$$

and another two-point NBC:

$$u(0) = \gamma u(\xi), \tag{4.3$_1$}$$
$$u(0) = \gamma u'(\xi), \tag{4.3$_2$}$$
$$u'(0) = \gamma u(\xi), \tag{4.3$_3$}$$
$$u'(0) = \gamma u'(\xi). \tag{4.3$_4$}$$

If we change argument $t = 1 - \tau$, then get SLP

$$-v'' = \lambda v, \quad \tau \in (0,1), \tag{4.4}$$
$$v(0) = 0, \tag{4.5}$$

with NBC

$$v(1) = \tilde{\gamma} v(\tilde{\xi}), \quad \tilde{\gamma} = \gamma, \tag{4.6$_1$}$$
$$v(1) = \tilde{\gamma} v'(\tilde{\xi}), \quad \tilde{\gamma} = -\gamma, \tag{4.6$_2$}$$
$$v'(1) = \tilde{\gamma} v(\tilde{\xi}), \quad \tilde{\gamma} = \gamma, \tag{4.6$_3$}$$
$$v'(1) = \tilde{\gamma} v'(\tilde{\xi}), \quad \tilde{\gamma} = -\gamma. \tag{4.6$_4$}$$

where $v(\tau) = u(1-\tau)$ and $\tilde{\xi} = 1 - \xi$.

We have received the problems we have previously investigated. So, corresponding Spectrum Curves will be the same, but in some cases the direction of them will be oppositive ($\tilde{\gamma} = -\gamma$). We have the same view of them for ξ and $\tilde{\xi}$.

Chapter 3

Discrete Sturm–Liouville Problems

1 Introduction

In this chapter we investigate *a discrete Sturm–Liouville Problems* (dSLP) corresponding to SLPs in Chapter 1 and Chapter 2:

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1.1}$$

with one classical (Dirichlet or Neumann) BC:

$$u(0) = 0 \text{ or } u'(0) = 0 \tag{1.2}$$

and another two-point NBC:

$$u(1) = \gamma u'(\xi), \tag{1.3$_1$}$$

$$u(1) = \gamma u(\xi), \tag{1.3$_2$}$$

with the parameter $\gamma \in \mathbb{R}$, $0 < \xi < 1$. For the differential SLP (1.1)–(1.3$_2$) with Diriclet BC real eigenvalues were analyzed in [125, Sapagovas 2007]. We analyze, how Spectrum Curves depend on parameters of grid and NBC. The main results of this Chapter 3 are published in [158, Skučaitė-Bingelė and Štikonas 2011] and [8, Bingelė, Bankauskienė and Štikonas 2019].

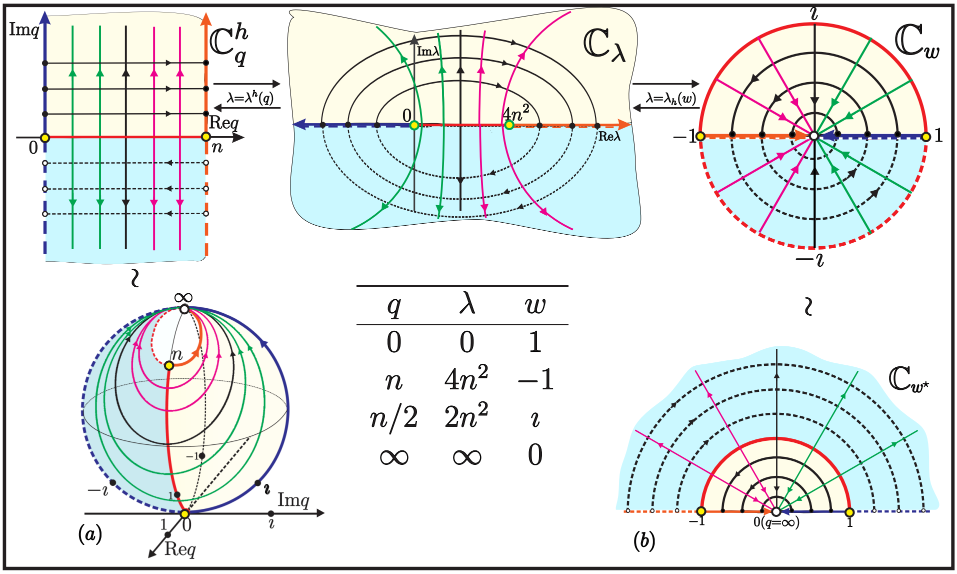


Fig. 3.1. Bijective mappings: $\lambda = \frac{4}{h^2}\sin^2(\frac{\pi q h}{2})$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q^h$; $\lambda = \frac{2}{h^2}(1 - \frac{w - w^{-1}}{2})$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w$, ○ – Branch Point, ● – Ramification Point; (a) $\overline{\mathbb{C}}_q^h$ on *Riemann sphere*; (b) Domain $\mathbb{C}_{w^\star}$ on the upper half-plane.

2 Discrete Problem

We introduce a uniform grids and we use notation $\overline{\omega}^h = \{t_j = jh, j = \overline{0,n}; nh = 1\}$ for $2 \leqslant n \in \mathbb{N}$, and $\mathbb{N}^h := (0, n) \cap \mathbb{N}$, $\overline{\mathbb{N}}^h := \mathbb{N}^h \cup \{0, n\}$. Also, we make an assumption, that ξ is located on the grid, i.e., $\xi = mh = m/n$, $0 \leqslant m \leqslant n$. Let us denote the greatest common divisor $K := \gcd(n, m)$ and $N := n/K$, $M := m/K$. Then $\xi = M/N$, too.

We approximate differential equation (1.1) by the following Finite-Difference Scheme (FDS)

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}. \tag{2.1}$$

The function $\lambda^h \colon \mathbb{C} \to \mathbb{C}$,

$$\lambda^h(z) := \tfrac{4}{h^2}\sin^2(\pi z h/2)$$

is holomorphic (entire) function. Inverse function is multivalued. Function $\lambda^h(z)$ has an infinite set of *Branch Points* (BP) of the second order at $\lambda = 0$ and $\lambda = 4n^2$ and a logarithmic BP at $\lambda = \infty$. Points $q = 0$, $q = n$ are *Ramification Points* (RP) of this function (see Figure 3.1). We make inverse function single-valued if we make branch cuts along the intervals $(-\infty, 0)$ and $(4n^2, +\infty)$.

Now we can consider a bijection

$$\lambda = \lambda^h(q) = \tfrac{4}{h^2}\sin^2(\pi q h/2) = \tfrac{2}{h^2}\bigl(1-\cos(\pi z h)\bigr) \tag{2.2}$$

between $\mathbb{C}_\lambda := \mathbb{C}$ and $\mathbb{C}_q^h$ (see Figure 3.1), where $\mathbb{C}_q^h := \mathbb{R}_y^- \cup \{0\} \cup \mathbb{R}_x^h \cup \{n\} \cup \mathbb{R}_q^{h+} \cup \mathbb{C}_y^{h+} \cup \mathbb{C}_q^{h-}$, $\mathbb{R}_x^h := \{q = x\colon 0 < x < n\}$, $\mathbb{R}_y^- := \{q = \imath y\colon y > 0\}$, $\mathbb{R}_y^{h+} := \{q = n + \imath y\colon y > 0\}$, $\mathbb{C}_q^{h+} := \{q = x + \imath y\colon 0 < x < n, y > 0\}$, $\mathbb{C}_q^{h-} := \{q = x + \imath y\colon 0 < x < n, y < 0\}$. We use notation $\overline{\mathbb{C}}_q^h = \mathbb{C}_q^h \cup \{\infty\}$ for domain on *Riemann sphere* (see Figure 3.1). Then for any eigenvalue $\lambda \in \mathbb{C}_\lambda$ there exists the *Eigenvalue Point* (EP) $q \in \mathbb{C}_q^h$. Let us denote $\mathring{\mathbb{C}}_q^h := \mathbb{C}_q^h \setminus \{0, n\}$ the relative complement of RP points in $\mathbb{C}_q^h$ and $\mathring{\mathbb{R}}_q^h := \mathbb{R}_y^- \cup \mathbb{R}_x^h \cup \mathbb{R}_y^{h+}$. It follows, that $\lambda < 0$ for $q \in \mathbb{R}_y^{h-}$, $0 < \lambda < 4/h^2$ for $q \in \mathbb{R}_x^h$, $\lambda > 4/h^2$ for $q \in \mathbb{R}_y^{h+}$. So, if $q \in \mathbb{R}_q^h := \mathring{\mathbb{R}}_q^h \cup \{0, n\}$, then corresponding eigenvalue is real. If EP is RP, then we call this point *Branch Eigenvalue Point* (BEP). For differential problems (1.1)–(1.3) eigenvalues are defined by the formula $\lambda = (\pi q)^2$, $q \in \mathbb{C}_q := \mathbb{R}_y^- \cup \{0\} \cup \{q = x + \imath y\colon x > 0, y \in \mathbb{R}\}$ [166, Štikonas and Štikonienė 2009]. We also use bijection $\lambda = \lambda_h(w) := \frac{2}{h^2}(1 - (w - w^{-1})/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w := \{w \in \mathbb{C} : |w| \leqslant 1, w \neq 0\}$ (see Figure 3.1). The domain $\mathbb{C}_w$ will be used for investigation of eigenvalues in the neighborhood of $\lambda = \infty$ ($w = 0$). This bijection maps $\lambda < 0$ to the interval $w \in (0, 1)$, $0 < \lambda < 4/h^2$ to the upper unit semi-circle, $\lambda > 4/h^2$ to the interval $(-1, 0)$, respectively. Complex λ points correspond to the points w, $\operatorname{Im} w \neq 0$, inside the unit circle (see Figure 3.1). The points $w = \pm 1$ are RPs in $\mathbb{C}_w$ of the function $\lambda_h(w)$ and correspond two BP of the second order at $\lambda = 0$ and $\lambda = 4n^2$. The function $w = \mathrm{e}^{\imath\pi h q}$ maps $\mathbb{C}_q^{h+}$ to the upper unit semi disk in $\mathbb{C}_w$ or $\mathbb{C}_{w^\star}$ and $\mathbb{C}_q^{h-}$ to outer part of the unit semi-disk in $\mathbb{C}_{w^\star}$. The corresponding points in the different domains are shown in the table (see Figure 3.1).

Using formula (2.2), the discrete equation (2.1) can be rewritten in form:

$$U_{j+1} - 2\cos(\pi q h) U_j + U_{j-1} = 0, \qquad j \in \mathbb{N}^h, \tag{2.3}$$

where $q = x + \imath y \in \mathbb{C}_q^h$.

The general solution U_j, $j \in \overline{\mathbb{N}}^h$, of the difference equation (2.3) is equal to:

$$U_j = C_1 \frac{\sin(\pi q t_j)}{(1 - hq)\pi q} + C_2 \cos(\pi q t_j). \tag{2.4}$$

In case $q = 0$ and $q = n$ this formula we can be written as

$$U_j = C_1 t_j + C_2 \text{ for } q = 0;$$
$$U_j = C_1 (-1)^{j+1} t_j + C_2 (-1)^j \text{ for } q = n.$$

3 Problems with a Two-Point NBC and one classical Dirichlet condition

We approximate SLP (1.1)–(1.3) with Diriclet BC by the following Finite-Difference Scheme (FDS) and get *a discrete Sturm–Liouville Problem* (dSLP):

$$\frac{U_{j-1}-2U_j+U_{j+1}}{h^2}+\lambda U_j=0,\quad j=\overline{1,n-1}, \tag{3.1}$$

$$U_0=0, \tag{3.2}$$

with NBC ($0<m<n$)

$$U_n=\tfrac{\gamma}{2h}(U_{m+1}-U_{m-1}), \tag{3.3$_1$}$$

$$U_n=\gamma U_m. \tag{3.3$_2$}$$

Remark 3.1. In Case 1 discrete NBC is three-point.

If $\gamma=0$, we have the classical BCs and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters ξ:

$$\lambda_k(0)=\lambda^h(q_k(0)),\ U_{k,j}(0)=\sin(\pi q_k(0)t_j),\ q_k(0)=k\in\mathbb{N}^h. \tag{3.4}$$

Remark 3.2. Problem (3.1)–(3.3$_1$) for $\xi=1/2$, $n=2$ and $\gamma=1$ is degenerated (the second BC is trivial $U_2=U_2$). If $\gamma\neq1$, then we have one real eigenvalue $\lambda=8$ ($q=1$). So, in Case 1 we suppose that $n>2$.

If $m=0$ in Case 2, then we have the same dSLP as $\gamma=0$. Problem (3.1)–(3.3$_2$) for $m=n$ and $\gamma=1$ is degenerated (the second BC is trivial $U_n=U_n$). If $\gamma\neq1$, then we have dSLP equivalent to dSLP with $\gamma=0$.

From classical BC (3.2) we get $C_1=0$ in formulae for general solution (2.4) of discrete equation (3.1). So, we are looking for solutions which have the following form

$$U_j=Ct_j \text{ for } q=0, \tag{3.5$_0$}$$

$$U_j=C(-1)^{j+1}t_j \text{ for } q=n, \tag{3.5$_n$}$$

$$U_j=C\frac{\sin(\pi q t_j)}{(1-hq)\pi q}\quad\text{(general case)}. \tag{3.5}$$

Let us substitute expressions (3.5) into NBC (3.3). First of all we consider two limit cases.

1) The case $q=0$ ($\lambda=0$). In this case we have equality

$$C=C\gamma, \tag{3.6$_1$}$$

$$C=C\gamma\xi. \tag{3.6$_2$}$$

Nontrivial ($C \neq 0$) solution (eigenfunction) for $\lambda = 0$ exists if

$$\gamma = 1, \tag{3.7$_1$}$$

$$\gamma = 1/\xi. \tag{3.7$_2$}$$

Note, that for differential SLP we have eigenvalue $\lambda = 0$ if condition (3.7) is valid.

2) The case $q = n$ ($\lambda = 4/h^2$). In this case we have equality

$$C(-1)^{n+1} = C\gamma(-1)^m, \tag{3.8$_1$}$$

$$C(-1)^{n+1} = C\gamma(-1)^{m+1}\xi. \tag{3.8$_2$}$$

Nontrivial ($C \neq 0$) solution (eigenfunction) for $\lambda = 4/h^2$ exists if

$$\gamma = (-1)^{n-m-1}, \tag{3.9$_1$}$$

$$\gamma = (-1)^{n-m}/\xi. \tag{3.9$_2$}$$

Now we consider general case. If we substitute (3.5) with $C \neq 0$ into NBC, then we get equation for $q \in \mathbb{C}_q^h$:

$$\frac{\sin(\pi q)}{(1-hq)\pi q} = \gamma\cos(\xi\pi q)\cdot\frac{\sin(\pi q h)}{(1-hq)\pi q h}, \tag{3.10$_1$}$$

$$\frac{\sin(\pi q)}{(1-hq)\pi q} = \gamma\frac{\sin(\pi q\xi)}{(1-hq)\pi q}. \tag{3.10$_2$}$$

The equation (3.10) can be rewritten in a more convenient form:

$$\frac{\sin(\pi q)}{\pi q}\cdot\frac{\pi q h}{\sin(\pi q h)} = \gamma\cos(\xi\pi q), \tag{3.11$_1$}$$

$$\frac{\sin(\pi q)}{\pi q}\cdot\frac{1}{1-hq} = \gamma\frac{\sin(\xi\pi q)}{\pi q}\cdot\frac{1}{1-hq}. \tag{3.11$_2$}$$

This equation is valid (as the limit cases) for $q = 0, n$, too. Roots of this equation are EPs for dSLP (3.1)–(3.3). The bijection (2.2) allows to find corresponding eigenvalues.

Remark 3.3. If $hq \ll 1$, then $\sin(\pi q h) \approx \pi q h$. So, in limit case, the equation (3.11) is the same as for differential SLPs in Chapter 1 (2.1)–(2.3), [166, Štikonas and Štikonienė 2009].

3.1 Constant Eigenvalues and Characteristic Function

We introduce functions:

$$Z^h(z) := Z(z)\cdot\frac{\pi z h}{\sin(\pi z h)}, \qquad Z(z) := \frac{\sin(\pi z)}{\pi z}, \tag{3.12$_1$}$$

$$Z^h(z) := Z(z)\cdot\frac{1}{\pi z(hz-1)}, \qquad Z(z) := \sin(\pi z); \tag{3.12$_2$}$$

$$P^h_\xi(z) = P_\xi(z) := \cos(\xi\pi z), \qquad (3.13_1)$$

$$P^h_\xi(z) = P_\xi(z) \cdot \frac{1}{\pi z(hz-1)}, \qquad P_\xi(z) := \sin(\xi\pi z). \qquad (3.13_2)$$

Zeroes points. All nonzero integers are zeroes of the function Z in Case 1, and all integers are zeroes of the function Z in Case 2. Zeroes of the functions $Z^h(q)$, $q \in \mathbb{C}^h_q$, coincide with EPs in the classical case $\gamma = 0$, i.e., a set of zeroes for this function is

$$\hat{\mathcal{Z}} := \mathbb{N}^h = \{1, \ldots, n-1\}. \qquad (3.14)$$

All zeroes are simple.

All zeroes of the function $P^h_\xi(q)$ in $\mathbb{C}^h_q$ are simple:

$$\overline{\mathcal{Z}}_\xi := \{p_k = (k-1/2)/\xi, \;\; k = \overline{1,m}\}, \qquad (3.15_1)$$

$$\overline{\mathcal{Z}}_\xi := \{p_k = k/\xi, \;\; k = \overline{1, m-1}\}. \qquad (3.15_2)$$

Zeroes exist always in Case 1, $1/2 < p_1 \leqslant n/2$ and $k_{\max} = m = n/(2p_1)$. In Case 2 zeroes of the function $P^h_\xi(q)$ do not exist for $m = 1$, and $1 < p_1 \leqslant n/2$ for $m => 1$, $k_{\max} = m - 1 = n/p_1 - 1$.

Remark 3.4. Zeroes of functions Z^h and P^h_ξ are the same as in differential case, but in the discrete case they are located in $(0, n)$.

We rewrite the equation (3.11) in the form:

$$Z^h(q) = \gamma P^h_\xi(q), \quad q \in \mathbb{C}^h_q. \qquad (3.16)$$

Constant Eigenvalues. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEP are roots of the system:

$$Z^h(q) = 0, \qquad P^h_\xi(q) = 0. \qquad (3.17)$$

We use notation $\xi = m/n = M/N$, $\gcd(n, m) = K$, $\gcd(N, M) = 1$.

Lemma 3.5. *For dSLP* (3.1)–(3.3$_1$) *Constant Eigenvalues exist only for* $\xi = M/N \in (0, 1)$, $M \in \mathbb{N}_o$, $N \in \mathbb{N}_e$, *values and those eigenvalues are equal to* $\lambda_s = \lambda^h(c_s)$, $c_s := (s - 1/2)N$, $s = \overline{1, K}$.

Proof. From Remark 3.4 it follows that CEP for this problem are the CEPs for corresponding differential SLP. We find formula (see Lemma 1.6) for CEPs in differential case: $c_s = (s - 1/2)N$, $s \in \mathbb{N}$, $M \in \mathbb{N}_o$, $N \in \mathbb{N}_e$. In $(0, n)$ are CEPs when $s = \overline{1, K}$. □

Lemma 3.6. *For dSLP* (3.1)–(3.3_2) *Constant Eigenvalues exist only for* $\xi = M/N \in (0,1)$, $M, N \in \mathbb{N}$, $K > 1$, *values and those eigenvalues are equal to* $\lambda_s = \lambda^h(c_s)$, $c_s := Ns$, $s = \overline{1, K-1}$.

Proof. From Remark 3.4 it follows that CEP for this problem are the CEPs for corresponding differential SLP. We find formula (see subsection 10.1 in Introduction) for CEPs in differential case: $c_s = sN$, $s \in \mathbb{N}$. In $(0, n)$ are CEPs when $s = \overline{1, K-1}$ and $K > 1$. □

We see that in Case 2 and $K = 1$ CEPs do not exist. For example we have such situation for $m = n - 1$ because $\gcd(n, n-1) = 1$.

Finally, we write expressions for CEPs $c_s = z_{l_s} = p_{k_s} = N(s - 1/2)$:

$$k_s = M(s-1/2) + 1/2, \qquad l_s = N(s-1/2), \qquad s = \overline{1, K}, \tag{3.18_1}$$

$$k_s = Ms, \qquad l_s = Ns, \qquad s = \overline{1, K-1}, \tag{3.18_2}$$

where conditions for N, M, K are given in Lemma 3.5 and Lemma 3.6. The notation $\mathcal{C}_\xi$ is used for the set of all CEPs.

Complex Characteristic Function. Let to consider *Complex Characteristic Function* $\gamma_c \colon \mathbb{C}^h_q \to \mathbb{C}$:

$$\gamma_c(q) = \gamma_c(q; \xi) := \frac{Z^h(q)}{P^h_\xi(q)}, \quad q \in \mathbb{C}^h_q, \tag{3.19}$$

where Z^h and P^h_ξ are functions (3.12)–(3.13).

For dSLP (3.1)–(3.3) we have meromorphic functions (Complex CF)

$$\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} \cdot \frac{\pi q h}{\sin(\pi q h)} = \frac{\sin(\pi q)}{\pi q \cos(\xi \pi q)} \cdot \frac{\pi q h}{\sin(\pi q h)}, \tag{3.20_1}$$

$$\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} = \frac{\sin(\pi q)}{\sin(\xi \pi q)}. \tag{3.20_2}$$

All zeroes and poles of meromorphic function $\gamma_c(q)$ lie in $(0, n) = \mathbb{R}^h_x$. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. All these sets are finite.

Remark 3.7. The first factor $Z(q)/P_\xi(q)$ in Complex CF formula (3.20_1) in Case 1 coincide with Complex CF in differential case. In Case 2 Complex CF are same for SLP and dSLP. So, in Case 2 $\gamma_c^{dSLP} = \gamma_c^{SLP}|_{\mathbb{C}^h_q}$ for $\xi \in \mathbb{Q}$.

Lemma 3.8. *Complex CFs for dSLP* (3.1)–(3.3) *have the property of symmetry:*

$$\gamma_c(n-q) = (-1)^{n-m-1} \gamma_c(q), \tag{3.21_1}$$

$$\gamma_c(n-q) = (-1)^{n-m} \gamma_c(q). \tag{3.21_2}$$

Proof. Elementary proof of this statement is obtained by using the properties of trigonometric functions. □

Remark 3.9. If the point $q \in \mathcal{C}_\xi$, then it is Removable Singularity Point of Complex CF.

The point $q = \infty \notin \mathbb{C}_q^h$. In the domain $\mathbb{C}_w$, this point corresponds to $w = 0$. Expression for Complex CF can be rewritten in the following form:

$$\gamma_c(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi} + e^{-\imath\pi q\xi}} \cdot \frac{2h}{e^{\imath\pi qh} - e^{-\imath\pi qh}}, \tag{3.22$_1$}$$

$$\gamma_c(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi} - e^{-\imath\pi q\xi}}, \tag{3.22$_2$}$$

So, we can investigate Complex CF in the neighborhood $w = 0$ ($w = e^{\imath\pi qh}$) in the domain $\mathbb{C}_w$:

$$\gamma_c(w) = \frac{w^n - w^{-n}}{w^m + w^{-m}} \cdot \frac{2h}{w - w^{-1}} = \frac{2h}{w^{n-m-1}}\big(1 + w^2 + \mathcal{O}(w^4)\big), \tag{3.23$_1$}$$

$$\gamma_c(w) = \frac{w^n - w^{-n}}{w^m - w^{-m}} = \frac{1}{w^{n-m}}\big(1 + \mathcal{O}(w^{2m})\big). \tag{3.23$_2$}$$

From these formulae two lemmas are follows.

Lemma 3.10. *Complex CF at the point $q = \infty$ for dSLP* (3.1)–(3.3$_2$) *has Pole Point of the $n - m$-order.*

Lemma 3.11. *Complex CF at the point $q = \infty$ for dSLP* (3.1)–(3.3$_1$) *has Pole Point of the $n-m-1$-order in case $m < n-1$. Complex CF at the point $q = \infty$ for dSLP* (3.1)–(3.3$_1$) *has Removable Singularity Point of Complex CF in case $m = n - 1$ and $\gamma_c(\infty) = 2h$. This point is Critical Point of the first order, too.*

Real Characteristic Function. *Real CF* $\gamma_r(q)$ is defined on the domain $\{q \in \mathbb{R}_q^h\}$ and Real CF describes only real eigenvalues. We plot the graph of Real CF for EPs $0 < x < n$ in the middle graph; $x = 0$, $y > 0$ in the left half plane and $x = n$, $y > 0$ in the right half plane. Two γ-axes correspond to RPs $q = 0, n$. So,

$$\gamma_r = \{\gamma_{r-}(y), \gamma_r(x), \gamma_{r+}(y)\} = \begin{cases} \gamma_{r-}(y) := \gamma_c(0 + \imath y), & y \geqslant 0; \\ \gamma_r(x) := \gamma_c(x + \imath 0), & 0 \leqslant x \leqslant n; \\ \gamma_{r+}(y) := \gamma_c(n + \imath y), & y \geqslant 0. \end{cases}$$

This function is useful for investigation of real negative, zero and positive eigenvalues:

$$\gamma_r = \Big\{ \frac{\sinh(\pi y)}{\cosh(\xi\pi y)} \cdot \frac{h}{\sinh(\pi y h)}, \frac{\sin(\pi x)}{\cos(\xi\pi x)} \cdot \frac{h}{\sin(\pi x h)}, \\ (-1)^{n-m-1} \frac{\sinh(\pi y)}{\cosh(\xi\pi y)} \cdot \frac{h}{\sinh(\pi y h)} \Big\}; \tag{3.24$_1$}$$

$$\gamma_r = \Big\{ \frac{\sinh(\pi y)}{\sinh(\xi\pi y)}, \frac{\sin(\pi x)}{\sin(\xi\pi x)}, (-1)^{n-m} \frac{\sinh(\pi y)}{\sinh(\xi\pi y)} \Big\}. \tag{3.24$_2$}$$

One can see the Real CF graphs in Figure 3.2 in Case 1 and Figure 3.3 in Case 2. Vertical blue solid and red dashes lines are added at the CEPs and PPs. We note, that in Case 2 for $m = n - 1$ there is only real eigenvalues (see Figure 3.3(c),(d)). In Case 1 only real eigenvalues exist for $\xi = 1/3$ (in this case we have two eigenvalues and one PP $p_1 = 3/2$) and $\xi = 2/4$ (in this case we have three eigenvalues and 2 CEP $c_1 = 1$, $c_2 = 3$).

Real CF in case Case 1, $m = n - 1$, has a horizontal asymptote $\gamma = 2h$ (corresponds to Removable Singularity Point of Complex CF, see Figure 3.2(d)).

We calculate values of Real CF and it's derivatives at CEP ($c_s := (s - 1/2)N$, $s = \overline{1, K}$ for $\xi = M/N \in (0, 1)$, $M \in \mathbb{N}_o$, $N \in \mathbb{N}_e$ in Cases 1 ; $c_s := Ns$, $s = \overline{1, K-1}$ for $\xi = M/N \in (0, 1)$, $M, N \in \mathbb{N}$, $K > 1$ in Case 2):

$$\gamma_s(\xi) := \gamma(c_s; \xi) = \frac{(-1)^{s+(N-M+1)/2}}{K \sin(\pi(2s-1)/K)}, \tag{3.25$_1$}$$

$$\gamma_s(\xi) := \gamma(c_s; \xi) = (-1)^{(1-\xi)c_s} \xi^{-1} = (-1)^{(N-M)s} \frac{N}{M}; \tag{3.25$_2$}$$

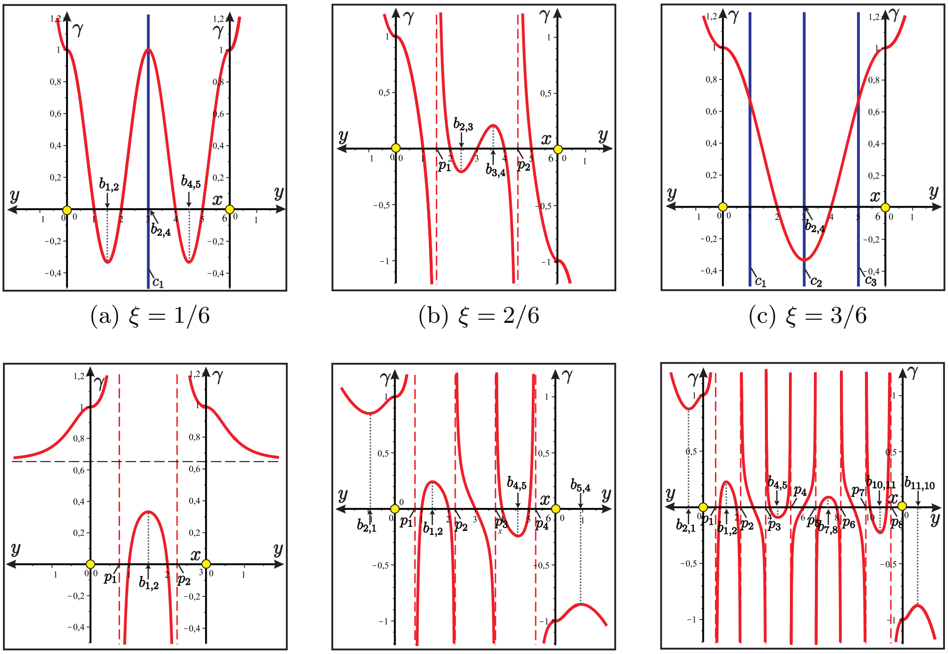
$$\gamma'_s(\xi) := \gamma'(c_s; \xi) = -\frac{\pi}{KN} \frac{\cos(\pi(2s-1)/K)}{\sin(\pi(2s-1)/K)} \gamma_s, \tag{3.26$_1$}$$

$$\gamma'_s(\xi) := \gamma'(c_s; \xi) = 0; \tag{3.26$_2$}$$

$$\gamma''_s(\xi) := \gamma''(c_s; \xi) = -\pi^2 \Big(\frac{1}{K^2 N^2} - \frac{2}{K^2 N^2 \sin^2\big(\pi(2s-1)/K\big)} \\ + \frac{N^2 - M^2}{3N^2} \Big) \gamma_s. \tag{3.27$_1$}$$

$$\gamma''_s(\xi) := \gamma''(c_s; \xi) = -\pi^2(1 - \xi^2)/3 \cdot \gamma_s = -\frac{\pi^2(N^2 - M^2)}{3N^2} \cdot \gamma_s. \tag{3.27$_2$}$$

We note that in Case 1 $\gamma'_s(\xi)$ has different sign for symmetric CEPs (see (3.26$_1$), Figure 3.2(c)). If CEP is at $x = n/2$ ($K \in \mathbb{N}_o$), then $\gamma'_s(\xi) = 0$ (see Figure 3.2(a),(c)).



(a) $\xi = 1/6$ (b) $\xi = 2/6$ (c) $\xi = 3/6$

(d) $\xi = 2/3$ (e) $\xi = 4/6$ (f) $\xi = 8/12$

Fig. 3.2. Real CF (dSLP Dirichlet BC) for various parameter ξ values in Case 1.

Complex-Real Characteristic Function. All *nonconstant eigenvalues* (which depend on the parameter γ) are γ-points of Complex-Real Characteristic Function (CF) [166, Štikonas and Štikonienė 2009, SLP]. CF $\gamma(q)$ is the restriction of Complex CF $\gamma_c(q)$ on a set $\mathcal{D}_{\boldsymbol{\xi}} := \{q \in \mathbb{C}_q^h \colon \operatorname{Im}\gamma_c(q) = 0\}$ [8, Bingelė, Bankauskienė and Štikonas 2019, dSLP]. CF $\gamma(q)$ describes the value of the parameter γ at the point $q \in \mathcal{D}_\xi$, such that there exist the Nonconstant Eigenvalue $\lambda = 4/h^2 \sin^2(\pi q h/2)$.

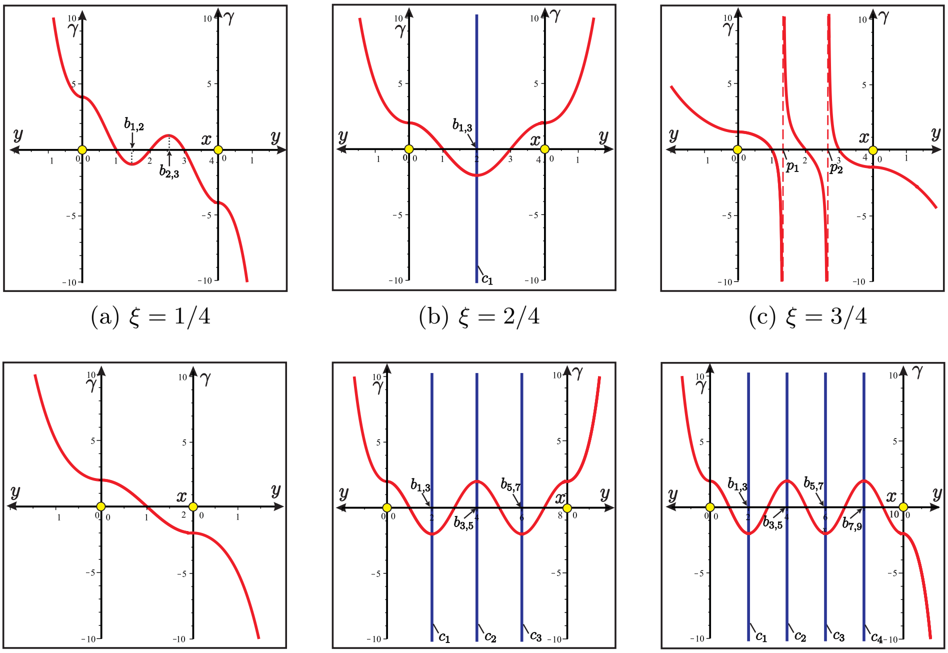
Ramification Points and Critical Points. Taylor series for $\gamma(q)$ at RP $q = 0$ is

$$\gamma(q) = 1 + \frac{\pi^2}{6n^2}\big(3m^2 - n^2 + 1\big)q^2 + \mathcal{O}\big(q^4\big), \qquad (3.28_1)$$

$$\gamma(q) = \frac{n}{m} + \frac{\pi^2}{6mn}\big(m^2 - n^2\big)q^2 + \mathcal{O}\big(q^4\big). \qquad (3.28_2)$$

In Case 2 a coefficient of the second term is negative ($m < n$). So, $q = 0$ (and $q = n$ by symmetry) is the CP of the first order in $\mathbb{C}_q^h$ but BP $\lambda = 0$ (and $\lambda = 4n^2$) is not CP.

If $n^2 \neq 3m^2 + 1$ in Case 1, then a coefficient of the second term is nonzero and $q = 0$ (and $q = n$ by symmetry) is the CP of the first order in $\mathbb{C}_q^h$ but



(a) $\xi = 1/4$ (b) $\xi = 2/4$ (c) $\xi = 3/4$

(d) $\xi = 1/2$ (e) $\xi = 4/8$ (f) $\xi = 5/10$

Fig. 3.3. Real CF (dSLP Dirichlet BC) for various parameter ξ values in Case 2.

this point is not CP in C_λ. Now we solve equation $n^2 = 3m^2 + 1$. We see that $n = 3k \pm 1$. Then it follows $3k^2 + \pm 2k = m^2 = k^2 l^2$, i.e., $3k + \pm 2 = kl^2$. From this equation we have $k = 1, 2$. Then $n = 4, 5, 7$. In the first and the second case equation $n^2 = 3m^2 + 1$ has not solutions in integers. The pair $n = 7$ and $m = 4$ satisfies this equation and

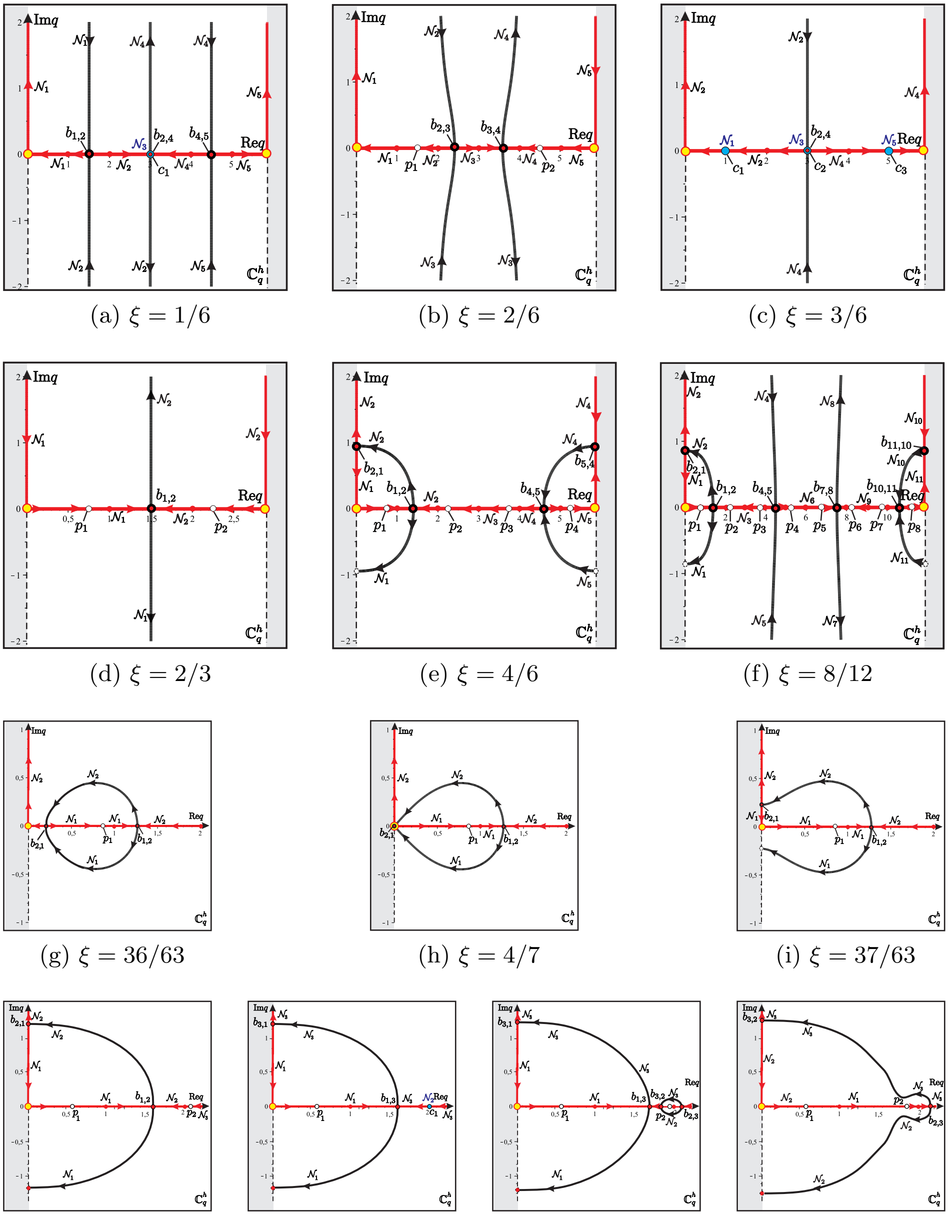
$$\gamma(q) = 1 + \frac{8\pi^4}{2401} q^4 + \mathcal{O}(q^6).$$

So, in Case 1 and $n = 7$, $m = 4$ we have CP of the first order in $\mathbb{C}_\lambda$ (and of the third order in $\mathbb{C}_q^h$). If $n^2 < 3m^2 + 1$ then there exist negative CP ($b = \imath y, y > 0$) and large positive CP ($b = n + \imath y, y > 0$). If $n^2 > 3m^2 + 1$ then there is not such CP.

CPs always exist if exist complex eigenvalues. The existence of the second order CP is open problem.

For the dSLP (3.1)–(3.3$_2$) CF is the same as for differential SLP. So, negative and large positive CPs ($b \in \mathbb{R}_q^- \cup \mathbb{R}_q^+$) do not exist [107, Pečiulytė *et al.* 2008]. The Case 1 is more complicated. Let to consider a function ($\xi, h \in \mathbb{R}$)

$$f(y) = \frac{\sinh(y)}{\cosh(\xi y)} \cdot \frac{h}{\sinh(hy)}, \quad y > 0,\ 0 < \xi < 1, 0 \leqslant h \leqslant 1 - \xi. \qquad (3.29)$$



(a) $\xi = 1/6$ (b) $\xi = 2/6$ (c) $\xi = 3/6$

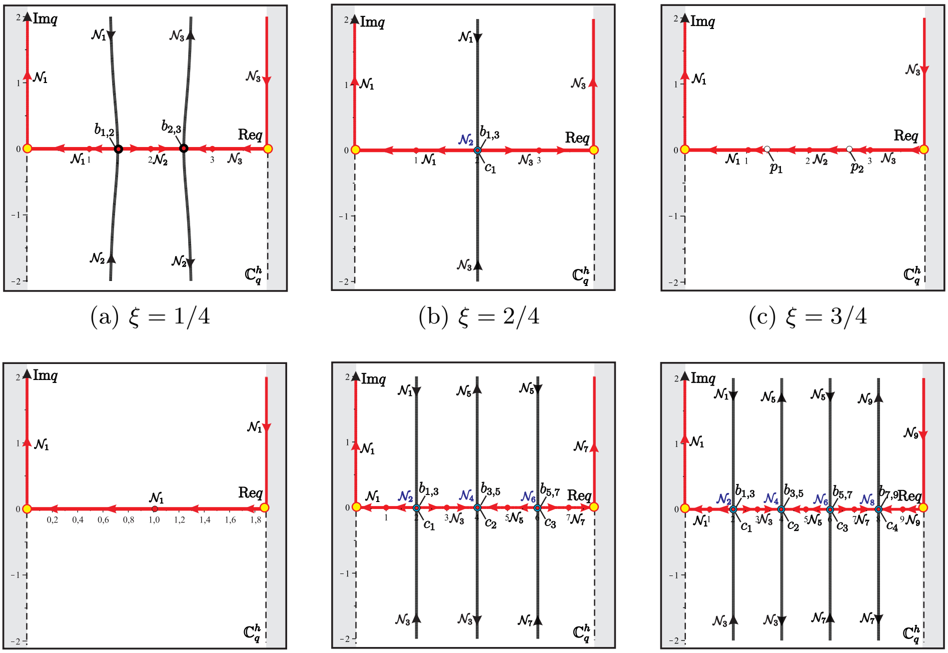
(d) $\xi = 2/3$ (e) $\xi = 4/6$ (f) $\xi = 8/12$

(g) $\xi = 36/63$ (h) $\xi = 4/7$ (i) $\xi = 37/63$

(j) $\xi = 299/400$ (k) $\xi = 300/400$ (l) $\xi = 301/400$ (m) $\xi = 302/400$

Fig. 3.4. (a)–(f) Spectrum Curves (dSLP Dirichlet BC) for various ξ values in Case 1; (g)-(m) Spectrum Curves near Ramification Point $q = 0$.

The limit case $h = 0$ corresponds CF for SLP in Chapter 1. CPs of this case are roots of equation $G(\tilde{y}, \xi) := \varphi_-(\tilde{y}) - \psi_-(\xi\tilde{y}) = 1$, where $\varphi_-(\tilde{y}) := \tilde{y}\coth\tilde{y}$, $\psi_-(\tilde{y}) := \tilde{y}\tanh\tilde{y}$, $\tilde{y} = \pi y$. The functions $\varphi_-(y)$, $\psi_-(y)$, $G(y, \xi)$



(a) $\xi = 1/4$ (b) $\xi = 2/4$ (c) $\xi = 3/4$

(d) $\xi = 1/2$ (e) $\xi = 4/8$ (f) $\xi = 5/10$

Fig. 3.5. Spectrum Curves (dSLP Dirichlet BC) for various ξ values in Case 2.

were investigated in [107]. For function (3.29) CPs are roots of equation

$$G(\tilde{y}, \xi) - \varphi_-(h\tilde{y}) = 0. \tag{3.30}$$

The graphs of the function $G(y, \xi) - \varphi_-(hy)$ are presented in Figure 3.6 for $h = 1/7$. Equation $G(y, \xi) = 1$ has one positive root for $\sqrt{3}/3 < \xi < 1$. We have

$$G(y, \xi) - \varphi_-(y) = -\frac{3\xi^2 - 1 + h^2}{3} y^2 + \mathcal{O}(y^4),$$

and

$$\lim_{y \to +\infty} G(y, \xi) - \varphi_-(y)/y = 1 - \xi - h.$$

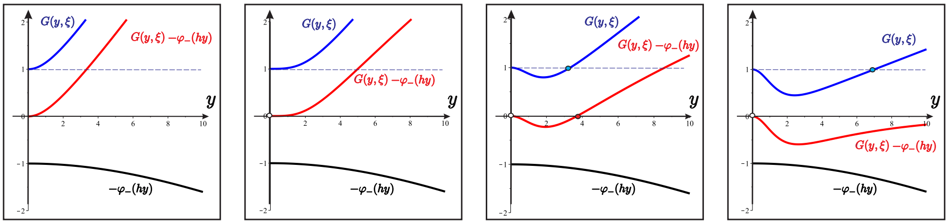
So, equation (3.30) has one positive root if

$$3\xi^2 - 1 + h^2 > 0, \quad \xi < 1 - h. \tag{3.31}$$

For dSLP (3.1)–(3.3_1) $h = 1/n$, $\xi = m/n$ and conditions (3.31) are

$$3m^2 - n^2 + 1 > 0, \quad m + 1 < n, \tag{3.32}$$

and it follows that we have one negative $b \in \mathbb{R}_q^{h-}$ (and by symmetry one large positive $b \in \mathbb{R}_q^{h+}$) CP for $3m^2 + 1 > n^2$ and $m + 1 < n$. If $m + 1 = n$, then CP is at $q = \infty$ (see Figure 3.6(d)). For $n = 7$ negative CP exists if $m = 5$. In the case $n = 2, 3, 4$ such CP do not exist for all m.



(a) $\xi = 3/7$ (b) $\xi = 4/7$ (c) $\xi = 5/7$ (d) $\xi = 6/7$

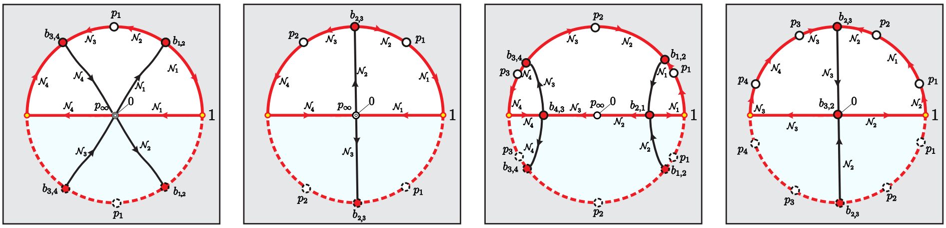
Fig. 3.6. Function $G(y, \xi) - \varphi_-(hy)$ for $h = 1/7$.

3.2 Spectrum Curves

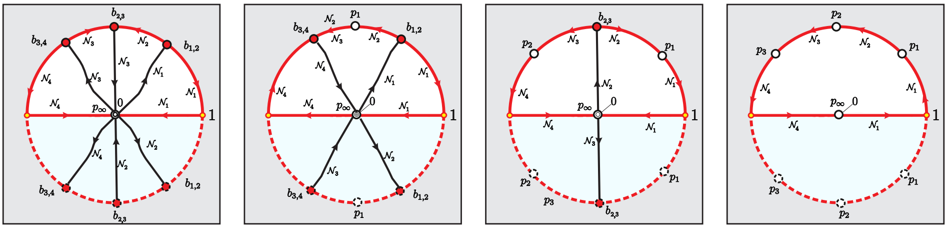
Spectrum Domain is the set $\mathcal{N}_{\boldsymbol{\xi}} = \mathcal{D}_{\boldsymbol{\xi}} \cup \mathcal{C}_{\boldsymbol{\xi}}$. Function γ_c has real values on $\mathcal{D}_{\boldsymbol{\xi}}$ except pole points. A set $\mathcal{E}_{\boldsymbol{\xi}}(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all nonconstant eigenvalue points for $\gamma_0 \in \mathbb{R}$. So, $\mathcal{D}_{\boldsymbol{\xi}} = \cup_{\gamma \in \mathbb{R}} \mathcal{E}_{\boldsymbol{\xi}}(\gamma)$. If $q \in \mathcal{D}_{\boldsymbol{\xi}}$ and $\gamma_c'(q) \neq 0$ (q is not a critical point of CF), then $\mathcal{E}_{\boldsymbol{\xi}}(\gamma)$ is smooth parametric curve $\mathcal{N}\colon \mathbb{R} \to \mathbb{C}_q^h$ locally and we can add arrow on this curve (arrows show the direction in which $\gamma \in \mathbb{R}$ is increasing). We call such curves *regular Spectrum Curves*. We can enumerate those Spectrum Curves for our problem by classical case ($\gamma = 0$): if $z_k = k \in \mathbb{N}^h$ belongs to Spectrum Curve, then the index of this Spectrum Curve is k. So, $\mathcal{N}_k(0) = z_k = k$. Few Spectrum Curves may intersect at the Critical Point. At this point, Spectrum Curves change direction and the angle between the old and the new direction is $\pi/(k+1)$ for the Critical Point of the kth order. We use the "right-hand rule". So, the Spectrum Curve turns to the right. For the $\gamma \to \pm\infty$, Spectrum Curve $\mathcal{N}_k(\gamma)$ approaches a pole point or the point ∞. The index of a Critical Point is formed of the indices of the Spectrum Curves which intersect at this Critical point. If Critical Point of the first order is real, then the left index coincides with the index of Spectral Curve which is defined by the smaller real λ values, and the right index is defined by greater λ values.

For every Constant Eigenvalue Point $c_j = j$, we define *nonregular Spectrum Curve* $\mathcal{N}_j = \{c_j\}$. We note that nonregular Spectrum Curves can overlap with a point of a regular Spectrum Curve. Finally, we have that $\mathcal{N}_{\boldsymbol{\xi}}$ is a finite union of Spectrum Curves $\mathcal{N}_l$, where $l = \overline{1, n-1}$. So, we have $\mathcal{D}_\xi = \cup_{l \in \mathcal{Z}_\xi} \mathcal{N}_l$ for all ξ.

One can see the Spectrum Curves in Figure 3.4(a)–(f) (Case 1) and Figure 3.5 (Case 2). We see that domains $\mathcal{N}_\xi$ are symmetrical with respect to vertical line $x = n/2$. In Case 2 for $m = n-1$ we have $n-2$ poles in $(0, n)$. So, all Spectrum Curves are regular and lie in $\mathbb{R}_q^h$ and only real eigenvalues exist (see Figure 3.5(c),(d)).



(a) Case 1, $m = 1$ (b) Case 1, $m = 2$ (c) Case 1, $m = 3$ (d) Case 1, $m = 4$



(e) Case 2, $m = 1$ (f) Case 2, $m = 2$ (g) Case 2, $m = 3$ (h) Case 2, $m = 4$

Fig. 3.7. Spectrum Curves in $\mathbb{C}_w$ (dSLP Dirichlet BC) for $n = 5$.

The point $q = 0$ is BEP. At this point, Spectrum Curve $\mathcal{N}_1$ (in Case 2) turns orthogonally to the right, i.e., the first positive eigenvalue point reaches $q = 0$ ($\gamma = 1/\xi$). The Case 1 is more complicated and we have situation similar as in differential case (see Chapter 1). In Case 1 CP of the third order exists at RP $q = 0$ for $\xi = 4/7$ (see Figure 3.4(g)–(i)), for SLP such CP exists in RP for $\xi = 1/\sqrt{3}$). At RP $q = n$, situation is symmetrical. If $n^2 < 3m^2 + 1$, then can exist two negative and two large positive ($\lambda > 4n^2$) eigenvalues, negative and large positive CPs (see Figure 3.2(e)–(f),Figure 3.4(e)–(f)).

Spectrum Curves in $\mathbb{C}_w$ are presented in Figure 3.7. The domain $\mathbb{C}_w$ is useful for investigation of Spectrum Curves near the point $q = \infty$ ($w = 0$). We see Spectrum Curves in $\mathbb{C}_w$ for $n = 5$, $m = 1, 2, 3, 4$. Formula (3.23) shows that this point is PP of the $(n - m)$-order in Case 2, PP of the $(n-m-1)$-order in Case 1 (if $n = m+1$ then we have Removable Singularity point with $\gamma = 2h$, and $w = 0$ is CP of the first order, see Figure 3.7(d)). So, two Spectrum Curves enters and leaves the point $w = 0$ (or $q = \infty$).

When the value n is increasing, then the Spectrum Curves of dSLP (3.1)–(3.3) become more similar to the Spectrum Curves of the differential problem. Every zero point, PP of CF in $\mathbb{R}^h_x$ and CEP for dSLP is zero point, PP and CEP for SLP, respectively. We note, that for large n the loop type Spectrum Curves part can be around zero and pole point as in differential case (see Figure 3.4(j)–(m)).

3.3 Some remarks and conclusions

We have $n-1$ Spectrum Curve for every $n \in \mathbb{N}$, $n \geqslant 2$. Nonregular Spectrum Curves are CEPs and belong to $\mathbb{R}_x^h = (0, n)$. The number of such Spectrum Curves is equal to

$$n_{ce} = K \text{ for } N \in \mathbb{N}_e, \quad n_{ce} = 0 \text{ for } N \in \mathbb{N}_o, \tag{3.33$_1$}$$

$$n_{ce} = K - 1, \tag{3.33$_2$}$$

where $K := \gcd(n, m)$ and $N := n/K$, $M := m/K$ for $\xi = m/n \in \mathbb{Q}$. Then the number of regular Spectrum Curves is equal to $n_{nce} = n - 1 - n_{ce}$.

The poles of CF belong to $\mathbb{R}_x^h \cup \{\infty\}$. The pole at $q = \infty$ is of

$$n_\infty = n - m - 1, \tag{3.34$_1$}$$

$$n_\infty = n - m \tag{3.34$_2$}$$

order. Then n_∞ Spectrum Curves go to this point and the same number of Spectrum Curves leave this point. Note that incoming Spectral Curves alternate with outgoing. If we denote the number of poles at $\mathbb{R}_x^h$ corresponding to function γ_c by n_p (all poles are of the first order), then $n_p + n_\infty = n_{nce}$. So, we have formula

$$n_p + n_{ce} = m, \tag{3.35$_1$}$$

$$n_p + n_{ce} = m - 1. \tag{3.35$_2$}$$

Particulary, $n_{ce}, n_p \leqslant m$ in Case 1 and $n_{ce}, n_p \leqslant m - 1$ in Case 2.

Remark 3.12. In Case 1 for $n = m + 1$ and $\gamma = 2h$ there are exist only $n - 3$ eigenvalues. For example, there are not eigenvalues in case $\xi = 2/3$, $\gamma = 2/3$ (see Figure 3.2(d)).

If n_c is the number of parts Spectrum Curves in the complex part of C_q^h between two CP (including $q = \infty$ for $n = m + 1$) and n_{cr} is number of CP in $\mathbb{R}_q^h$, then the following relation is valid:

$$n_{cr} - n_c = n - m - 2, \tag{3.36$_1$}$$

$$n_{cr} = n - m - 1 \tag{3.36$_2$}$$

($n_c = 0$ in Case 2).

Remark 3.13. Let us consider dSLP with NBC:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \tag{3.37}$$

$$U_n = 0, \tag{3.38}$$

$$U_0 = \gamma U_m. \tag{3.39}$$

If we change index $j = n - i$, then get dSLP

$$\frac{V_{i-1} - 2V_i + V_{i+1}}{h^2} + \lambda V_i = 0, \quad i = \overline{1, n-1}, \tag{3.40}$$

$$V_0 = 0, \tag{3.41}$$

$$V_n = \gamma U_{n-m}. \tag{3.42}$$

where $V_i = U_{n-i}$. We have received the problem we have previously investigated in this section. So, corresponding Spectrum Curves will be the same for m in dSLP (3.37)–(3.39) and $n - m$ in dSLP (3.40)–(3.42).

Example 3.14. For dSLP (3.37)–(3.39) in case $m = 1$ eigenvalues are the same as for dSLP (3.1)–(3.3$_2$) in case $m = n - 1$ for all γ. They are real and Spectrum Curves are regular, i.e., we can find by solving equation $\gamma_r(q) = \gamma$. For example, if $\gamma = 1$, then we have equation

$$\frac{\sin(\pi q)}{\sin(\pi q(n-1)/n)} = 1 \quad \Leftrightarrow \quad \cos\big(\pi q(1 - h/2)\big) = 0.$$

In $(0, n)$ we have all $n - 1$ eigenvalues

$$\lambda_k = \frac{4}{h^2}\sin^2(\pi x_k h/2), \quad x_k = \frac{2k-1}{2n-1}n = \frac{k - 1/2}{1 - h/2}, \quad k = \overline{1, n-1} \tag{3.43}$$

and eigenfunctions

$$U_{k,j} = \frac{\sin\big(\pi x_k(1 - t_j)\big)}{(1 - x_k h)\pi x_k} = \frac{\sin(\pi x_k)}{(1 - x_k h)\pi x_k} \cdot \frac{\cos\big(\pi x_k(t_j - h/2)\big)}{\cos(\pi x_k h/2)}, \tag{3.44}$$

$k = \overline{1, n-1}$.

4 Problems with a Two-Point NBC and one Neumann type condition

We approximate SLP (1.1)–(1.3) with Neumann type BC by FDS and get dSLP:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \tag{4.1}$$

$$U_0 = U_1, \tag{4.2}$$

with NBC ($0 < m < n$)

$$U_n = \tfrac{\gamma}{2h}(U_{m+1} - U_{m-1}), \tag{4.3$_1$}$$

$$U_n = \gamma U_m. \tag{4.3$_2$}$$

If $\gamma = 0$, then we have BCs

$$U_0 = U_1, \quad U_n = 0 \tag{4.4}$$

as in Example 3.14. So, for $\gamma = 0$ we have EP

$$q_k(0) = \frac{k - 1/2}{1 - h/2}, \quad k = \overline{1, n-1}, \tag{4.5}$$

and we have eigenvalues and discrete eigenfunctions

$$\lambda_k(0) = \frac{4}{h^2} \sin^2(\pi q_k(0) h/2), \quad U_{k,j}(0) = \frac{\cos\big(\pi q_k(0)(t_j - h/2)\big)}{\cos(\pi q_k(0) h/2)}, \tag{4.6}$$

$k = \overline{1, n-1}$.

Remark 3.15. Problem (4.1)–(4.3_2) for $m = n$ and $\gamma = 1$ is degenerated (the second BC is trivial $U_n = U_n$). If $\gamma \neq 1$, then the second BC is equivalent to $U_n = 0$ and we have dSLP in Remark 3.13. Problem (4.1)–(4.3_2) for $m = 0$ is the same as for $m = 1$.

From BC (4.2) and formula for general solution (2.4) of FDS (4.1) ve get

$$C_2 = C_1 \frac{\sin(\pi q h)}{(1 - qh)\pi q} + C_2 \cos(\pi q h).$$

We express

$$C_1 = C_2 \frac{\sin(\pi q h/2)(1 - qh)\pi q}{\cos(\pi q h/2)}$$

and substitute into general solution (2.4).

So, we are looking for solutions which have the following form

$$U_j = C \frac{\cos\big(\pi q (t_j - h/2)\big)}{\cos(\pi q h/2)}. \tag{4.7}$$

In the limit cases we have:

$$U_j = C \quad \text{for } q = 0, \tag{4.8_0}$$

$$U_j = C(-1)^{j+1}(2t_j/h - 1) \quad \text{for } q = n. \tag{4.8_n}$$

Let us substitute expressions (4.7) into NBC (4.3). First of all we consider two limit cases.

1) The case $q = 0$ ($\lambda = 0$). In this case we have equality

$$C = C\gamma 0 = 0, \tag{4.9_1}$$

$$C = C\gamma. \tag{4.9_2}$$

In Case 1 all solutions are trivial. So, $\lambda = 0$ is not eigenvalue. In Case 2 nontrivial ($C \neq 0$) solution (eigenfunction) for $\lambda = 0$ exists if $\gamma = 1$.

2) The case $q = n$ ($\lambda = 4/h^2$). In this case we have equality

$$C(-1)^{n+1}(2/h - 1) = C\gamma 2/h(-1)^{m+2}, \tag{4.10$_1$}$$
$$C(-1)^{n+1}(2/h - 1) = C\gamma(-1)^{m+1}(2\xi/h - 1). \tag{4.10$_2$}$$

Nontrivial ($C \neq 0$) solution (eigenfunction) for $\lambda = 4/h^2$ exists if

$$\gamma = (-1)^{n-m-1}(1 - h/2) = (-1)^{n-m-1}\frac{2n-1}{2n}, \tag{4.11$_1$}$$
$$\gamma = (-1)^{n-m}\frac{1 - h/2}{\xi - h/2} = (-1)^{n-m}\frac{2n-1}{2m-1}. \tag{4.11$_2$}$$

Now we consider general case. If we substitute (4.7) with $C \neq 0$ into NBC, then we get equation for $q \in \mathbb{C}_q^h$:

$$-\frac{\cos\big(\pi q(1 - h/2)\big)}{\cos(\pi q h/2)} = \gamma \sin\big(\pi q(\xi - h/2)\big) \cdot \frac{\sin(\pi q h/2)}{h/2}, \tag{4.12$_1$}$$
$$\frac{\cos\big(\pi q(1 - h/2)\big)}{\cos(\pi q h/2)} = \gamma\frac{\cos\big(\pi q(\xi - h/2)\big)}{\cos(\pi q h/2)}. \tag{4.12$_2$}$$

Remark 3.16. If $hq \ll 1$, then $\sin(\pi q h/2) \approx \pi q h/2$, $\cos(\pi q h/2) \approx 1$. So, in limit case, the equation (4.12) is the same as for differential SLPs in Chapter 1 (2.1)–(2.3$_3$), (2.1)–(2.3$_4$).

4.1 Constant Eigenvalues and Characteristic Function

We introduce functions:

$$Z^h(z) := \frac{\cos\big(\pi z(1 - h/2)\big)}{\cos(\pi z h/2)}; \tag{4.13}$$

$$P_\xi^h(z) := -\sin\big(\pi z(\xi - h/2)\big) \cdot \frac{\sin(\pi z h/2)}{h/2}, \tag{4.14$_1$}$$
$$P_\xi^h(z) := \frac{\cos\big(\pi z(\xi - h/2)\big)}{\cos(\pi z h/2)}. \tag{4.14$_2$}$$

Zeroes points. Zeroes of the functions $Z^h(q)$, $q \in \mathbb{C}_q^h$, coincide with EPs in the classical case $\gamma = 0$, i.e., a set of zeroes for this function is

$$\hat{\mathcal{Z}} := \Big\{ z_l = \frac{l - 1/2}{1 - h/2} = \frac{2l-1}{2n-1}n, \;\; l = \overline{1, n-1} \Big\}. \tag{4.15}$$

All zeroes are simple.

A set of zeroes of the function $P^h_\xi(q)$ in $\mathbb{C}^h_q$ is

$$\overline{\mathcal{Z}}_\xi := \Big\{p_k = \frac{k}{\xi - h/2} = \frac{2k}{2m-1}n, \ \ k = \overline{0, m-1}\Big\}, \tag{4.16$_1$}$$

$$\overline{\mathcal{Z}}_\xi := \Big\{p_k = \frac{k-1/2}{\xi - h/2} = \frac{2k-1}{2m-1}n, \ \ k = \overline{1, m-1}\Big\}. \tag{4.16$_2$}$$

All zeroes p_k, $k = \overline{1, m-1}$ are simple. Zero p_0 in Case 1 is of the second order. If $m = 0$ in Case 2, then $\overline{\mathcal{Z}}_\xi = \varnothing$. If $m = 1$ in Case 1, then $\overline{\mathcal{Z}}_\xi = \{0\}$.

We rewrite the equation (4.12) in the form:

$$Z^h(q) = \gamma P^h_\xi(q), \quad q \in \mathbb{C}^h_q. \tag{4.17}$$

Constant Eigenvalues. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}^h_q$. CEP are roots of the system:

$$Z^h(q) = 0, \qquad P^h_\xi(q) = 0. \tag{4.18}$$

Lemma 3.17. *For dSLP* (4.1)–(4.3$_1$) *Constant Eigenvalues do not exist.*

Proof. CEP exists if $z_l = p_k$ for some l and k. So, we have equation

$$n\frac{2l-1}{2n-1} = n\frac{2k}{2m-1}.$$

We can rewrite this equation in the following form:

$$2(2n-1)k + 2(1-2m)l = 1 - 2m.$$

Left hand side of this equation is even number and right hand side is odd number. So, this equation has not solution. □

If $\xi = m/n \in \mathbb{Q}$ and $\gcd(2n-1, 2m-1) = K$, then we will use notation $N = \big((2n-1)/K+1\big)/2$, $M = \big((2m-1)/K+1)\big)$, So, $\gcd(2N-1, 2M-1) = 1$ and $K \in \mathbb{N}_o$.

Lemma 3.18. *For dSLP* (4.1)–(4.3$_2$) *Constant Eigenvalues exist only for $\xi = m/n \in (0, 1)$, $K > 1$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := n/K \cdot (2s-1)$, $s = \overline{1, (K-1)/2}$.*

Proof. From (4.15) and (4.3$_2$) we have equation for l, k

$$(2n-1)k - (2m-1)l = n - m.$$

This equation we can rewrite

$$(2N-1)k - (2M-1)l = N - M, \qquad \gcd(2N-1, 2M-1) = 1. \tag{4.19}$$

It is easy check that pair

$$k_0 = 1 - M, \quad l_0 = 1 - N$$

satisfies this equation. So, we have

$$l = (2N-1)s + 1 - N, \; k = (2M-1)s + 1 - M, \; s \in \mathbb{N}. \tag{4.20}$$

CEPs are in $(0, n)$ if $s = \overline{1, (K-1)/2}$. $\square$

We see that in Case 2 and $K = 1$ CEPs do not exist. For example, such situation is for $m = 0, 1, n-4, n-2, n-1$ $(n > 3)$ because for such m we have $\gcd(2n-1, 2m-1) = 1$.

Example 3.19. If $\xi = \frac{5}{8}$ then there is one CEP $c_1 = \frac{8}{3} = 2.6(6)$ (see Figure 3.9(b)).

Finally, In Case 2 we write expressions for CEPs $c_s = z_{l_s} = p_{k_s} = n/K \cdot (2s-1) = \big((N-1/2)K + 1/2\big)/K(2s-1)$, $s = \overline{1, (K-1)/2}$:

$$l_s = \frac{(2N-1)(2s-1)+1}{2}, \quad k_s = \frac{(2M-1)(2s-1)+1}{2}. \tag{4.21}$$

Complex Characteristic Function. Let to consider *Complex Characteristic Function* $\gamma_c \colon \mathbb{C}_q^h \to \mathbb{C}$:

$$\gamma_c(q) = \gamma_c(q;\xi) := \frac{Z^h(q)}{P_\xi^h(q)}, \quad q \in \mathbb{C}_q^h, \tag{4.22}$$

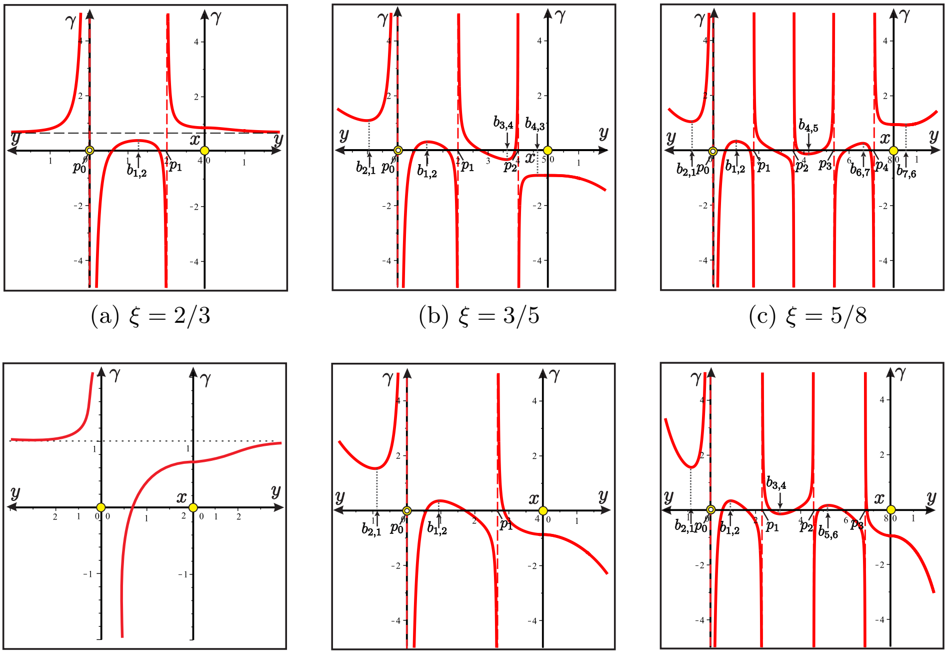
where Z^h and P_ξ^h are functions (4.13)–(4.14).

For dSLP (4.1)–(4.3) we have meromorphic functions (Complex CF)

$$\gamma_c(q) := -\frac{\cos\big(\pi q(1-h/2)\big)}{\sin\big(\pi q(\xi - h/2)\big)} \cdot \frac{h}{\sin(\pi q h)}, \tag{4.23$_1$}$$

$$\gamma_c(q) := \frac{\cos\big(\pi q(1-h/2)\big)}{\cos\big(\pi q(\xi - h/2)\big)}. \tag{4.23$_2$}$$

In Case 2 all zeroes and poles of meromorphic function $\gamma_c(q)$ lie in $(0, n) = \mathbb{R}_x^h$. In Cace 1 we have additional pole of the second order in $q = 0$. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. The set of zeroes for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. All these sets are finite.



(a) $\xi = 2/3$ (b) $\xi = 3/5$ (c) $\xi = 5/8$

(d) $\xi = 1/2$ (e) $\xi = 2/4$ (f) $\xi = 4/8$

Fig. 3.8. Real CF γ_r (dSLP Neumann BC) for various parameter ξ values in Case 1.

The point $q = \infty \notin \mathbb{C}_q^h$. In the domain $\mathbb{C}_w$, this point corresponds to $w = 0$. So, we can investigate Complex CF in the neighborhood $w = 0$ ($w = \mathrm{e}^{\imath \pi q h}$) in the domain $\mathbb{C}_w$:

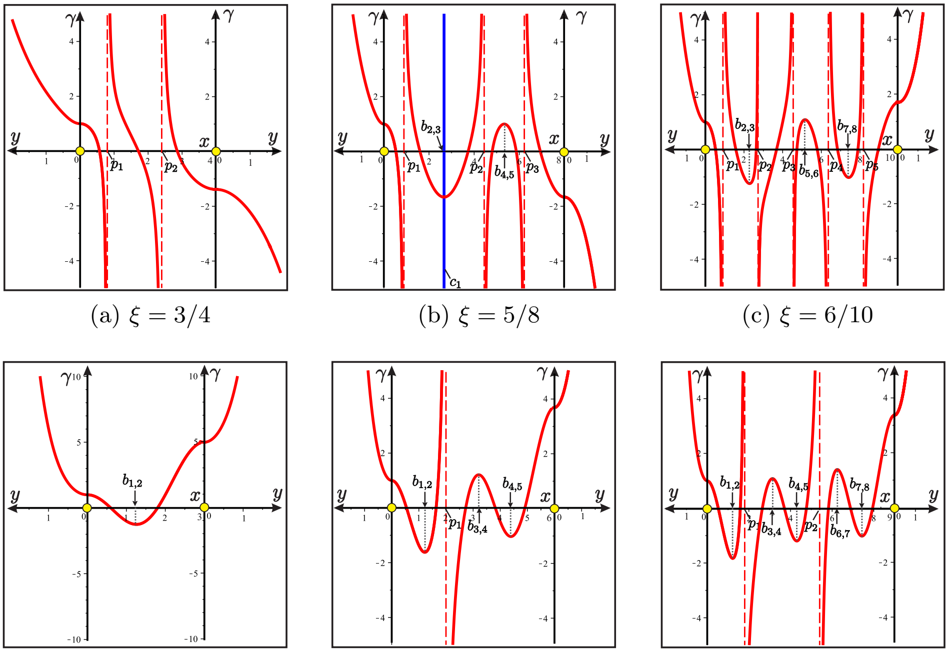
$$\gamma_c(w) = \frac{2h}{w^{n-m-1}}\big(1 + w^{\min(m,2)} + \mathcal{O}(w^2)\big), \qquad (4.24_1)$$

$$\gamma_c(w) = \frac{1}{w^{n-m}}\big(1 + \mathcal{O}(w^{2m-1})\big). \qquad (4.24_2)$$

From these formulae two lemmas are follows.

Lemma 3.20. *Complex CF at the point $q = \infty$ for dSLP (4.1)–(4.3$_1$) has Pole Point of the $n-m-1$-order in case $m < n-1$. Complex CF at the point $q = \infty$ for dSLP (4.1)–(4.3$_1$) has Removable Singularity Point of Complex CF in case $m = n - 1$ and $\gamma_c(\infty) = 2h$. This point is Critical Point of the first order in case $n > 2$, too.*

Lemma 3.21. *Complex CF at the point $q = \infty$ for dSLP (4.1)–(4.3$_2$) has Pole Point of the $n - m$-order.*



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 1/3$ (e) $\xi = 2/6$ (f) $\xi = 3/9$

Fig. 3.9. Real CF γ_r (dSLP Neumann BC) for various parameter ξ values in Case 2.

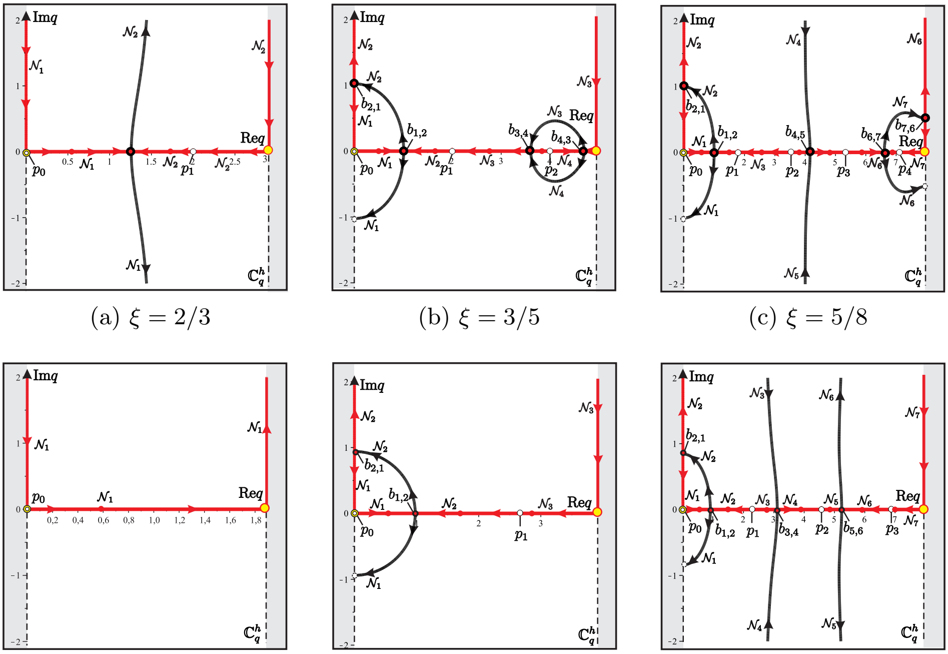
Real Characteristic Function. *Real CF* $\gamma_r(q)$ is useful for investigation of real negative, zero and positive eigenvalues:

$$\gamma_r = \left\{ \frac{\cosh\left(\pi y(1-h/2)\right)}{\sinh\left(\pi y(\xi-h/2)\right)} \cdot \frac{h}{\sinh(\pi y h)}, -\frac{\cos\left(\pi x(1-h/2)\right)}{\sin\left(\pi x(\xi-h/2)\right)} \cdot \frac{h}{\sin(\pi x h)}, \right.$$
$$\left. (-1)^{n-m-1}\frac{\sinh\left(\pi y(1-h/2)\right)}{\cosh\left(\pi y(\xi-h/2)\right)} \cdot \frac{h}{\sinh(\pi y h)} \right\}; \qquad (4.25_1)$$

$$\gamma_r = \left\{ \frac{\cosh\left(\pi y(1-h/2)\right)}{\cosh\left(\pi y(\xi-h/2)\right)}, \frac{\cos\left(\pi x(1-h/2)\right)}{\cos\left(\pi x(\xi-h/2)\right)}, \right.$$
$$\left. (-1)^{n-m}\frac{\sinh\left(\pi y(1-h/2)\right)}{\sinh\left(\pi y(\xi-h/2)\right)} \right\}. \qquad (4.25_2)$$

One can see the Real CF graphs in Figure 3.8 in Case 1 and Figure 3.9 in Case 2. We note, that in Case 2 for $m = n-1$ there is only real eigenvalues (see Figure 3.9(a)). In Case 1 only real eigenvalues exist for $\xi = 1/2$ (in this case we have one real eigenvalues and one PP $p_0 = 0$).

Real CF in case Case 1, $m = n-1$, has a horizontal asymptote $\gamma = 2h$ (corresponds to Removable Singularity Point of Complex CF, see Figure 3.8(a),(d)).



(a) $\xi = 2/3$ (b) $\xi = 3/5$ (c) $\xi = 5/8$

(d) $\xi = 1/2$ (e) $\xi = 2/4$ (f) $\xi = 4/8$

Fig. 3.10. Spectrum Curves (dSLP Neumann BC) for various ξ values in Case 1.

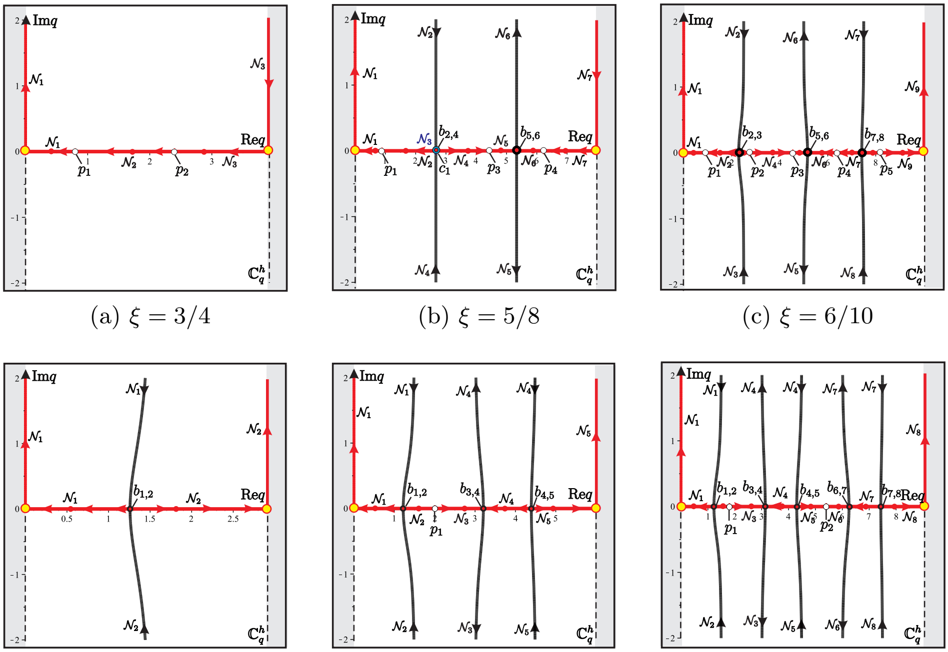
We calculate values of Real CF and it's derivatives at CEP in Case 2 ($c_s := n/K(2s-1) = \big((N-1/2)K+1/2\big)/K(2s-1)$, $s \in \overline{1,(K-1)/2}$, $\xi = m/n = \big((2M-1)K+1\big)/\big((2N-1)K+1\big) \in (0,1)$, $K > 1$):

$$\gamma_s(\xi) := \gamma(c_s;\xi) = (-1)^{N-M}\frac{2N-1}{2M-1}, \tag{4.26$_2$}$$

$$\gamma_s'(\xi) := \gamma'(c_s;\xi) = 0, \tag{4.27$_2$}$$

$$\begin{aligned}\gamma_s''(\xi) &:= \gamma''(c_s;\xi) = \frac{\pi^2}{3}(1-\xi)(1+\xi-h)\gamma_s \\ &= \frac{2\pi^2}{3}\frac{(N-M)\big(2(N+M-1)K+1\big)}{\big((2N-1)K+1\big)^2}\gamma_s.\end{aligned} \tag{4.28$_2$}$$

Complex-Real Characteristic Function. All *nonconstant eigenvalues* (which depend on the parameter γ) are γ-points of Complex-Real Characteristic Function (CF). CF $\gamma(q)$ is the restriction of Complex CF $\gamma_c(q)$ on a set $\mathcal{D}_\xi := \{q \in \mathbb{C}_q^h\colon\ \mathrm{Im}\gamma_c(q) = 0\}$ [8, Bingelė, Bankauskienė and Štikonas 2019, dSLP]. CF $\gamma(q)$ describes the value of the parameter γ at the point $q \in \mathcal{D}_\xi$, such that there exist the Nonconstant Eigenvalue $\lambda = 4/h^2\sin^2(\pi qh/2)$.



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 1/3$ (e) $\xi = 2/6$ (f) $\xi = 3/9$

Fig. 3.11. Spectrum Curves (dSLP Neumann BC) for various ξ values in Case 2.

Ramification Points and Critical Points. Taylor (Laurent) series for $\gamma(q)$ at RP $q = 0$ is

$$\gamma(q) = -\frac{2n}{2m-1}\pi^{-2}q^{-2} + \frac{6n^2 - 2m^2 - 6n + 2m - 1}{6n(2m-1)} + \mathcal{O}(q^2), \quad (4.29_1)$$

$$\gamma(q) = 1 - \frac{\pi^2}{2n^2}(n-m)(n+m-1)q^2 + \mathcal{O}(q^4); \quad (4.29_2)$$

In Case 1 RP $q = 0$ is PP of the second order in $\mathbb{C}_q^h$ and BP $\lambda = 0$ is PP of the first order. In Case 2 RP $q = 0$ is the CP of the first order in $\mathbb{C}_q^h$ and BP $\lambda = 0$ is not CP in $\mathbb{C}_\lambda$.

Taylor series for $\gamma(q)$ at RP $q = n$ is

$$\begin{aligned}\gamma(q) =& (-1)^{n-m-1}(1 - 1/(2n)) \\ &+ (-1)^{n-m-1}\frac{\pi^2(2n-1)}{24n^3}(6m^2 - 2n^2 - 6m + 2n + 3)(q-n)^2 \\ &+ \mathcal{O}((q-n)^4), \quad (4.30_1)\end{aligned}$$

$$\begin{aligned}\gamma(q) =& (-1)^{n-m}\frac{2n-1}{2m-1} - (-1)^{n-m}\frac{\pi^2(2n-1)}{6n^2(2m-1)}(n-m)(n+m-1)(q-n)^2 \\ &+ \mathcal{O}((q-n)^4). \quad (4.30_2)\end{aligned}$$

RP $q = n$ is the CP of the first order in $\mathbb{C}_q^h$ ($6m^2 - 2n^2 - 6m + 2n + 3$ is odd integer, i.e. nonzero) and BP $\lambda = 4n^2$ is not CP in $\mathbb{C}_\lambda$ in Case 1 and in Case 2.

In Case 2 negative CP is the root of equation $\varphi_-((1-h/2)\pi y) = \varphi_-((\xi - h/2)\pi y)$ and large positive CP is the root of equation $\psi_-((1-h/2)\pi y) = \psi_-((\xi - h/2)\pi y)$. Functions $\varphi_-(y) = y\coth(y)$ and $\psi_-(y) = y\tanh(y)$ are strongly monotonic functions for $y > 0$ [107] and $\pi(1-h/2) \neq \pi(\xi - h/2)$. So, for the dSLP (4.1)–(4.3$_2$) negative (large positive) CPs $b \in \mathbb{R}_q^-$ ($b \in \mathbb{R}_q^+$) do not exist.

Let to consider Case 1. Real CF for large positive λ is defined by the function γ_{r+} (see (4.25$_1$)) which has the form of the function (3.29). We have equation (3.30) ($\tilde{y} = \pi y(1-h/2)$ and $\tilde{h}$ instead of h). So, conditions (3.31) for CPs are valid

$$3\tilde{\xi}^2 - 1 + \tilde{h}^2 > 0, \quad \tilde{\xi} < 1 - \tilde{h},$$

where $\tilde{\xi} = (\xi - h/2)/(1-h/2)$, $\tilde{h} = h/(1-h/2)$, $h = 1/n$, $\xi = m/n$. We can rewrite these conditions as

$$6m^2 - 2n^2 + 6m - 2n + 3 > 0, \quad m + 1 < n. \tag{4.31}$$

So, we have one large positive CP $b \in \mathbb{R}_q^{h+}$ for $6m^2 - 2n^2 + 6m - 2n + 3 > 0$ and $m + 1 < n$ (see Figure 3.12). The first inequality agree with formula (4.30$_1$).

In Case 1 the function γ_{r-} (see (4.25$_1$)) has the form

$$y = \frac{\cosh(\delta x)}{\sinh(\alpha x)\sinh(\beta x)}, \quad x > 0, \quad \alpha, \beta, \delta > 0. \tag{4.32}$$

Lemma 3.22. *The function* (4.32) *has one CP* $b \in (0, +\infty)$ *if* $\delta > \alpha + \beta$, *otherwise CPs do not exist.*

Proof. A derivative of this function is zero if

$$f_1(x) := \delta\tanh(\delta x) = \alpha\coth(\alpha x) + \beta\coth(\beta x) =: f_2(x). \tag{4.33}$$

The function $\coth(x)$ is decreasing function, the function $\tanh(x)$ is increasing function, and $\lim_{x\to+\infty}\coth(x) = \lim_{x\to+\infty}\tanh(x) = 1$. We have

$$\lim_{x\to+\infty} \delta\tanh(\delta x) = \delta,$$
$$\lim_{x\to+\infty} (\alpha\coth(\alpha x) + \beta\coth(\beta x)) = \alpha + \beta.$$

So, graphs of the functions f_1 and f_2 have one common point if $\delta > \alpha + \beta$, and $f_1(x) < f_2(x)$ if $\delta \leqslant \alpha + \beta$. □

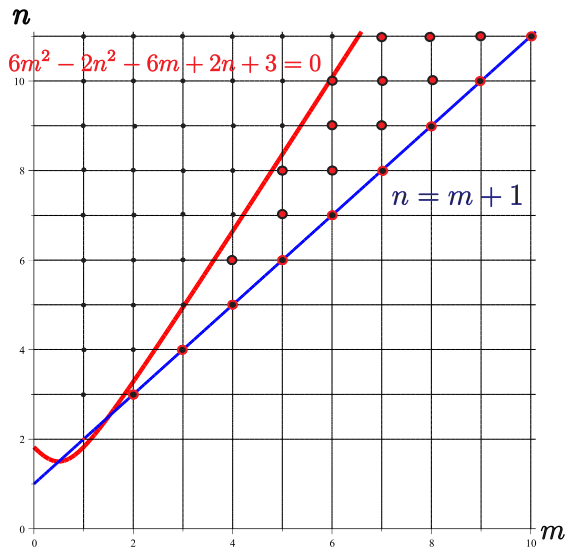


Fig. 3.12. Critical Points in $\mathbb{R}_y^{h+}$ and at $q = \infty$ (on line $n = m + 1$) in Case 1.

In Case 1 for function γ_{r-} we have $\delta = \pi(1 - h/2)$, $\alpha = \pi(\xi - h/2)$, $\beta = \pi h$. Then we have condition

$$\pi(1 - h/2) > \pi(\xi - h/2) + \pi h$$

or

$$m + 1 < n. \tag{4.34}$$

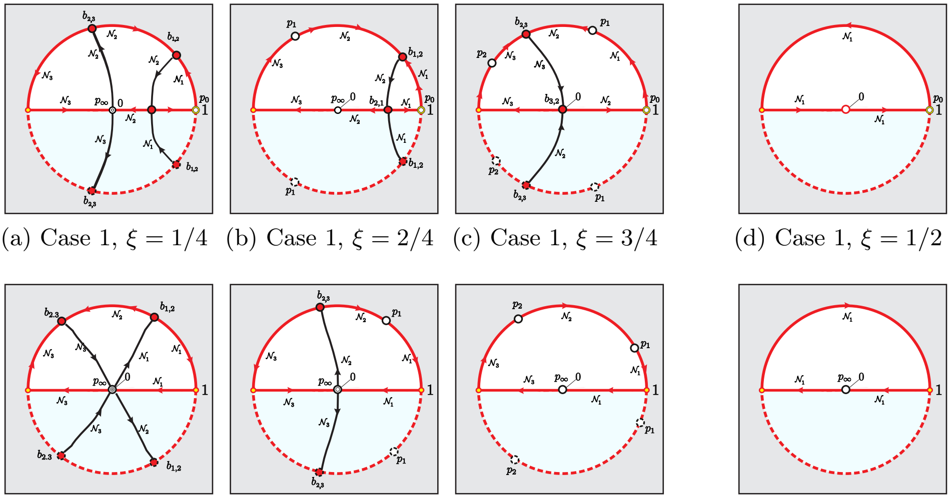
We have one negative CP $b \in \mathbb{R}_q^{h-}$ for $m + 1 < n$. If $m + 1 = n$, then there are not negative CP. If $n = m + 1 > 2$ we have CP of the first order at $q = \infty$, if $\xi = 1/2$ at $q = \infty$ we have a removable singularity point.

4.2 Spectrum Curves

Spectrum Domain is the set $\mathcal{N}_{\boldsymbol{\xi}} = \mathcal{D}_{\boldsymbol{\xi}} \cup \mathcal{C}_{\boldsymbol{\xi}}$. We can enumerate those Spectrum Curves for our problem by classical case ($\gamma = 0$): if $z_k = (k - 1/2)/(1 - h/2)$ belongs to Spectrum Curve, then the index of this Spectrum Curve is k. So, $\mathcal{N}_k(0) = z_k$. We have that $\mathcal{N}_{\boldsymbol{\xi}}$ is a finite union of Spectrum Curves $\mathcal{N}_l$, where $l = \overline{1, n-1}$. So, we have $\mathcal{D}_\xi = \cup_{l \in \mathcal{Z}_\xi} \mathcal{N}_l$ for all $\xi = m/n$.

One can see the Spectrum Curves in Figure 3.10 (Case 1) and Figure 3.11 (Case 2). In Case 2 for $m = n - 1$ we have $n - 2$ poles in $(0, n)$ and pole of the first order in $q = \infty$. So, all Spectrum Curves are regular and lie in $\mathbb{R}_q^h$ and only real eigenvalues exist (see Figure 3.9(a), Figure 3.11(a)).

Spectrum Curves in $\mathbb{C}_w$ are presented in Figure 3.13. We see Spectrum Curves in $\mathbb{C}_w$ for $\xi = 1/4, 2/4, 3/4, 1/2$. Formula (4.24) shows that this



(a) Case 1, $\xi = 1/4$ (b) Case 1, $\xi = 2/4$ (c) Case 1, $\xi = 3/4$ (d) Case 1, $\xi = 1/2$

(e) Case 2, $\xi = 1/4$ (f) Case 2, $\xi = 2/4$ (g) Case 2, $\xi = 3/4$ (h) Case 2, $\xi = 1/2$

Fig. 3.13. Spectrum Curves in $\mathbb{C}_w$ (dSLP Neumann BC) for $n = 4$ and $n = 2$.

point is PP of the $(n-m)$-order if $m+1 < n$ in Case 2 (see Figure 3.13(e)–(f)). In Case 1 this point is PP of the $(n-m-1)$-order if $m+1 < n$. If $n = m+1$, then we have Removable Singularity point with $\gamma = 2h$ (see Figure 3.7(e)–(d)). Additionally, in Case 1 $w = 0$ is CP of the first order if $n = m+1 > 2$ (see Figure 3.7(e)).

When the value n is increasing, then the Spectrum Curves of dSLP (4.1)–(4.3) become more similar to the Spectrum Curves of the differential problem.

4.3 Some remarks and conclusions

We have $n-1$ Spectrum Curve for every $n \in \mathbb{N}$, $n \geqslant 2$. Nonregular Spectrum Curves are CEPs and belong to $\mathbb{R}^h_x = (0, n)$. The number of such Spectrum Curves is equal to

$$n_{ce} = 0, \qquad (4.35_1)$$

$$n_{ce} = (K-1)/2, \qquad (4.35_2)$$

where $K := \gcd(2n-1, 2m-1) \in \mathbb{N}_o$ and $N = \big((2n-1)/K + 1\big)/2$, $M = \big((2m-1)/K + 1\big)$ for $\xi = m/n \in \mathbb{Q}$. Then the number of regular Spectrum Curves is equal to $n_{nce} = n - 1 - n_{ce}$.

The poles of CF belong to $\mathbb{R}^h_x \cup \{\infty\}$ in Case 2 and $\mathbb{R}^h_x \cup \{0\} \cup \{\infty\}$ in Case 1. In the Case 1 eigenvalue $\lambda = 0$ do not exist (we have PP $p_0 = 0$).

The pole at $q = \infty$ is of

$$n_\infty = n - m - 1, \tag{4.36$_1$}$$

$$n_\infty = n - m \tag{4.36$_2$}$$

order. Then n_∞ Spectrum Curves go to this point and the same number of Spectrum Curves leave this point. Note that incoming Spectral Curves alternate with outgoing. If we denote the number of poles at $\mathbb{R}_x^h \cup \{0\}$ corresponding to function γ_c by n_p, then $n_p + n_\infty = n_{nce}$. So, we have formula

$$n_p = m, \tag{4.37$_1$}$$

$$n_p + n_{ce} = m - 1. \tag{4.37$_2$}$$

Particulary, $n_{ce}, n_p \leqslant m - 1$ in Case 2. If $m = 1$ in Case 1, then we have only one PP $p_0 = 0$.

Remark 3.23. In Case 1 for $n = m + 1 > 2$ and $\gamma = 2h$ there are exist only $n - 3$ eigenvalues. If $\xi = 1/2$, $\gamma = 1$ or $\xi = 2/3$, $\gamma = 2/3$ there are not eigenvalues (see Figure 3.8(a),(d)).

If n_c is the number of parts Spectrum Curves in the complex part of C_q^h between two CP (including $q = \infty$ for $n = m + 1$) and n_{cr} is number of CP in $\mathbb{R}_q^h$, then the following relation is valid:

$$n_{cr} - n_c = n - m - 2, \quad n > 2, \tag{4.38$_1$}$$

$$n_{cr} = n - m - 1 \tag{4.38$_2$}$$

($n_c = 0$ in Case 2). Formula (4.38$_1$) is valid in case $\xi = 1/2$ if $n_c = 1$ by definition.

Conclusions

During the doctoral studies at Vilnius University we have studied the Sturm–Liouville problems with one classical and another two-point NBC. From the results obtained in the previous chapters we derive the following conclusions:

- In Chapter 1 we investigate the spectrum of SLP with one classical (Dirichlet BC) and two-point NBC depending on two parameters γ and ξ. The dependence on parameter $\gamma \in \mathbb{R}$ is described by Characteristic Function in domain $\mathbb{C}_q$. We find special points of this meromorphic function: Zero Points, Pole Points, Critical Points and Constant Eigenvalue Points. We find values of the first two derivatives of Characteristic Function at Constant Eigenvalues points and at Ramification Point $q = 0$. We get quite new results about complex part of SLP spectrum.

- For fixed parameter ξ in NBC Spectrum points define the Spectrum Curves. Each Spectrum Curve is the trajectory of a Spectrum Point in domain $\mathbb{C}_q$. We have regular Spectrum Curves ($\gamma \in \mathbb{R}$), semi-regular Spectrum Curves ($\gamma \in (0, +\infty)$) and non-regular Spectrum Curves (Constant Eigenvalues Points).

- The global view of Spectrum Curves (Spectrum Domain) depends on parameter ξ. At bifurcation points this view undergoes qualitative changes. We find bifurcation types for these SLP.

- In Chapter 2 we investigate the spectrum for problems with another NBCs: two SLP with one classical Dirichlet BC and two-points NBCs, four SLP with one classical Neumann BC and two-points NBCs and SLP with symmetrical NBC. For all these SLP we get similar results about Spectrum Curves as in Chapter 1.

- In Chapter 3 we investigate the spectrum discrete Problems: two dSLP with one classical Dirichlet BC and two-points or three-point NBCs, two dSLP with one classical Neumann type BC and two-points or three-point NBCs. Discrete SLP depends on additional parameter h. Characteristic Function for dSLP is defined on domain $\mathbb{C}_q^h$ and a relation between eigenvalues and $q \in \mathbb{C}_q^h$ is more complicated. For discrete case we can investigate Spectrum Curves near point $q = \infty$. This point is either Pole Point or Removable Singularity Point. For dSLP we have two Ramification Points $q = 0$ and $q = n$. In this chapter we get new results about negative and large positive ($\lambda > 4/h^2$) eigenvalues.

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