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Value distribution theorems for the Lerch zeta-function

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Reikšmių pasiskirstymo teoremos Lercho dzeta funkcijai

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Chapter 1

Introduction

1.1 Research topic

Throughout the thesis, the value distribution of the Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, is investigated with emphasis to universality of $L(\lambda, \alpha, s)$, i.e, to approximation of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$.
We start with the definition of the function $L(\lambda, \alpha, s)$. Let $\lambda \in \mathbb{R}$ and α, $0 < \alpha \leqslant 1$, be fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.$$

If the parameter λ is an integer, then $L(\lambda, \alpha, s)$ becomes the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \ \sigma > 1.$$

Therefore, with $\lambda \in \mathbb{Z}$ the function $L(\lambda, \alpha, s)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. If $\lambda \notin \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ is entire [27]. In general, the function $L(\lambda, \alpha, s)$, differently from the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 1,$$

has no the Euler product over primes p, except for the cases $L(k, 1, s) = \zeta(s)$,

$$L\Big(\frac{1}{2}, 1, s\Big) = \Big(1 - 2^{1-s}\Big)\zeta(s)$$

and
$$L\Big(k, \frac{1}{2}, s\Big) = \Big(2^s - 1\Big)\zeta(s),$$
where $k \in \mathbb{Z}$. Thus, the Lerch zeta-function $L(\lambda, \alpha, s)$ is a generalization of the classical zeta-functions $\zeta(s)$ and $\zeta(s, \alpha)$. The function $L(\lambda, \alpha, s)$ was introduced independently by M. Lerch [36] and R. Lipschitz [37]. Lerch also obtained the analytic continuation and proved the functional equation for the function $L(\lambda, \alpha, s)$. Denote by $\Gamma(s)$ the Euler gamma-function. Then the function $L(\lambda, \alpha, s)$ satisfies the functional equation

$$L(\lambda, \alpha, 1-s) =$$
$$\frac{\Gamma(s)}{(2\pi)^s}\Big(\exp\{\frac{\pi i s}{2} - 2\pi i \alpha\lambda\}L(-\alpha, \lambda, s) +$$
$$\exp\{-\frac{\pi i s}{2} + 2\pi i \alpha(1-\lambda)\}L(\alpha, 1-\lambda, s)\Big),$$

here $0 < \lambda \leqslant 1$. In general, $L(\lambda, \alpha, s)$ is an interesting analytic object depending on two parameters. Analytic theory of the Lerch zeta-function is given in [27].

1.2 Aims and problems

The function $L(\lambda, \alpha, s)$, as the majority of other zeta and L-functions, is universal in the sense that its shifts $L(\lambda, \alpha, s+i\tau)$, $\tau \in \mathbb{R}$, for some classes of the parameters α and λ, approximate a wide class of analytic functions. The aim of the thesis is the extension of the universality for the Lerch zeta-function for other classes of parameters α and λ. The problems of the thesis are the following:

1. The extension of a continuous universality theorem for the Lerch zeta-function with transcendental parameter α.
2. The extension of a discrete universality theorem for the Lerch zeta-function with transcendental parameter.
3. The extension of a joint continuous universality theorem for Lerch zeta-functions with algebraically independent parameters.
4. A joint discrete universality theorem for Lerch zeta functions.
5. The extension of the functional independence for the Lerch zeta-function with transcendental parameter α.
6. A joint functional independence theorem for Lerch zeta-functions.

1.3 Actuality

Approximation of analytic functions is one of the central problems of the function theory. By the famous Mergelyan theorem [38], every analytic function can be approximated uniformly on compact sets with connected complements by a polynomial. Thus, for each analytic function a polynomial with approximating property exists. The advantage of universality theorems for zeta-functions as to compare to the Mergelyan theorem is that the whole class of analytic functions is approximated by the shifts of the same zeta-function. Zeta-functions, as polynomials , are compatitively simple because, by approximate functional equations, they are approximated by Dirichlet polynomials. Thus, universality theorems for zeta-functions is a powerful instrument in the approximation theory. Since the Lerch zeta-function depends on two parameters, it is possible to choose the most convenient approximations. For this, it is important to extend the classes of parameters α and λ for which the Lerch zeta-function remains universal.

Universality of zeta-functions, including the Lerch zeta-function, also has serious theoretical applications. One of these applications comes back to famous Hilbert problems and is related to the independence of functions, more precisely, universality theorems imply the functional independence of zeta-functions. Moreover, universality theorems for zeta-functions without Euler product (the function $L(\lambda, \alpha, s)$ has no Euler product) keep the information on the zero-distribution. These and other properties of universal zeta-functions make universality one of the urgent problems of modern analytic number theory.

1.4 Methods

In the thesis, the universal probabilistic method is used for the proof of universality theorems. This method is based on limit theorems for weakly convergent probability measures in the space of analytic functions with explicitly given limit measure. Proofs of these theorems use elements of Fourier analysis, Dirichlet series and Prokhorov theory connecting the tightness and relative compactness of families of probability measures. Universality theorems follow from limit theorems and the Mergelyan theorem.

1.5 Novelty

The results of the thesis are new. Universality theorems for the Lerch zeta-function were known only with transcendental parameter α. In the thesis, the transcendence of α is replaced by a weaker condition. Joint universality of Lerch zeta-functions was known only for algebraically independent parameters $\alpha_1, ..., \alpha_r$.

1.6 History of the problem and results

The Lerch zeta-function was forgotten for a long time. Some authors only gave different proofs of the functional equation [1], [2], [4], [5], [39], [47]. Some of these proofs also can be found in [27]. Moreover, D. Klush obtained [20], [21] some mean-value results for the function $L(\lambda, \alpha, s)$. For example, in [20], it was obtained that

$$\int_0^T \left|L(\lambda, \alpha, \sigma + it)\right|^2 \sim \begin{cases} T \log T & \text{if } \sigma = \frac{1}{2}, \\ T\zeta(2\sigma, \alpha) & \text{if } \frac{1}{2} < \sigma < 1 \end{cases}$$

as $T \to \infty$.
The next progress in the theory of the Lerch zeta-function is related to the names of R. Garunkštis, M. Katsurada, A. Laurinčikas, K. Matsumoto and J. Steuding.
The first results for the Lerch zeta-function were devoted to probabilistic limit theorems. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $\left(\mathbb{X}, \mathcal{B}(\mathbb{X})\right)$. We recall that, by the definition, P_n converges weakly to P as $n \to \infty$, if, for every real bounded continuous function g on $\mathbb{X}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}} g dP_n = \int_{\mathbb{X}} g dP.$$

R. Garunkštis and A. Laurinčikas proved [12] a limit theorem in the sense of weak convergence of probability measures on the complex plane. For $A \in \mathcal{B}(\mathbb{C})$, define

$$P_{T,\sigma}(A) = \frac{1}{T} \operatorname{meas}\left\{t \in [0, T] : L(\lambda, \alpha, \sigma + it) \in A\right\}.$$

In [12], the following statement has been obtained.

Theorem A. *Suppose that $\sigma > \frac{1}{2}$ is fixed. Then, on $\left(\mathbb{C}, \mathcal{B}(\mathbb{C})\right)$, there exists a probability measure P_σ such that $P_{T,\sigma}$ converges weakly to P_σ as $T \to \infty$.*

Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic functions on G endowed with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n(s)\} \subset H(G)$ converges to the function $g(s) \in H(G)$ if and only if, for every compact set $K \subset G$,

$$\lim_{n\to\infty} \sup_{s\in K} \left|g_n(s) - g(s)\right| = 0.$$

In [23] , Theorem A was extended to the space of analytic functions for $G = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$.
B. Bagchi in his thesis [3] proposed a new method how to identify limit measures in limit theorems for some zeta-functions. In [24], the Bagchi method was applied for the Lerch zeta-function. Denote by γ the unit circle $\{s \in \mathbb{C} : |s| = 1\}$ on the complex plane, and define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $\Big(\Omega, \mathcal{B}(\Omega)\Big)$, the probability Haar measure m_H can be defined. We recall that the Haar measure m_H differs from other probability measures by its invariance, i.e., for all $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega$,

$$m_H(A) = m_H(\omega A) = m_H(A\omega).$$

This gives the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Denote by $\omega(m)$ the mth component of an element $\omega \in \Omega$, $m \in \mathbb{N}_0$, and on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$, define the $H(D)$-valued random element $L(\lambda, \alpha, s, \omega)$ by

$$L(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m} \omega(m)}{(m+\alpha)^s}, \omega \in \Omega.$$

Let P_L be the distribution of the random element $L(\lambda, \alpha, s, \omega)$, i.e,

$$P_L(A) = m_H\Big\{\omega \in \Omega : L(\lambda, \alpha, s, \omega) \in A\Big\}, A \in \mathcal{B}(H(D)).$$

Then in [24], the identification of the limit measure in Theorem of [23] was given, namely, the following theorem was proved. We recall that a number α is transcendental if, for any polynomial $p(s) \not\equiv 0$ with rational coefficients, the inequality $p(\alpha) \neq 0$ is true.

Theorem B. *Let $\lambda \notin \mathbb{Z}$, and α be a transcendental number. Then*

$$P_T(A) = \frac{1}{T} \operatorname{meas} \Big\{\tau \in [0, T] : L(\lambda, \alpha, s + i\tau) \in A\Big\}, A \in \mathcal{B}(H(D)),$$

converges weakly to the measure P_L as $T \to \infty$.

R. Garunkštis and A. Laurinčikas proved [11] a weighted limit theorem in the space of analytic functions. Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty]$, $T_0 > 0$,such that the variation $V_a^b w$ on $[a, b]$ satisfies the inequality

$V_a^b w \leqslant cw(a)$ with some $c > 0$ for all $[a, b] \subset [T_0, \infty]$. Moreover, let

$$U = U(T, w) = \int_{T_0}^{T} w(t)dt,$$

and $\lim_{T\to\infty} U(T, w) = +\infty$. For $A \in \mathcal{B}(H(D))$, define

$$P_{T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau)I\Big(\Big\{\tau \in [T_0, T] : L\Big(\lambda, \alpha, s + i\tau) \in A\Big\}\Big)d\tau,$$

here $I(A)$ denotes the indicator function of the set A. Then a limit theorem on the complex plane has the following form.

Theorem C. *Suppose that $\lambda \notin \mathbb{Z}$ and α is a transcendental number. Then, on $\big(H(D), \mathcal{B}(H(D))\big)$, there exists a probability measure P_w such that $P_{T,w}$ converges weakly to P_w as $T \to \infty$.*

In his thesis [10], R. Garunkštis identified the limit measure in Theorem C, however, under the additional condition on the weight function $w(t)$ that

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)X(t + \tau, \omega)dt = \mathbb{E}X(0, \omega) + O(1 + |t|)^{\beta}, T \to \infty.$$

almost surely for all $t \in \mathbb{R}$ with some $\beta > 0$. Here $X(\tau, \omega)$ is an ergodic process, $\mathbb{E}|X(\tau, \omega)| < \infty$, with sample paths integrable almost surely in the Riemann sense over every finite interval.

Now, we pass to universality results which are the subject of our thesis.

The universality of the Riemann zeta-function $\zeta(s)$ was discovered by S. M. Voronin in [53]. He proved that if $0 < r < \frac{1}{4}$, $f(s)$ is a continuous non-vanishing function in the disc $|s| \leqslant r$, and analytic in $|s| < r$, then, for every $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s|\leqslant r}\Big|\zeta(s + \frac{3}{4} + i\tau) - f(s)\Big| < \varepsilon.$$

This interesting Voronin's theorem was observed by number theorists, and slightly improved. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the modern version of the Voronin theorem is the following statement, see, for example, [22].

Theorem D. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K}\left|\zeta(s+i\tau) - f(s)\right| < \varepsilon\right\} > 0.$$

Theorem D shows that there are infinitely many shifts $\zeta(s+i\tau)$ approximating a given function $f(s) \in H_0(K)$.
The first universality theorem for the Lerch zeta-function was obtained in [25]. Denote by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Thus, $H_0(K) \subset H(K)$.

Theorem E. *Suppose that $0 < \lambda < 1$, and α is a transcendental number. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - f(s)\right| < \varepsilon\right\} > 0.$$

Chapter 2 of the thesis is devoted to the extension of Theorem E for a new class of parameters α. Define the set

$$L(\alpha) = \left\{\log(m+\alpha) : m \in \mathbb{N}_0\right\}.$$

The main result of the chapter is the following continuous universality theorem for the function $L(\lambda,\alpha,s)$.

Theorem 2.1. *Suppose that the set $L(\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - f(s)\right| < \varepsilon\right\} > 0.$$

We observe that if the number α is transcendental, then the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Actually, suppose that α is transcendental, however, the set $L(\alpha)$ is linearly dependent over $\mathbb{Q}$. Then there exist $m_1, ..., m_r \in \mathbb{N}_0$ and $k_1, ..., k_r \in \mathbb{Z} \setminus \{0\}$ such that

$$k_1 \log(m_1+\alpha) + ... + k_r \log(m_r+\alpha) = 0.$$

Hence,

$$(m_1+\alpha)^{k_1} ... (m_r+\alpha)^{k_r} = 1.$$

Therefore, using the Newton binomial expansions, we find that there exists a polynomial $p(s) \not\equiv 0$ such that $p(\alpha) = 0$. However, this contradicts the transcendence of α. Thus, the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$.

We recall that, by the definition, the number α is algebraic if there exists a polynomial $p(s) \not\equiv 0$ with rational coefficients such that $p(\alpha) = 0$. For example, $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{\sqrt{2}}$ are algebraic numbers because they are roots of the polynomials $2s = 1$ and $2s^2 = 1$, respectively.

By the famous Cassels theorem [7], at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linearly independent over $\mathbb{Q}$. Therefore, it can happen that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with algebraic irrational α, thus, Theorem 2.1 can be valid also for algebraic irrational α. On the other hand, at the moment we do not know any algebraic irrational α such that the set $L(\alpha)$ would be linearly independent over $\mathbb{Q}$. Theorem 2.1 has the following modification.

Theorem 2.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left| L(\lambda, \alpha, s+i\tau) - f(s) \right| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proofs of Theorems 2.1 and 2.2 are based on a limit theorem in the space of analytic functions for the function $L(\lambda, \alpha, s)$. Let P_T and P_L be the same as in Theorem B.

Theorem 2.3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Then P_T converges weakly to P_L as $T \to \infty$.*

Theorems 2.1-2.3 are published in [41].

In place of the shifts $L(\lambda, \alpha, s + i\tau)$, where τ is an arbitrary real number, one can consider the shifts $L(\lambda, \alpha, s + i\varphi(k))$, where $\varphi(t)$ is a certain function, and k runs over non-negative integers. Limit and universality theorems for shifts $L(\lambda, \alpha, s + i\tau)$ are called continuous, while with shifts $L(\lambda, \alpha, s + i\varphi(k))$ are called discrete theorems. The simplest function $\varphi(k)$ is of the type kh, $k \in \mathbb{N}_0$, with a fixed $h > 0$.

Discrete limit theorems for the Lerch zeta-function were obtained by J. Ignatavičiūtė in her thesis [18]. In [16], a discrete analogue of Theorem A was proved.

Theorem F. *Suppose that α is transcendental, the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational and $\sigma > \frac{1}{2}$ is fixed. Then*

$$\frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : L(\lambda, \alpha, \sigma + ikh) \in A\right\}, A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the complex-valued random element

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^{\sigma}}$$

as $N \to \infty$.

Here N runs over non-negative integers, and $\#A$ is the cardinality of the set A.

In [17], Theorem F was extended to the space of analytic and even meromorphic functions. Let $\hat{D} = \left\{s \in \mathbb{C} : \sigma > \frac{1}{2}\right\}$. Let

$$\Omega_1 = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. Denote by $\omega_1(p)$ the p th component of an element $\omega_1 \in \Omega$, and on the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, where m_{1H} is the Haar measure on $(\Omega, \mathcal{B}(\Omega_1))$, define the $H(\hat{D})$ valued random element

$$L(\lambda, \alpha, s, \omega_1) = \omega_1(b) b^s e^{-2\pi i\lambda \frac{a}{b}} \sum_{m=0; m\equiv a(\mathrm{mod} b)}^{\infty} \frac{e^{2\pi i\lambda m}\omega_1(m)}{m^s}.$$

Then we have the statement [17].

Theorem G. *Suppose that* $\lambda \notin \mathbb{Z}$, α *is a transcendental number, and* $h > 0$ *is such that* $\exp\{\frac{2\pi}{h}\}$ *is a rational number. Then*

$$P_N(A) = \frac{1}{N+1}\#\Big\{0 \leqslant h \leqslant N : L(\lambda, \alpha, s+ikh) \in A\Big\}, A \in \mathcal{B}(H(\hat{D})),$$

converges weakly to the measure P_L *as* $N \to \infty$. *If* $\alpha = \frac{a}{b}$, $a, b \in \mathbb{N}$, $1 \leqslant a \leqslant b$, $(a, b) = 1$, *and* $h > 0$ *is such that* $\exp\left\{\frac{2\pi k}{h}\right\}$, $k \in \mathbb{N}$, *is an irrational number, then* P_N *converges weakly to the distribution of the random element* $L(\lambda, \alpha, s, \omega_1)$ *as* $N \to \infty$.

A discrete version of the Voronin universality theorem was proposed by A. Reich [51], and is of the following form.

Theorem H. *Suppose that* $K \in \mathcal{K}$ *and* $f(s) \in H_0(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K}\Big|\zeta(s+ikh) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Actually, Reich proved a discrete universality theorem for Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ of algebraic number fields $\mathbb{K}$. If $\mathbb{K} = \mathbb{Q}$, then we have the Riemann zeta-function. A different proof from that of Reich was given by Bagchi in

[3]. Discrete universality theorems are also known for the Hurwitz zeta-function $\zeta(s,\alpha)$ which is the case of the function $L(\lambda,\alpha,s)$ with $\lambda \in \mathbb{Z}$. The first theorem of such a type belongs to Bagchi [3].

Theorem I. *Suppose that α is a rational number, $\alpha \neq \frac{1}{2}$, $\alpha \neq 1$. Let K be a compact simply connected and locally path connected subset of D, and let $f(s)$ be a continuous function on K that is analytic in the interior of K. Then, for all $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} \Big|\zeta(s+ikh,\alpha) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Theorem I follows from the representation

$$\zeta(s,\frac{a}{b}) = b^s \sum_{m=0; m\equiv a (\mathrm{mod} b)}^{\infty} \frac{1}{m^s}, 1 \leqslant a \leqslant b, (a,b) = 1, b \geqslant 3,$$

and joint properties of the pair of functions

$$(b^s, \sum_{m=0; m\equiv a (\mathrm{mod} b)}^{\infty} \frac{1}{m^s}).$$

It turns out that the case of transcendental α is more complicated and that of rational, and an analogue of Theorem I for all $h > 0$ is not known. For example, in [28], an analogue of Theorem I with transcendental α was obtained for $h > 0$ such that $\exp\left\{\frac{2\pi}{h}\right\}$ is rational number, and with $K \in \mathcal{K}$ and $f(s) \in H(K)$.

Chapter 3 of the thesis is devoted to discrete universality theorems for the function $L(\lambda,\alpha,s)$. Define the set

$$L(\alpha,h,\pi) = \Big\{\Big(\log(m+\alpha) : m \in \mathbb{N}_0\Big), \frac{2\pi}{h}\Big\}, h > 0.$$

The latter set consists of all logarithms $\log(m+\alpha)$ and the number $\frac{2\pi}{h}$. The main results of the chapter are the following theorems.

Theorem 3.1. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$ and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} \Big|L(\lambda,\alpha,s+ikh) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Theorem 3.1 admits the following modification.

Theorem 3.2. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$ and*

$0 < \lambda \leqslant 1$. *Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} \Big| L(\lambda, \alpha, s+ikh) - f(s) \Big| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorems 3.1 and 3.2 can be generalized for composite functions. In the thesis, we give one example.

Theorem 3.3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, $0 < \lambda \leqslant 1$ and that $F : H(D) \to H(D)$ is a continuous operator such that, for every set $G \subset H(D)$, the pre-image $F^{-1}G$ is non empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} \Big| F(L(\lambda, \alpha, s+ikh)) - f(s) \Big| < \varepsilon \Big\} > 0.$$

Theorem 3.3 is an analogue of Theorem 3.2 for the function $F(L(\lambda, \alpha, s + ikh))$. The proofs of universality theorems of Chapter 3 are based on a limit theorem for

$$\frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : L(\lambda, \alpha, s+ikh) \in A \Big\}, A \in \mathcal{B}(H(D)),$$

as $N \to \infty$. They are published in [33].

The first joint value-distribution theorems for Lerch zeta-function were obtained in [29]. The first theorem of the latter paper is a multidimensional generalization of Theorem A.

Theorem J. *Suppose that $\min\limits_{1\leqslant j\leqslant r} \sigma_j > \frac{1}{2}$. Then there exists a probability measure P on $\big(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r)\big)$ such that the measure*

$$\frac{1}{T} \operatorname{meas}\Big\{t \in [0,T] : \Big(L(\lambda_1, \alpha_1, \sigma_1+it), ..., L(\lambda_r, \alpha_r, \sigma_r+it)\Big) \in A\Big\}, A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to P as $T \to \infty$.

In Theorem J, the limit measure P is not explicitly given.
In the second theorem of [31] the latter gap was removed, and a joint limit theorem for Lerch zeta-function in the space $H^r(D)$ was obtained. The further statistical investigations of the joint value-distribution of Lerch zeta-functions were continued in [32]. There also some correction of the paper [30] are given. Voronin also introduced the joint universality of zeta and L-functions: in [54], he obtained the joint universality of Dirichlet L-functions $L(s, \chi_1), ..., L(s, \chi_r)$ with

non-equivalent Dirichlet characters (not generated by the same primitive character). We recall that the Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma > 1$, by the series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and by analytic continuation elsewhere, and the character $\chi(m)$ is a periodic with period q completely multiplicative function $\big(\chi(mn) = \chi(m)\chi(n), m, n \in \mathbb{N}\big)$, $\chi(m) = 0$ for $(m,q) > 1$, and $\chi(m) \neq 0$ for $(m,q) = 1$. In the case of joint universality for Dirichlet L-functions, a collection of analytic functions from the classes $H(K_1), ..., H(K_r)$ with $K_1, ..., K_r \in \mathcal{K}$ are simultaneously approximated by shifts $L(s+i\tau,\chi_1), ..., L(s+i\tau,\chi_r)$. The joint universality of Lerch zeta-functions was considered by various authors. We mention the papers [35], [42],[45],[46]. In these papers, the algebraic independence over $\mathbb{Q}$ of the parameters $\alpha_1, ..., \alpha_r$ was required. We recall that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$ if there is no a polynomial $p(s_1, ..., s_r) \not\equiv 0$ with rational coefficients such that $p(\alpha_1, ..., \alpha_r) = 0$. We recall a joint universality theorem from [30].

Theorem K. *Suppose that $\alpha_1, ..., \alpha_r$ are algebraically independent numbers over $\mathbb{Q}$, $\lambda_1 = \dfrac{a_1}{q_1}, ..., \lambda_n = \dfrac{a_r}{q_r}$, $(a_1, q_1) = 1, ..., (a_r, q_r) = 1$, where $q_1, ..., q_r$ are distinct positive integers and $a_1, ..., a_r$ are positive integers with $a_1 < q_1, ..., a_r < q_r$. Let $K_1, ..., K_r \in \mathcal{K}$ and $f_1(s) \in H(K_1), ..., f_r(s) \in H(K_r)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \Big|L(\lambda_j, \alpha_j, s+i\tau) - f_j(s)\Big| < \varepsilon\Big\} > 0.$$

In [42] and [35], the case of $\alpha = \alpha_1 = ... = \alpha_r$ with transcendental α was discussed.

In chapter 4 of the thesis, we prove joint universality theorems for Lerch zeta functions without using the algebraic independence of the parameters $\alpha_1, ..., \alpha_r$. Also, we do not use any conditions for the parameters $\lambda_1, ..., \lambda_r$.

Let

$$L(\alpha_1, ..., \alpha_r) = \Big\{(\log(m+\alpha_1 : m \in \mathbb{N}_0), ..., (\log(m+\alpha_r) : m \in \mathbb{N}_0)\Big\}.$$

Theorem 4.1. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then, for every*

$\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left|L(\lambda_j, \alpha_j, s+i\tau) - f_j(s)\right| < \varepsilon\right\} > 0.$$

As other universality theorems, Theorem 4.1 has the following modified version.

Theorem 4.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left|L(\lambda_j, \alpha_j, s+i\tau) - f_j(s)\right| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

It is not difficult to see that the linear independence over $\mathbb{Q}$ of the set $L(\alpha_1, ..., \alpha_r)$ is a weaker condition that the algebraic independence of the numbers $\alpha_1, ..., \alpha_r$. Actually, suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, however, the set $L(\alpha_1, ..., \alpha_r)$ is linearly dependent. Then there exists the numbers $m_{1\alpha_1}, ..., m_{r\alpha_r}, ..., m_{l_1\alpha_1}, ..., m_{l_r\alpha_r} \in \mathbb{N}_0$ and
$k_{1\alpha_1}, ..., k_{r\alpha_r}, ..., k_{l_1\alpha_1}, ..., k_{l_r\alpha_r} \in \mathbb{Z} \setminus \{0\}$ such that

$$\begin{aligned} &k_{1\alpha_1} \log(m_{1\alpha_1} + \alpha_1) + ... + k_{1\alpha_r} \log(m_{1\alpha_r} + \alpha_r) + ...+ \\ &k_{l_1\alpha_1} \log(m_{l_1\alpha_1} + \alpha_1) + ... + k_{l_r\alpha_r} \log(m_{l_r\alpha_r} + \alpha_r) = 0. \end{aligned}$$

Hence,

$$(m_{1\alpha_1} + \alpha_1)^{k_{1\alpha_1}} ...(m_{1\alpha_r} + \alpha_r)^{k_{1\alpha_r}} ...(m_{l_1\alpha_1} + \alpha_1)^{k_{l_1}\alpha_1} ...(m_{l_r\alpha_r} + \alpha_r)^{k_{l_r}\alpha_r} = 1.$$

Therefore, using of the Newton binomial theorem gives that there is a polynomial $p(s_1, ..., s_r)$ with integers coefficients such that $p(\alpha_1, ..., \alpha_r) = 0$, i.e., the numbers $\alpha_1, ..., \alpha_r$ are algebraically dependent. Thus, the contradiction shows that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$.

Chapter 5 of the thesis contains joint discrete universality theorems for Lerch zeta-functions. To our knowledge, earlier, the theorems of such a kind were not known. For $h > 0$, define the set

$$L(\alpha_1, ..., \alpha_r; h, \pi) = \left\{(\log(m+\alpha_1) : m \in \mathbb{N}_0), ..., (\log(m+\alpha_r) : m \in \mathbb{N}), \frac{2\pi}{h}\right\}.$$

The main results of the chapter are the following two theorems.

Theorem 5.1. *Suppose that the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \Big|L(\lambda_j, \alpha_j, s+ikh) - f_j(s)\Big| < \varepsilon\Big\} > 0.$$

Theorem 5.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \Big|L(\lambda_j, \alpha_j, s+ikh) - f_j(s)\Big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorems 5.1 and 5.2 are published in [34].

We already have mentioned that one of theoretical applications of universality for zeta-functions is closely related to the functional independence of these functions, and comes back to Hilbert. It is well known that in the International Congress of Mathematicians in Paris (1900) Hilbert presented the list [15] of the most important problems in mathematics that would be solved in the next century. In his 18 th problem, Hilbert observed that the Riemann zeta-function $\zeta(s)$ cannot satisfy any algebraic-differential equation, i.e., there is no any polynomial $p(s_1, ..., s_n) \not\equiv 0$ such that

$$p(\zeta(s), \zeta'(s), ..., \zeta^{(n-1)}(s)) = 0,$$

and that this follows from an analogous result for the Euler gamma-function $\Gamma(s)$ and the functional equation for the function $\zeta(s)$

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

Moreover, Hilbert conjectured that the function

$$\zeta(s, \chi) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}$$

has also an algebraic-differential independence property. This Hilbert conjuncture was proved by Ostrowski [48]. Similar problems for Dirichlet L-functions were studied by A.G.Postnikov [49]. Voronin generalized significantly the above results.

In [52], see also [19], he obtained the functional independence of the Riemann zeta-function. More precisely, he proved the following theorem.

Theorem L. *For $j = 0, ..n$, let $V_j : \mathbb{C}^k \to \mathbb{C}$ be a continuous function, and let*

$$\sum_{j=0}^{n} s^j V_j\Big(\zeta(s), \zeta'(s), ..., \zeta^{(k-1)}(s)\Big) = 0$$

identically for s. Then $V_j \equiv 0$ for $j = 0, ..., n$.

In [54], Voronin extended theorem L for Dirichlet L-functions with non-equivalent Dirichlet characters. In [13], the analogue of Theorem L was obtained for the Lerch zeta-function $L(\lambda, \alpha, s)$ with transcendental parameter α.

In the last chapter of the thesis, the functional independence of the function $L(\lambda, \alpha, s)$ was proved under a weaker condition than the transcendence of the parameter α. The following statement is true. We recall that $L(\alpha) = \big\{ \log(m + \alpha) : m \in \mathbb{N}_0 \big\}$.

Theorem 6.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. For $j = 0, ..., n$, let $V_j : \mathbb{C}^k \to \mathbb{C}$ be a continuous function , and let*

$$\sum_{j=0}^{n} s^j V_j\Big(L(\lambda, \alpha, s), L'(\lambda, \alpha, s), ..., L^{(k-1)}(\lambda, \alpha, s)\Big) = 0$$

identically for s. Then $V_j \equiv 0$ for $j = 0, ..., n$.

In other words, Theorems 6.1 asserts that if $V_0, V_1, ..., V_n : \mathbb{C}^k \to \mathbb{C}$ are continuous functions not all identically zero, then

$$\sum_{j=0}^{n} s^j V_j\Big(L(\lambda, \alpha, s), L'(\lambda, \alpha, s), ..., L^{(k-1)}(\lambda, \alpha, s)\Big) \neq 0$$

for some $s \in \mathbb{C}$.

The Lerch zeta-function also have a joint functional independence property, i.e., the following theorem is valid.

Theorem 6.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda_j \leqslant 1$. Let $V_j : \mathbb{C}^{k_1+...+k_r} \to \mathbb{C}$, $j = 0, ..., n$, be a continuous function, and let the equality*

$$\sum_{j=0}^{n} s^j V_j\Big(L(\lambda_1, \alpha_1, s), L'(\lambda_1, \alpha_1, s), ..., L^{(k_1-1)}(\lambda_1, \alpha_1, s), ...,$$
$$L(\lambda_r, \alpha_r, s), L'(\lambda_r, \alpha_r, s), ..., L^{(k_r-1)}(\lambda_r, \alpha_r, s)\Big) = 0$$

hold identically for s. Then $V_j \equiv 0$ for $j = 0, ..., n$.

For the proof of Theorems 6.1 and 6.2, the universality theorems (Theorems 2.1 and 4.1) are applied.
Theorems 6.1 and 6.2 are published in [41] and [40], respectively.

1.7 Approbation

The results of the thesis were presented at the International MMA (Mathematical Modelling and Analysis) conferences (MMA2016, June 1-4, 2016, Tartu, Estonia), (MMA2017, May 30 – June 2, 2017, Druskininkai), (MMA2018, May 29 – June 1, 2018, Sigulda, Latvia), at the 16th International Conference (May 13-18, 2019, Tula, Russia), the Conferences of Lithuanian Mathematical Society (2016, 2017, 2018, 2019), as well as at the Number Theory Seminar of Vilnius University.

1.8 Principal publications

1. A. Laurinčikas, A. Mincevič, Discrete universality theorems for the Lerch zeta-functions, in: Anal. Probab. Methods Number Theory, A. Dubickas et al. Eds Vilnius University, Vilnius, 2017, 87-95.

2. Laurinčikas A., Mincevič A. Joint discrete universality for Lerch zeta-functions. Chebyshevskii Sbornik. **19** (1) 2018, 138-151.

3. A. Mincevič, Value distribution theorems for the Lerch zeta-function, in: Algebra, Number Theory and Discrete Geometry: Modern Problems and Appl., XVI intern. Conf., Tula, TGPU im. L. N. Tolstogo, 2019, pp. 197-199.

4. A. Mincevič, D. Mochov, On the discrete universality of the periodic Hurwitz zeta-function, Šiauliai Math. Semin. **10** (**18**) (2015), 81-89.

5. A. Mincevič, D. Šiaučiūnas, Joint universality theorems for Lerch zeta-functions, Šiauliai Math. Semin **12** (**20**), 2017, 31-47.

6. A. Mincevič, A. Vaiginytė, Remarks on the Lerch zeta – function, Šiauliai Math. Semin **11** (**19**), 2016, 65-73.

1.9 Abstracts of conferences

1. A. Mincevič, On universality of the Lerch zeta-function, Abstracts of MMA2016, June 1-4, 2016, Tartu,Inst. of Math. Statist. of Univ. Tartu, Tartu, Estonia. Abstracts, p.52.

2. A. Mincevič, On the discrete universality of the Lerch zeta-function, Abstracts of MMA 2017, May 30 - June 2, 2017, Druskininkai, VGTU, 2017, p.43.

3. D. Šiaučiūnas, A. Mincevič, Joint universality for the Lerch zeta-functions, Abstracts of MMA 2019, May 29 - June 1, 2018, Sigulda, University of Latvia, Ryga, 2018, p. 73.

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Chapter 2

Continuous universality theorems for the Lerch zeta-function

In this chapter, we consider the approximation of analytic function defined on the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ by continuous shifts of the Lerch zeta-function $L(\lambda, \alpha, s + i\tau)$, where τ is an arbitrary real number. We recall that

$$L(\alpha) = \Big\{ \log(m + \alpha) : m \in \mathbb{N}_0 \Big\}.$$

In some sense, the set $L(\alpha)$ controls the dependence of the terms $e^{2\pi i\lambda m}(m+\alpha)^{-s}$ of Dirichlet series from the definition of the function $L(\lambda, \alpha, s)$. $\mathcal{K}$ is the class of compact subsets of the strip D with connected complements, and $H(K)$, $K \in \mathcal{K}$, is the class of continuous functions on K that are analytic in the interior of K.

2.1 Statements of the theorems

We will prove the following two universally theorems.

Theorem 2.1. *Suppose that the set $L(\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0, T] : \sup_{s\in K} \Big| L(\lambda, \alpha, s + i\tau) - f(s) \Big| < \varepsilon \Big\} > 0.$$

The inequality of the theorem means that the set of shifts $L(\lambda, \alpha, s + i\tau)$ satisfying the inequality

$$\sup_{s\in K} \Big| L(\lambda, \alpha, s + i\tau) - f(s) \Big| < \varepsilon \tag{2.1}$$

has a positive lower density. Hence, we have that the above set of shifts $L(\lambda, \alpha, s+i\tau)$ is infinite.
On the other hand, deeper results on the properties of sets are usually related to their density. In the next theorem, the universality of the function $L(\lambda, \alpha, s)$ is described by terms of the density of shifts $L(\lambda, \alpha, s+i\tau)$ satisfying (2.1).

Theorem 2.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0, T] : \sup_{s\in K} \Big| L(\lambda, \alpha, s+i\tau) - f(s) \Big| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Unfortunately, the inequality of the theorem is true not for all $\varepsilon > 0$. However, a countable set of values of ε is narrow, thus, it remains sufficiently many values of ε that can be used in Theorem 2.1.
Proofs of Theorems 2.1 and 2.2 use probabilistic limit the theorems for probability measures in the space of analytic function.

2.2 A continuous limit theorem

Denote by $H(D)$ the space of analytic functions on the strip D equipped with the topology of uniform convergence on compacta. Recall that $\mathcal{B}(\mathbb{X})$ denotes the Borel σ-field of the space $\mathbb{X}$, and, for $A \in \mathcal{B}(H(D))$, define

$$P_T(A) = \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0, T] : L(\lambda, \alpha, s+i\tau) \in A \Big\}.$$

This section is devoted to the weak convergence of P_T as $T \to \infty$. Proofs of universality theorems require the explicit form of the limit measure. For this, the following topological structure is applied. Let $\gamma = \big\{ s \in \mathbb{C} : |s| = 1 \big\}$, and

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the definition of the Cartesian product, the set Ω consists of all functions $\omega : \mathbb{N}_0 \to \gamma$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Hence, on $\big(\Omega, \mathcal{B}(\Omega)\big)$, the probability Haar measure m_H exists, and we have the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Denote by $\omega(m)$ the mth component of an element $\omega \in \Omega$,

$m \in \mathbb{N}_0$, and, for $s \in D$, define

$$L\Big(\lambda, \alpha, s, \omega\Big) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m} \omega(m)}{(m+\alpha)^s}.$$

Lemma 2.1. *For all α and λ, $L(\lambda, \alpha, s, \omega)$ is an $H(D)$-valued random element defined on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$.*

The lemma is Lemma 5.2.1 of [27]. Its proof is based on the orthogonality of the random variables $\omega(m)$, that is, that

$$\mathbb{E}\omega(m)\overline{\omega(n)} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

and on the Rademacher theorem (Theorem 1.2.9 of [22]). The latter theorem implies that the series

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m} \omega(m)}{(m+\alpha)^s}$$

is almost surely convergent with respect to m_H uniformly on compact subsets of the strip D. From this the lemma follows.
Let P_L be the distribution of the random element $L(\lambda, \alpha, s, \omega)$ i.e., for $A \in \mathcal{B}(H(D))$,

$$P_L(A) = m_H\Big\{\omega \in \Omega : L(\lambda, \alpha, s, \omega) \in A\Big\}.$$

Now, we are ready to state a limit theorem for P_T.

Theorem 2.3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Then P_T converges weakly to P_L as $T \to \infty$.*

We start the proof of Theorem 2.3 with a limit theorem for probability measures on $\big(\Omega, \mathcal{B}(\Omega)\big)$. Before that, we recall some classical result on probability measures on compact groups. Thus, let $\mathcal{G}$ be a compact group. A character χ of the group $\mathcal{G}$ is a function $\chi : \mathcal{G} \to \gamma$ which is multiplicative, i.e., $\chi(g_1, g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in \mathcal{G}$. All characters of the group $\mathcal{G}$ form the group $\mathcal{D}$ which is called dual (or character) group of $\mathcal{G}$. Let P be a probability measure on $\Big(\mathcal{G}, \mathcal{B}(\mathcal{G})\Big)$. The Fourier transform $g_P(\chi)$ of P is defined by

$$g_P(\chi) = \int_{\mathcal{G}} \chi(\})dP, \quad \chi \in \mathcal{D}.$$

For probability measures on $\Big(\mathcal{G}, \mathcal{B}(\mathcal{G})\Big)$, the following theorem is valid [14].

Lemma 2.2. *Let P_n, $n \in \mathbb{N}_0$, be probability measures on $\Big(\mathcal{G}, \mathcal{B}(\mathcal{G})\Big)$, and $g_{P_n}(\chi)$ be the corresponding Fourier transforms. Suppose that $g_{P_n}(\chi)$ converges to a certain continuous function $g(\chi)$ as $n \to \infty$. Then, on $\Big(\mathcal{G}, \mathcal{B}(\mathcal{G})\Big)$, there exists a probability measure P such that P_n converges weakly to P. Moreover, $g(\chi)$ is the Fourier transform of the measure P.*

Now, we return to the group Ω. It is well known that the dual group of Ω is isomorphic to

$$\mathcal{D} = \bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}_0$. An element $k_1 = (k_m : k_m \in \mathbb{Z}, m \in \mathbb{N}_0)$ acts on Ω by

$$\omega \to \omega^{\underline{k}} = \prod_{m=0}^{\infty}{}' \omega^{k_m}(m),$$

where `$'$" means that only a finite number of integers k_m are distinct from zero. Therefore, the characters of the group Ω are of the form

$$\prod_{m=0}^{\infty}{}' \omega^{k_m}(m),$$

and the Fourier transform $g_P(k)$ of the measure P on $\Big(\Omega, \mathcal{B}(\Omega)\Big)$ is defined by

$$g_P(\underline{k}) = \int_{\Omega} \Big(\prod_{m=0}^{\infty}{}' \omega^{k_m}(m)\Big) dP. \tag{2.2}$$

Now, we apply the above remarks for

$$Q_T = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : ((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0) \in A\Big\}, A \in \mathcal{B}(\Omega).$$

Lemma 2.3. *On $\Big(\Omega, \mathcal{B}(\Omega)\Big)$, there exists a probability measure Q such that Q_T converges weakly to Q as $T \to \infty$.*

Proof. We will apply Lemma 2.2. Let $g_T(\underline{k})$ be the Fourier transform of Q_T. Then, in view of (2.2),

$$g_T(\underline{k}) = \int_{\Omega} \Big(\prod_{m=0}^{\infty}{}' \omega^{k_m}(m)\Big) dQ_T,$$

and, by the definition of Q_T, we obtain that

$$g_T(\underline{k}) = \frac{1}{T}\int\limits_0^T {\prod_{m=0}^{\infty}}'(m+\alpha)^{-ik_m\tau}d\tau = \frac{1}{T}\int\limits_0^T \exp\{-i\tau{\sum_{m=0}^{\infty}}' k_m\log(m+\alpha)\}d\tau. \tag{2.3}$$

If

$${\sum_{m=0}^{\infty}}' k_m\log(m+\alpha) = 0,$$

then, obviously,

$$g_T(\underline{k}) = \frac{1}{T}\int\limits_0^T d\tau = 1. \tag{2.4}$$

If

$${\sum_{m=0}^{\infty}}' k_m\log(m+\alpha) \neq 0,$$

then, after integration in (2.3), we find that

$$g_T(\underline{k}) = \frac{\exp\{-i\tau{\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha)\}}{-iT{\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha)}\Bigg|_0^T = \frac{1-\exp\{-iT{\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha)\}}{iT{\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha)}.$$

Hence, in this case,

$$\lim_{T\to\infty} g_T(\underline{k}) = 0.$$

This equality together with (2.4) shows that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } {\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha) = 0, \\ 0 & \text{if } {\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha) \neq 0. \end{cases}$$

$\square$

The function

$$g(\underline{k}) = \begin{cases} 1 & \text{if } {\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha) = 0, \\ 0 & \text{if } {\sum\limits_{m=0}^{\infty}}' k_m\log(m+\alpha) \neq 0 \end{cases} \tag{2.5}$$

is continuous in the discrete topology. Therefore, by Lemma 2.2, we obtain that Q_T, as $T \to \infty$, converges weakly to the measure Q defined by the Fourier transform $g(\underline{k})$.

Lemma 2.3 is valid for all α, however, the limit measure Q is not given explicitly. To have the explicit form of the measure Q, we must use a certain restriction for the parameter α, and this restriction is the linear independence of the set $L(\alpha)$.

Lemma 2.4. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then Q_T converges weakly to the Haar measure m_H.*

Proof. If the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, then

$$\sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) = 0$$

if and only if $k_m = 0$ for all m. Thus, in view of (2.5), the Fourier transform of the measure Q is of the form

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Since the latter function is the Fourier transform of the Haar measure m_H, the lemma follows from Lemmas 2.2 and 2.3. □

We continue with a limit theorem in the space of analytic functions. We note that, differently from [27], we deduce this theorem directly from Lemma 2.4, while in [27], first a limit theorem in the space of analytic functions is proved for a Dirichlet polynomial, and then for absolutely convergent Dirichlet series. Thus, we start with the definition of absolutely convergent Dirichlet series.

Let $\hat{\sigma} > \frac{1}{2}$ be a fixed number, and, for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$v_n(m,\alpha) = \exp\left\{ -\left(\frac{m+\alpha}{n+\alpha}\right)^{\hat{\sigma}} \right\}.$$

Define the series

$$L_n(\lambda,\alpha,s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)}{(m+\alpha)^s}.$$

Moreover, let, for $n \in \mathbb{N}$,

$$l_n(\alpha,s) = \frac{s}{\hat{\sigma}}\Gamma\left(\frac{s}{\hat{\sigma}}\right)\left(n+\alpha\right)^s.$$

Lemma 2.5. *The Dirichlet series for $L_n(\lambda, \alpha, s)$ converges absolutely for $\sigma > \frac{1}{2}$. Moreover, the integral representation*

$$L_n(\lambda, \alpha, s) = \frac{1}{2\pi i} \int\limits_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} L(\lambda, \alpha, s+z) l_n(\alpha, z) \frac{dz}{z}$$

holds.

Proof. First we observe that

$$v_n(m, \alpha) = \frac{1}{2\pi i} \int\limits_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} \frac{l_n(\alpha, z)}{z(m+\alpha)^z} dz. \tag{2.6}$$

Actually, applying the well-known formula

$$\frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} \Gamma(s) b^{-s} ds = e^{-b}, \quad a, b > 0,$$

we find

$$v_n(m, \alpha) = \frac{1}{2\pi i} \int\limits_{1-i\infty}^{1+i\infty} \Gamma(z) \Big(\frac{m+\alpha}{n+\alpha}\Big)^{-z\hat{\sigma}} dz =$$

$$\frac{1}{2\pi i} \int\limits_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} \Gamma\Big(\frac{z}{\hat{\sigma}}\Big) \Big(\frac{m+\alpha}{n+\alpha}\Big)^{-z} d\Big(\frac{z}{\hat{\sigma}}\Big) =$$

$$\frac{1}{2\pi i} \int\limits_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} \frac{z}{\hat{\sigma}} \Gamma\Big(\frac{z}{\hat{\sigma}}\Big) \frac{(n+\alpha)^z}{(m+\alpha)^z} \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} \frac{l_n(\alpha, z)}{z(m+\alpha)^z} dz.$$

Therefore, by (2.6) and the definition of $l_n(\alpha, z)$,

$$v_n(m, \alpha) \ll_n (m+\alpha)^{-\hat{\sigma}} \int\limits_{-\infty}^{+\infty} \Gamma(\hat{\sigma} + it) dt \ll_n (m+\alpha)^{-\hat{\sigma}} \tag{2.7}$$

because of the estimate $\Gamma(\sigma + it) \ll_n \exp\{-c|t|\}$, $c > 0$, uniform for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. From (2.7), it follows that

$$\frac{e^{2\pi i \lambda m} v_n(m, \alpha)}{(m+\alpha)^{\sigma}} \ll_n \frac{1}{(m+\alpha)^{\sigma+\hat{\sigma}}}.$$

Since $\hat{\sigma} > \frac{1}{2}$, hence, we have that the series

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)}{(m+\alpha)^s}$$

is absolutely convergent for $\sigma > \frac{1}{2}$.
The series

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^{s+z}}$$

is absolutely convergent for $\sigma > \frac{1}{2}$ and $Rez = \hat{\sigma}$. Therefore, using (2.6), we obtain that

$$L_n(\lambda,\alpha,s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)}{(m+\alpha)^s} =$$
$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} \frac{1}{2\pi i} \int_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} \frac{l_n(\alpha,z)}{z(m+\alpha)^z} dz =$$
$$\frac{1}{2\pi i} \int_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} \Big(\frac{l_n(\alpha,z)}{z} \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^{s+z}} \Big) dz =$$
$$\frac{1}{2\pi i} \int_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty} L(\lambda,\alpha,s+z) \frac{l_n(\alpha,z)}{z} dz,$$

and the lemma is proved. □

Additionally to $L_n(\lambda,\alpha,s)$, define

$$L_n(\lambda,\alpha,s,\omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} \omega(m) v_n(m,\alpha)}{(m+\alpha)^s}, \quad \omega \in \Omega.$$

Obviously, the latter series also converges absolutely for $\sigma > \frac{1}{2}$, because $|e^{2\pi i\lambda m}\omega(m)| = 1$.

In what follows, we will consider the measures defined by means of $L_n(\lambda,\alpha,s)$ and $L_n(\lambda,\alpha,s,\omega)$. Let, for $A \in \mathcal{B}(H(D))$ and $\hat{\omega} \in \Omega$,

$$P_{T,n}(A) = \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0,T] : L_n(\lambda,\alpha,s+i\tau) \in A \Big\}$$

and

$$\hat{P}_{T,n}(A) = \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0,T] : L_n(\lambda,\alpha,s+i\tau,\hat{\omega}) \in A \Big\}.$$

For the proof of the weak convergence for $P_{T,n}$ and $\hat{P}_{T,n}$, we will apply Lemma 2.4 and one property of weak convergence of probability measures involving certain mappings. For convenience, we recall some notions. Suppose that $\mathbb{X}_1$ and $\mathbb{X}_2$ are two spaces, and $u : \mathbb{X}_1 \to \mathbb{X}_2$. Then the mapping u is called $\Big(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2)\Big)$ - measurable if $u^{-1}A \subset \mathcal{B}(\mathbb{X}_1)$ for every $A \in \mathcal{B}(\mathbb{X}_2)$. If u is $\Big(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2)\Big)$ - measurable, then every probability measure P on the space $\Big(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1)\Big)$ define the unique probability measure Pu^{-1} on $\Big(\mathbb{X}_2, \mathcal{B}(\mathbb{X}_2)\Big)$, where

$$Pu^{-1}(A) = P(u^{-1}A)$$

for all $A \in \mathcal{B}(\mathbb{X}_2)$, and $u^{-1}A$ is the pre-image of the set A.

Lemma 2.6. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $\Big(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1)\Big)$, $u : \mathbb{X}_1 \to \mathbb{X}_2$ is a continuous mapping, and P_n converges weakly to P as $n \to \infty$. Then $P_n u^{-1}$ converges weakly to Pu^{-1} as $u \to \infty$.*

Proof. The lemma is proved in [6]. We only remark that every continuous mapping $u : \mathbb{X}_1 \to \mathbb{X}_2$ is $\Big(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2)\Big)$-measurable, thus $P_n u^{-1}$ and Pu^{-1} are correctly defined. □

Now, we state a lemma for $P_{T,n}$ and $\hat{P}_{T,n}$. Define the function $u_n : \Omega \to H(D)$ by the formula

$$u_n(\omega) = L_n(\lambda, \alpha, s, \omega), \quad \omega \in \Omega.$$

Since the series for $L_n(\lambda, \alpha, s, \omega)$ is absolutely convergent for $\sigma > \frac{1}{2}$, the function u_n is continuous. Define, on $\Big(H(D), \mathcal{B}(H(D))\Big)$, the measure $V_n \overset{def}{=} m_H u_n^{-1}$.

Lemma 2.7. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to the measure V_n as $T \to \infty$.*

Proof. The definitions of $L_n(\lambda, \alpha, s)$ and u_n show that

$$u_n\Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0)\Big) = L_n(\lambda, \alpha, s + i\tau).$$

Therefore, in view of the definition of $P_{T,n}$, we have that, for $A \in \mathcal{B}(H(D))$,

$$P_{T,n}(A) = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : ((m+\alpha)^{-i\tau}, m \in \mathbb{N}_0) \in u^{-1}A\Big\} = Q_T(u^{-1}A),$$

where Q_T is the measure of Lemma 2.4. In other words, the equality $P_{T,n} = Q_T u^{-1}$ is true. By Lemma 2.4, Q_T converges weakly to the Haar measure m_H as $T \to \infty$. Therefore, Lemma 2.7 implies that $P_{T,n}$ converges weakly to $m_H u^{-1} =$

V_n as $T \to \infty$.
It remains to prove the same for $\hat{P}_{T,n}$. For this, we use the mapping $\hat{u} : \Omega \to H(D)$ defined by the formula

$$\hat{u}_n(\omega) = L_n(\lambda, \alpha, s, \omega\hat{\omega}), \quad \omega \in \Omega.$$

Then, repeating the arguments used in the case of $P_{T,n}$, we obtain that $\hat{P}_{T,n}$ converges weakly to the measure $\hat{V}_n \overset{def}{=} m_H \hat{u}_n$ as $T \to \infty$. We must prove that $\hat{V}_n = V_n$. For this, we use an auxiliary mapping $u : \Omega \to \Omega$ defined by

$$u(\omega) = \omega\hat{\omega}, \quad \omega \in \Omega.$$

From these definitions, it follows that $\hat{u}_n = u_n(u)$. At this moment, we apply the invariance of the Haar measure m_H with respect to translations by points from Ω, and obtain that

$$m_H \hat{u}_n^{-1} = m_H(u_n(u))^{-1} = (m_H u^{-1}) u_n^{-1} = m_H u_n^{-1}.$$

Thus, $\hat{V}_n = V_n$, and the lemma is proved. □

The pass from the function $L_n(\lambda, \alpha, s)$ to the function $L(\lambda, \alpha, s)$ requires a certain approximate result. For this, we need a metric of the space $H(D)$.
It is known [8] that there exists a sequence $\left\{K_l : l \in \mathbb{N}\right\} \subset D$ of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then K lies in some K_l.
Let for $g_1, g_2 \in H(D)$,

$$\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ϱ is the metric of the space $H(D)$ that induces the topology of uniform convergence on compacta.

Lemma 2.8. *For all λ and α, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho(L(\lambda, \alpha, s + i\tau), L_n(\lambda, \alpha, s + i\tau)) d\tau = 0$$

is true.

Proof. From the definition of the metric ϱ, it follows that it is sufficient to prove the equality

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}|(L(\lambda,\alpha,s+i\tau)-L_n(\lambda,\alpha,s+i\tau))|d\tau=0 \quad (2.8)$$

for an arbitrary compact set $K\subset D$.

We use the integral representation of Lemma 2.5, i.e., for $\sigma>\frac{1}{2}$,

$$L_n(\lambda,\alpha,s)=\frac{1}{2\pi i}\int\limits_{\hat{\sigma}-i\infty}^{\hat{\sigma}+i\infty}(L(\lambda,\alpha,s+z)l_n(\alpha,z))\frac{dz}{z},\quad \hat{\sigma}>\frac{1}{2}. \quad (2.9)$$

Suppose that $K\subset D$ is a fixed compact set, $\varepsilon>0$ is such that $\frac{1}{2}+2\varepsilon\leqslant Rez\leqslant 1-\varepsilon$ for points $z\in K$. We take $\theta>0$. Then, by (2.9) and the residue theorem,

$$L_n(\lambda,\alpha,s)-L(\lambda,\alpha,s)=\frac{1}{2\pi i}\int\limits_{-\theta-i\infty}^{-\theta+i\infty}(L(\lambda,\alpha,s+z)l_n(\alpha,z))\frac{dz}{z}+R_n(s), \quad (2.10)$$

where

$$R_n(s)=\begin{cases}0 & \text{if } 0<\lambda<1,\\ \dfrac{l_n(\alpha,1-s)}{1-s} & \text{if } \lambda=1.\end{cases}$$

For convenience, denote the points of the set K by $s=\sigma+iv$, and take

$$\theta=\sigma-\varepsilon-\frac{1}{2}>0,\quad \hat{\sigma}_0=\frac{1}{2}+\varepsilon.$$

Then in view of (2.10),

$$|L_n(\lambda,\alpha,s)-L(\lambda,\alpha,s)|\leqslant$$
$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|L(\lambda,\alpha,s+i\tau-\theta+it)|\frac{|l_n(\alpha,-\theta+it)|}{|-\theta+it|}dt+|R_n(s+i\tau)|.$$

Now, in the latter integral, we replace $t+v$ by t. This gives the inequality

$$|L_n(\lambda,\alpha,s)-L(\lambda,\alpha,s)|\leqslant$$
$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|L(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+\tau))|\frac{|l_n(\alpha,\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}dt+|R_n(s+i\tau)|.$$

Then,

$$\frac{1}{T}\int\limits_0^T \sup_{s\in K}|(L(\lambda,\alpha,s+i\tau)-L_n(\lambda,\alpha,s+i\tau))|d\tau \leqslant I_1+I_2, \tag{2.11}$$

where

$$I_1=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big(\frac{1}{T}\int\limits_0^T\Big|L\Big(\lambda,\alpha,\frac{1}{2}+\varepsilon+i\Big(t+\tau\Big)\Big)\Big|\sup_{s\in K}\frac{|l_n(\alpha,\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}dt$$

and

$$I_2=\frac{1}{T}\int\limits_0^T\sup_{s\in K}|R_n(s+i\tau)|d\tau.$$

It is well known that, for the gamma-function, the estimate

$$\Gamma\Big(\sigma+it\Big)\ll\exp\{-c|t|\},\quad c>0,$$

holds uniformly in σ, $\sigma_1\leqslant\sigma\leqslant\sigma_2$, for all $\sigma_1<\sigma_2$. Therefore, for $s\in K$, by the definition of $l_n(\alpha,s)$, we obtain

$$\begin{gathered}\frac{|l_n(\alpha,\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}\ll\\ \frac{(n+\alpha)^{\frac{1}{2}+\varepsilon-\sigma}}{\hat{\sigma}}\Big|\Gamma\Big(\frac{\frac{1}{2}+\varepsilon-\sigma}{\hat{\sigma}}+\frac{i(t-v)}{\hat{\sigma}_0}\Big)\Big|\ll\\ (n+\alpha)^{-\varepsilon}\exp\Big\{\frac{c}{\hat{\sigma}_0}|t-v|\Big\}\ll_k\\ (n+\alpha)^{-\varepsilon}\exp\{-c|t|\}.\end{gathered} \tag{2.12}$$

For the Lerch zeta-function, for $\sigma>\frac{1}{2}$, the mean square estimate

$$\int\limits_0^T|L(\lambda,\alpha,\sigma+it)|^2dt\ll_\sigma T$$

is true. Therefore, an application of the Cauchy inequality gives

$$\frac{1}{T}\int\limits_0^T |L(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+\tau))|dt \ll$$

$$\Big(\frac{1}{T}\int\limits_0^T |L(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+\tau))|^2 dt\Big)^{\frac{1}{2}} \ll 1+|\tau|.$$

This and estimates (2.12) and (2.11) show that

$$I_1 \ll_k (n+\alpha)^{-\varepsilon}\int\limits_{-\infty}^{\infty}(1+|t|)\exp\{-c|t|\}dt \ll_k (n+\alpha)^{-\varepsilon}. \tag{2.13}$$

Similarly, using the definition of $l_n(\alpha,s)$, we find that, for $s\in K$,

$$R_n(s+i\tau) \ll \frac{|l_n(\alpha,1-s-i\tau)|}{|1-s-i\tau|} \ll_k (n+\alpha)^{1-\sigma}\exp\{-c|\tau|\}.$$

Hence,

$$I_2 \ll_k (n+\alpha)^{\frac{1}{2}-2\varepsilon}\frac{1}{T}\int\limits_0^T \exp|-c|\tau|d\tau \ll_k \frac{(n+\alpha)^{\frac{1}{2}-2\varepsilon}}{T}.$$

Therefore, in view of (2.13) and (2.11),

$$\frac{1}{T}\int\limits_0^T \sup_{s\in K}|(L(\lambda,\alpha,s+i\tau)-L_n(\lambda,\alpha,s+i\tau))|d\tau \ll_k (n+\alpha)^{-\varepsilon}+\frac{(n+\alpha)^{\frac{1}{2}-2\varepsilon}}{T}.$$

Taking $T\to\infty$, and then $n\to\infty$, we obtain that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T \sup_{s\in K}|(L(\lambda,\alpha,s+i\tau)-L_n(\lambda,\alpha,s+i\tau))|d\tau = 0.$$

The lemma is proved. □

We also need the analogue of Lemma 2.8 for the functions $L(\lambda,\alpha,s,\omega)$ and $L_n(\lambda,\alpha,s,\omega)$. This case is more complicated than that of Lemma 2.8 because we have not the mean square estimate for the function $L(\lambda,\alpha,s,\omega)$. To obtain that estimate, the ergodic theory on the torus Ω is applied. We consider the family

$\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ of transformation on Ω defined by

$$\varphi_{\tau,\alpha}(\omega) = a_{\tau,\alpha}\omega, \quad \omega \in \Omega,$$

where

$$a_{\tau,\alpha} = \Big\{(m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big\}.$$

Since the Haar measure m_H is invariant, we have that $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is the one-parameter group of measurable measure preserving transformations on the probability space $\Big(\Omega, \mathcal{B}(\Omega), m_H\Big)$. We recall the notion of ergodicity. A set $A \in \mathcal{B}(\Omega)$ is called invariant with respect to the group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ if the sets A and $A_\tau = \varphi_{\tau,\alpha}(A)$ can differ one from another by at most a set of zero m_H-measure. The group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is called ergodic if the σ-field of its invariant sets consists only from the sets of m_H-measure zero or one.

Lemma 2.9. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is ergodic.*

The proof of the lemma is given in [26] $(r = 1)$ and in [27].
Also, we will use the Birkhoff-Khintchine ergodic theorem. We recall that a strongly stationary process (all finite dimensional distributions are invariant with respect to translations) is called ergodic if its σ-field of invariant sets consists only of the sets of the measure, defined by finite-dimensional distributions, zero or one.

Lemma 2.10. *(Birkhoff-Khintchine theorem) Suppose that $X(t,\omega)$ is an ergodic process, $\mathbb{E}|X(t,\omega)| < \infty$, having the sample paths integrable almost surely in the Riemann sense over every finite interval. Then, for almost all ω, the equality*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,\omega)dt = \mathbb{E}X(0,\omega)$$

holds.

Proof of the lemma can be found in [9].
Now we are in position to prove the mean square estimate for the function $L(\lambda, \alpha, s, \omega)$.

Lemma 2.11. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $\sigma > \frac{1}{2}$. Then, for almost all $\omega \in \Omega$ with respect to the Haar measure m_H,*

$$\int_0^T |L(\lambda, \alpha, \sigma + it, \omega)|^2 dt \ll_{\alpha,\sigma} T.$$

Proof. From the definition of the random variables $\omega(m)$, their orthogonality follows [27], i.e.,

$$\int\limits_{\Omega} \omega(m)\overline{\omega(n)} dm_H = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Therefore,

$$\mathbb{E}\Big| \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^{\sigma}} \Big|^2 = \sum_{m=0}^{\infty} \frac{|e^{2\pi i\lambda m}|^2}{(m+\alpha)^{2\sigma}} = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} < \infty,$$

because $\sigma > \frac{1}{2}$. Moreover, by the definition of the transformation $\varphi_{\tau,\alpha}$,

$$\Big| \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\varphi_{\tau,\alpha}(\omega)}{(m+\alpha)^{\sigma}} \Big|^2 = \Big| \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^{\sigma+it}} \Big|^2 = \Big| L(\lambda, \alpha, \sigma + it, \omega) \Big|^2.$$

Lemma 2.9 implies the ergodicity of the random process $\Big| L(\lambda, \alpha, \sigma + it, \omega) \Big|^2$. Therefore, in view of Lemma 2.10 and (2.14), we have that, for almost all $\omega \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \Big| L(\lambda, \alpha, \sigma + it, \omega) \Big|^2 dt = \mathbb{E}\Big| \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^{\sigma}} \Big|^2 < \infty.$$

Thus,

$$\int\limits_0^T \Big| L(\lambda, \alpha, \sigma + it, \omega) \Big|^2 dt \ll T_{\alpha,\sigma}$$

for almost all $\omega \in \Omega$. □

Using Lemma 2.11 and repeating the proof of Lemma 2.8, we obtain the analogue of Lemma 2.8 for the functions $L(\lambda, \alpha, s, \omega)$ and $L_n(\lambda, \alpha, s, \omega)$.

Lemma 2.12. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, for all λ, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho\Big(L(\lambda, \alpha, s + i\tau, \omega), L_n(\lambda, \alpha, s + i\tau, \omega)\Big) d\tau = 0$$

holds for almost all $\omega \in \Omega$.

We see that, differently from Lemma 2.8, in Lemma 2.12 we need the linear independence over $\mathbb{Q}$ of the set $L(\alpha)$.

Lemmas 2.8 and 2.12 are important ingredients for the proof of Theorem 2.3. Before to do that, we recall some notions and auxiliary probabilistic results. The family of probability measures $\{P\}$ on $\Big(\mathbb{X}, \mathcal{B}(\mathbb{X})\Big)$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$, and the family $\{P\}$ is called relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence to a certain probability measure on $\Big(\mathbb{X}, \mathcal{B}(\mathbb{X})\Big)$.
The following statement proved by Prohkorov [50], see also [6], plays an important role in the theory of weak convergence of probability measures.

Lemma 2.13. *If the family of probability measures is tight, then it is relatively compact. If the space $\mathbb{X}$ is complete and separable, and the family is relatively compact, then it is tight.*

Denote by $\xrightarrow{\mathcal{D}}$ the convergence of random elements in distribution. Then the following statement is valid.

Lemma 2.14. *Suppose that $\big(\mathbb{X}, d\big)$ is a separable metric space, X_{kn} and Y_n, $n \in \mathbb{N}$, $k \in \mathbb{N}$, are $\mathbb{X}$-valued random elements defined on the same probability space with the measure P. Let, for every $k \in \mathbb{N}$,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

If, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\Big(d(X_{kn}, Y_n) \geqslant \varepsilon\Big) = 0,$$

then

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Proof of the lemma can be found in [6]. Theorem 4.2.
Together with P_T, we will consider

$$\hat{P}_T(A) = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : L(\lambda, \alpha, s + i\tau, \omega) \in A\Big\}, \quad A \in \mathcal{B}(H(D)).$$

The next lemma gives actually the weak convergence for P_T, however, the limit measure is not given explicitly.

Lemma 2.15. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the measures P_T and $\hat{P}_T$ both converge weakly to the same probability measure P on $\Big(H(D), \mathcal{B}(H(D))\Big)$ as $T \to \infty$.*

Proof. We will consider the sequence of probability measure $\{V_n : n \in \mathbb{N}\}$, where V_n is the limit measure in Lemma 2.7, and prove that this sequence is relatively compact. In virtue of Lemma 2.13, it is sufficient to prove the tightness of $\{V_n\}$. Let a random variable ξ be distributed uniformly in the interval $[0, 1]$, and defined on a probability space with the measure ν. Define

$$X_{T,n} = X_{T,n}(s) = L_n(\lambda, \alpha, s + iT\xi).$$

Moreover, let Y_n be the $H(D)$-valued random element having the distribution V_n, $n \in \mathbb{N}$. Then the assertion of Lemma 2.7 can be written in form

$$X_{T,n} \underset{T\to\infty}{\xrightarrow{\mathcal{D}}} Y_n. \tag{2.15}$$

Since the series for the function $L_n(\lambda, \alpha, s)$ is absolutely convergent for $\sigma > \frac{1}{2}$, in the latter half-plane

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T |L_n(\lambda, \alpha, \sigma + it)|^2 dt = \tag{2.16}$$
$$\sum_{m=0}^{\infty} \frac{v_n^2(m, \alpha)}{(m + \alpha)^{2\sigma}} \leqslant \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^{2\sigma}} \leqslant C_\sigma < \infty.$$

Let K_l, $l \in \mathbb{N}$, be compact sets from the definition of the metric ϱ. Then an application of the Cauchy integral formula implies the estimate

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \sup_{s\in K_l} |L_n(\lambda, \alpha, s + i\tau)| d\tau \ll \tag{2.17}$$
$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \Big|\frac{1}{2\pi} \int_{L_l} \frac{L_n(\lambda, \alpha, z + i\tau)}{(z - s)} dz\Big| d\tau,$$

where L_l is a closed contour lying in D and enclosing the set K_l. Then we deduce from (2.16) and (2.17) that

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \sup_{s\in K_l} |L_n(\lambda, \alpha, s + i\tau)| d\tau \leqslant R_{\alpha,l} < \infty. \tag{2.18}$$

We fix $\varepsilon > 0$, and define $M_l = M_{\alpha,l}(\varepsilon) = R_{\alpha,l}2^l\varepsilon^{-1}$. Then, we obtain by (2.18) that

$$\limsup_{T\to\infty} \nu\Big(\sup_{s\in K_l}\Big|X_{T,n}(s)| > M_l\Big) =$$
$$\limsup_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\sup_{s\in K_l}\Big|L_n(\lambda,\alpha,s+i\tau)\Big| > M_l\Big\} \leqslant$$
$$\limsup_{T\to\infty}\frac{1}{M_lT}\int_0^T\sup_{s\in K_l}\Big|L_n(\lambda,\alpha,s+i\tau)\Big|d\tau \leqslant \frac{\varepsilon}{2^l}$$

for all $l\in\mathbb{N}$ and $n\in\mathbb{N}$. Therefore, by the continuity of probability measures and (2.15), we find that

$$\nu\Big(\sup_{s\in K_l}\Big|Y_n(s)\Big| > M_l\Big) \leqslant \frac{\varepsilon}{2^l} \tag{2.19}$$

for all $l\in\mathbb{N}$ and $n\in\mathbb{N}$. Let

$$K = K(\varepsilon) = \Big\{g\in H(D):\sup_{s\in K_l}|g(s)| \leqslant M_{\alpha,l}(\varepsilon),\quad l\in\mathbb{N}\Big\}.$$

Since the set is uniformly bounded on compact sets of D, by the compactness principle, see, for example, [22], it is a compact set of the space $H(D)$. Moreover, by (2.19),

$$\nu\Big(Y_n\in K\Big) \geqslant 1-\varepsilon\sum_{l=1}^{\infty}2^{-l} = 1-\varepsilon$$

for all $n\in\mathbb{N}$. In other words,

$$V_n(K)\geqslant 1-\varepsilon$$

for all $n\in\mathbb{N}$, and this shows that the sequence $\Big\{V_n : n\in\mathbb{N}\Big\}$ is tight, thus, relatively compact.

By the relative compactness of $\Big\{V_n\Big\}$, there exists a subsequence $\Big\{V_{n_r}\Big\}$ such that V_{n_r} converges weakly to a certain probability measure P on $\Big(H(D),\mathcal{B}(H(D))\Big)$ as $r\to\infty$. This also can be written in the form

$$Y_{n_r}\xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{2.20}$$

Let

$$Z_T = Z_T(s) = L(\lambda,\alpha,s+iT\xi).$$

Then Lemma 2.8 shows that with every $\varepsilon > 0$

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \rho\Big(L(\lambda,\alpha,s+i\tau), L_n(\lambda,\alpha,s+i\tau)\Big) \geqslant \varepsilon\Big\} \leqslant$$
$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \rho\Big(L(\lambda,\alpha,s+i\tau), L_n(\lambda,\alpha,s+i\tau)\Big) d\tau = 0.$$

This, and relations (2.15) and (2.20) together with Lemma 2.14 lead to the relation

$$Z_T \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{2.21}$$

In view of the definition of Z_T, this means that P_T converges weakly to P as $T \to \infty$. Also, (2.21) shows that the measure P does not depend of the subsequence V_{n_r}. This remark together with relative compactness of $\Big\{V_n\Big\}$, gives the relation

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{2.22}$$

Now consider the measure $\hat{P}_T$. Analogically to the case of P_T, define, for $\omega \in \Omega$, two $H(D)$-valued random elements

$$\hat{X}_{T,n} = \hat{X}_{T,n}(s) = L_n(\lambda,\alpha,s+iT\xi,\omega)$$

and

$$\hat{Z}_{T,n} = \hat{Z}_{T,n}(s) = L(\lambda,\alpha,s+iT\xi,\omega).$$

Then, similarly as in the case of P_T, we obtain by using Lemma 2.12 and (2.22) that $\hat{P}_T$ also converges weakly to the measure P as $T \to \infty$. The lemma is completely proved. $\square$

Proof of Theorem 2.3. In view of Lemma 2.15, it remains to identify the limit measure P, i.e., to prove that P coincides with the measure P_L. For this, we recall the equivalent of weak convergence of probability measures in terms of continuity sets. A set $A \in \mathcal{B}(\mathbb{X})$ is a continuity set of the measure P on $\Big(\mathbb{X}, \mathcal{B}(\mathbb{X})\Big)$ if $P(\partial A) = 0$, where ∂A is the boundary of A.

Lemma 2.16. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $\Big(\mathbb{X}, \mathcal{B}(\mathbb{X})\Big)$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every continuity set A of P, the equality*

$$\lim_{n\to\infty} P_n = P(A)$$

holds.

The lemma is a part of Theorem 2.1 from [6].

Let A be a fixed continuity set of the limit measure P in Lemma 2.15. On the probability space $\Big(\Omega, \mathcal{B}(\Omega), m_H\Big)$, define the random variable X by the formula

$$X(\omega) = \begin{cases} 1 & \text{if } L(\lambda, \alpha, s, \omega) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemmas 2.15 and 2.16, we have that

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : L(\lambda, \alpha, s + i\tau, \omega) \in A\Big\} = P(A). \tag{2.23}$$

The definition of the random variable X implies the equality

$$\mathbb{E}X = \int_\Omega X dm_H = m_H\Big\{\omega \in \Omega : L(\lambda, \alpha, s, \omega) \in A\Big\} = P_L(A), \tag{2.24}$$

where P_L is the distribution of the random element $L(\lambda, \alpha, s, \omega)$.

Now, we return to ergodic theory. In view of Lemma 2.9, the random process $X\Big(\varphi_{\tau,\alpha}(\omega)\Big)$, $\tau \in \mathbb{R}$, is ergodic. Thus, by Lemma 2.10, we have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^\sigma X\Big(\varphi_{\tau,\alpha}(\omega)\Big) d\tau = \mathbb{E}X \tag{2.25}$$

for almost all $\omega \in \Omega$. Moreover, by the definitions of X and $\varphi_{\tau,\alpha}$,

$$\frac{1}{T} \int_0^T X\Big(\varphi_{\tau,\alpha}(\omega)\Big) d\tau = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : L(\lambda, \alpha, s, \omega) \in A\Big\}.$$

Therefore, this equality together with (2.24) and (2.25) yields

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : L(\lambda, \alpha, s, \omega) \in A\Big\} = P_L(A).$$

Hence, in virtue of (2.23), we have that $P(A) = P_L(A)$ for all continuity sets A of P. Since all continuity sets form a determining class [6], this shows that $P(A) = P_L(A)$ for all $A \in \mathcal{B}\Big(H(D)\Big)$, in other words, P coincides with P_L. The theorem is proved. □

2.3 Support

For the proof of universality theorems, we additionaly need the support of the measure P_L. Since the space $H(D)$ is separable, the support of P_L, by the definition, is a minimal closed set $S \subset H(D)$ such that $P_L(S) = 1$. The set S consists of all elements $g \in H(D)$ such that, for every open neighbourhood G of g, the inequality $P_L(G) > 0$ is true.

Theorem 2.4. *The support of the measure P_L is the whole of $H(D)$.*

Before the proof of Theorem 2.4, we present some auxiliary probabilistic and exponential function results. We recall that the support S_X of the random element is the support of the distribution of X.

Lemma 2.17. *Let $\Big\{X_n : n \in \mathbb{N}\Big\}$ be a sequence of independent $H(D)$-valued random elements such that the series*

$$\sum_{n=1}^{\infty} X_n$$

is almost surely convergent. Then the support of the sum of the latter series is the closure of the set of all $g \in H(D)$ which may be written as a convergent series

$$g = \sum_{n=1}^{\infty} g_n, \quad g \in S_{X_n}.$$

Proof of lemma is given in [22], Theorem 1.7.10.
Now, we recall the definition of an entire function of exponential type. Let $0 < \theta_0 \leqslant \pi$. A function $g(s)$ analytic in the closed region $|\arg s| \leqslant \theta_0$ is called of exponential type if

$$\limsup_{r \to \infty} \frac{|\log(g(re^{i\theta}))|}{r} < \infty$$

uniformly in θ, $|\theta| \leqslant \theta_0$.

Lemma 2.18. *Suppose that $g(s)$ is an entire function of exponential type and*

$$\limsup_{r \to \infty} \frac{\log|g(r)|}{r} > -1.$$

Then

$$\sum_p |g(\log p)| = \infty.$$

The lemma is Theorem 6.4.14 from [22].

Lemma 2.19. *Suppose that μ is a complex Borel measure on $\Big(\mathbb{C}, \mathcal{B}(\mathbb{C})\Big)$ with compact support contained in the half-plane $\sigma > \sigma_0$, and*

$$g(s) = \int_{\mathbb{C}} e^{sz} d\mu(z) \not\equiv 0.$$

Then
$$\limsup_{r\to\infty} \frac{\log|g(r)|}{r} > \sigma_0.$$

The lemma is Lemma 6.4.10 from [22].

The next lemma gives sufficient conditions for the denseness of some series in $H(D)$.

Lemma 2.20. *Let* $\{g_n : n \in \mathbb{N}\} \subset H(D)$ *satisfy the conditions:*
1. If μ *is a complex Borel measure on* $\Big(\mathbb{C}, \mathcal{B}(\mathbb{C})\Big)$ *with compact support in* D *such that*
$$\sum_{n=1}^{\infty} \Big| \int_{\mathbb{C}} g_n d\mu \Big| < \infty,$$
then
$$\int_{\mathbb{C}} s^r d\mu(s) = 0$$
for all $r \in \mathbb{N}_0$*;*
2. The series
$$\sum_{n=1}^{\infty} g_n$$
is convergent in $H(D)$*;*
3. For any compact set $K \subset D$*,*
$$\sum_{n=1}^{\infty} \sup_{s\in K} \big|g_n(s)\big|^2 < \infty.$$

Then the set of all convergent series
$$\sum_{n=1}^{\infty} a_n g_n$$
with $|a_n| = 1$*,* $n \in \mathbb{N}$*, is dense in* $H(D)$*.*

The lemma is Theorem 6.3.10 from [22].

Proof of Theorem 2.4. By the definition of Ω, we have the $\{\omega(m) : m \in \mathbb{N}_0\}$ is a sequence of independent random variables on the probability space $\Big(\Omega, \mathcal{B}(\Omega), m_H\Big)$. Therefore,
$$\Big\{\frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s} : m \in \mathbb{N}_0\Big\}$$
is a sequence of independent $H(D)$-valued random elements on the probability space $\Big(\Omega, \mathcal{B}(\Omega), m_H\Big)$. The support of each random element $\omega(m)$ is the unit

circle γ. Therefore, the support of the $H(D)$-valued random element

$$\frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}, \quad m \in \mathbb{N}_0,$$

is the set

$$\Big\{g \in H(D) : g(s) = \frac{e^{2\pi i\lambda m}a}{(m+\alpha)^s}, \quad |a| = 1\Big\}.$$

Thus, in view of Lemma 2.17, the support of the random element

$$L(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}$$

is the closure of all convergent series

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}a_m}{(m+\alpha)^s} \tag{2.26}$$

with $|a_m| = 1, \quad m \in \mathbb{N}_0$. Now, let μ be a complex Borel measure on $\Big(\mathbb{C}, \mathcal{B}(\mathbb{C})\Big)$ with compact support lying in D such that

$$\sum_{m=0}^{\infty} \Big| \int\limits_{\mathbb{C}} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} d\mu(s) \Big| < \infty, \tag{2.27}$$

For $z \in \mathbb{C}$, define

$$g(z) = \int\limits_{\mathbb{C}} e^{-sz} d\mu(s).$$

Then (2.27) is equivalent to

$$\sum_{m=0}^{\infty} \Big| g\Big(\log(m+\alpha)\Big) \Big| < \infty. \tag{2.28}$$

Let $\nu = \mu h^{-1}$, where the function $h : \mathbb{C} \to \mathbb{C}$ is defined by $h(s) = -s$. Then ν is again a complex measure on $\Big(\mathbb{C}, \mathcal{B}(\mathbb{C})\Big)$ with support lying in the strip $-1 < \sigma < -\frac{1}{2}$. Moreover,

$$g(z) = \int\limits_{\mathbb{C}} e^{sz} d\nu(s).$$

Since the function $g(z)$ is of exponential type, by Lemma 2.19 we have that if $g(z) \not\equiv 0$, then

$$\limsup_{r\to\infty} \frac{\log |g(r)|}{r} > 1.$$

Therefore, Lemma 2.19 shows that

$$\sum_p |g(\log p)| = \infty, \tag{2.29}$$

where p runs over prime numbers. Clearly,

$$\log(m+\alpha) = \log m\Big(1+\frac{\alpha}{m}\Big) = \log m + \log\Big(1+\frac{\alpha}{m}\Big) = \log m + O\Big(\frac{1}{m}\Big).$$

Thus, for $m \geqslant 2$,

$$g\Big(\log(m+\alpha)\Big) =$$
$$\int\limits_{\mathbb{C}} e^{-s\log m} d\mu(s) + \Big(\Big|\int\limits_{\mathbb{C}} e^{-s\log m} O\Big(\frac{1}{m}\Big) d\mu(s)\Big|\Big) = g\Big(\log m\Big) + O\Big(m^{-\frac{3}{2}}\Big)$$

because $\sigma > \frac{1}{2}$. This estimate together with (2.28) shows that

$$\sum_{m=2}^{\infty} \Big|g(\log m)\Big| < \infty.$$

However, this gives a contradiction to (2.29). Hence, we obtain that $g(z) \equiv 0$. Differentiating r times the equality $g(s) = \int\limits_{\mathbb{C}} e^{-sz} d\mu(s)$ in s and then taking $z = 0$, we find that

$$\int_{\mathbb{C}} s^r d\mu(s) = 0, \quad r \in \mathbb{N}_0.$$

This shows that condition 1 of Lemma 2.20 is fulfilled. Obviously, for every compact set $K \subset D$,

$$\sum_{m=0}^{\infty} \sup_{s\in K} \Big|\frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}\Big|^2 = \sum_{m=0}^{\infty} \sup_{s\in K} \frac{1}{(m+\alpha)^{2\sigma}} < \infty.$$

Moreover, we have mentioned below Lemma 2.1 that the series

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}$$

is almost surely convergent uniformly on compact subsets of D, i.e., the series is convergent in $H(D)$. Therefore, all conditions of Lemma 2.20 are fulfilled. Hence, the set of all convergent series (2.26) is dence in $H(D)$. Thus, the closure of that set coincides with $H(D)$. The theorem is proved because P_L is the distribution of the random element $L(\lambda, \alpha, s, \omega)$.

$\square$

We observe that, in the proof of Theorem 2.4, the linear independence of the $L(\alpha)$ is not used.

2.4 Proofs of universality theorems

The proofs of Theorems 2.1 and 2.2 are based on Theorems 2.3 and 2.4 as well as on the Mergelyan theorem on the approximation of analytic functions by polynomials. For convenience, we present it as the next lemma.

Lemma 2.21. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $f(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - g(s)| < \varepsilon.$$

Proof of the lemma can be found in [38]. Examples show that the conditions of the lemma can't be replaced by weaker ones. The set K can't be, for example, a ring, because the complement of a ring is not connected.
For the proof of Theorem 2.1, we also need the equivalent of weak convergence of probability measures in terms of open sets.

Lemma 2.22. *Suppose that $P_n, \quad n \in \mathbb{N}$, and P are probability measures on $\Big(\mathbb{X}, \mathcal{B}(\mathbb{X})\Big)$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every open set $G \subset \mathbb{X}$, the inequality*

$$\liminf_{n \to \infty} P_n(G) \geqslant P(G)$$

holds.

The lemma is a part of Theorem 2.1 from [6].

Proof of Theorem 2.1. Define the set

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \Big\},$$

where $p(s)$ is a polynomial such that

$$\sup_{s \in K} \Big| f(s) - p(s) \Big| < \frac{\varepsilon}{2}. \tag{2.30}$$

The existence of the polynomial $p(s)$ follows from Lemma 2.21. The set G_ε is open in the space $H(D)$. Therefore, by Theorem 2.3 and Lemma 2.2,

$$\liminf_{T\to\infty} P_T(G_\varepsilon) \geqslant P_L(G_\varepsilon). \tag{2.31}$$

In view of Theorem 2.4, the polynomial $p(s)$ is an element of the support of the measure P_L. Hence, G_ε is an open neighbourhood of an element of the support of P_L. Therefore, by properties of the support,

$$P_L(G_\varepsilon) > 0. \tag{2.32}$$

This, (2.31) and definitions of P_T and G_ε show that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - p(s)\right| < \frac{\varepsilon}{2}\right\} > 0. \tag{2.33}$$

It remains to replace $p(s)$ by $f(s)$ in the above inequality. If $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - p(s)\right| < \frac{\varepsilon}{2},$$

then, in virtue of (2.30), we find that

$$\sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - f(s)\right| \leqslant$$
$$\sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - p(s)\right| + \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - p(s)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that

$$\left\{\tau \in [0,T] : \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - p(s)\right| <\right.$$
$$\left.\frac{\varepsilon}{2}\right\} \subset \left\{\tau \in [0,T] : \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - f(s)\right| < \varepsilon\right\}.$$

From this and (2.33), the inequality of the theorem

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K}\left|L(\lambda,\alpha,s+i\tau) - f(s)\right| < \varepsilon\right\} > 0$$

follows. □

Proof of Theorem 2.2. Define the set

$$\hat{G}_\varepsilon = \left\{g \in H(D) : \sup_{s\in K}\left|g(s) - f(s)\right| < \varepsilon\right\}.$$

The boundary $\partial\hat{G}_\varepsilon$ of the set $\hat{G}_\varepsilon$ lies in the set

$$\Big\{g \in H(D) : \sup_{s\in K}\Big|g(s) - f(s)\Big| = \varepsilon\Big\}.$$

Therefore, the boundaries $\partial\hat{G}_{\varepsilon_1}$ and $\partial\hat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. From this remark, it follows that the set $\hat{G}_\varepsilon$ is a continuity set of the measure P_L for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 2.3 together with Lemma 2.16 shows that the limit

$$\lim_{T\to\infty} P_T(\hat{G}_\varepsilon) = P_L(\hat{G}_\varepsilon) \tag{2.34}$$

exists for all but at most countably many $\varepsilon > 0$. It remains to prove that $P_L(\hat{G}_\varepsilon) > 0$. Suppose that $g \in H(D)$ satisfies the inequality

$$\sup_{s\in K}\Big|g(s) - p(s)\Big| < \frac{\varepsilon}{2},$$

where the polynomial $p(s)$ is from (2.30). Then we find that, for this g,

$$\sup_{s\in K}\Big|g(s) - f(s)\Big| \leqslant \sup_{s\in K}\Big|g(s) - p(s)\Big| + \sup_{s\in K}\Big|g(s) - f(s)\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $G_\varepsilon \subset \hat{G}_\varepsilon$, Therefore, in view of (2.31), we have that $P_L(\hat{G}_\varepsilon) > 0$. Therefore, by (2.33), the limit

$$\lim_{T\to\infty} P_T(\hat{G}_\varepsilon) > 0$$

exists for all but at most countably many $\varepsilon > 0$. It remains to use the definition of P_T and $\hat{G}_\varepsilon$ to obtain that the limit

$$\lim_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau \in [0,T] : \sup_{s\in K}\Big|L(\lambda, \alpha, s + i\tau) - f(s)\Big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$. The theorem is proved. □

The results of Chapter 2 are published in [40].

Chapter 3

Discrete universality theorems for the Lerch zeta-function

In this chapter, we prove the discrete versions of Theorems 2.1 and 2.2. Thus, we will consider the approximation of analytic functions defined on the strip D by discrete shifts of the Lerch zeta-function $L(\lambda, \alpha, s + ikh), \quad k \in \mathbb{N}_0, \quad h > 0$. We will deal with the set

$$L(\alpha, h, \pi) = \Big\{ \Big(\log(m + \alpha) : m \in \mathbb{N}_0 \Big), \frac{2\pi}{h} \Big\}, \quad h > 0.$$

3.1 Statements of the discrete theorems

We recall that $\#A$ means the cardinality of the set $A \subset \mathbb{N}_0$,and N runs over the set $\mathbb{N}_0$.

In the chapter, we will prove the following discrete universality theorems.

Theorem 3.1. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N + 1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{s \in K} \Big| L(\lambda, \alpha, s + ikh) - f(s) \Big| < \varepsilon \Big\} > 0.$$

The inequality of the theorem shows that the set of discrete shifts $L(\lambda, \alpha, s + ikh)$ satisfying the inequality

$$\sup_{s \in K} \Big| L(\lambda, \alpha, s + ikh) - f(s) \Big| < \varepsilon \tag{3.1}$$

has a positive lover density in the set $\mathbb{N}_0$. From this, we have that the set of the above shifts is infinite.

Theorem 3.1 admits a modification in which the positivity of the lower density of the set of shifts $L(\lambda, \alpha, s+ikh)$ satisfying (3.1) is replaced by a density, however, with some exception for $\varepsilon > 0$.

Theorem 3.2. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ and $0 < \lambda \leqslant 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{s\in K} \Big| L(\lambda, \alpha, s+ikh) - f(s) \Big| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Proofs of Theorems 3.1 and 3.2 are based on probabilistic discrete limit theorems for probability measures in the space of analytic functions $H(D)$.
We observe that proofs of discrete theorems 3.1 and 3.2 in a certain sense are more complicated than those of Theorems 2.1 and 2.2.

3.2 A discrete limit theorem

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,h}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : L(\lambda, \alpha, s+ikh) \in A \Big\}.$$

In this section, we will study the weak convergence for $P_{N,h}$ as $N \to \infty$.
We preserve the notation of Section 2.2 for Ω and for the $H(D)$-valued random element $L(\lambda, \alpha, s, \omega)$. Moreover, P_L is the distribution of the random element $L(\lambda, \alpha, s, \omega)$.

The main result of this section is the following discrete limit theorem for $P_{N,h}$.

Theorem 3.3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$.Then $P_{N,h}$ converges weakly to P_L as $N \to \infty$.*

As in section 2.2, we start with a limit theorem for probability measures on the space $(\Omega, \mathcal{B}(\Omega))$. For $A \in \mathcal{B}(\Omega)$, define

$$Q_{N,h}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \Big((m+\alpha)^{-ikh} : m \in \mathbb{N}_0 \Big) \in A \Big\}.$$

Lemma 3.1. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $Q_{N,h}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We consider the Fourier transform $g_{N,h}(\underline{k})$ of $Q_{N,h}$. We have by (2.2) that

$$g_{N,h}(\underline{k}) = \int\limits_{\Omega} \Big(\prod_{m=0}^{\infty}{}' \omega^{k_m}(m) \Big) dQ_{N,h}.$$

Thus, by the definition of $Q_{N,h}$,

$$g_{N,h}(k) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{m=0}^{\infty}{}' (m+\alpha)^{-ikh} = \tag{3.2}$$
$$\frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ikh \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) \Big\}.$$

Suppose that $\underline{k} = \underline{0}$. Then

$$\sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) = 0,$$

and it is easily seen that

$$g_{N,h}(\underline{k}) = \frac{1}{N+1} \sum_{k=0}^{N} 1 = 1. \tag{3.3}$$

Now, let $\underline{k} \neq \underline{0}$. Then

$$\exp\Big\{ -ih \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) \neq 1. \tag{3.4}$$

Actually, if the latter inequality is not true, then

$$\exp\Big\{ -ih \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) = e^{2\pi il}$$

with some $l \in \mathbb{Z}$. Hence, we find that

$$\sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) - \frac{2\pi l_1}{h} = 0$$

with some $l_1 \in \mathbb{Z}$. However, this equality contradicts the linear independence of the set $L(\alpha, h, \pi)$ because not all $k_m = 0$. Thus, in the case $\underline{k} \neq \underline{0}$, inequality (3.4) is true. Therefore, the application of the formula for the sum of the geometric progression, in view of 3.2, yields

$$g_{N,h}(\underline{k}) = \frac{1 - \exp\Big\{ - ih(N+1) \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) \Big\}}{(N+1)\Big(1 - \exp\Big\{ - ih \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) \Big\}\Big)}.$$

This and (3.3) show that

$$\lim_{N\to\infty} g_{N,h}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

The right-hand side of the latter equality is the Fourier transform of the Haar measure m_H. Therefore, by a continuity theorem for probability measures on compact groups (Lemma 2.2), we find that $Q_{N,h}$ converges weakly to the Haar measure m_H as $N \to \infty$. We see that the limit measure is independent of h. □

As in Chapter 2, we proceed with a limit theorem for absolutely convergent Dirichlet series $L_n(\lambda, \alpha, s)$ and $L_n(\lambda, \alpha, s, \omega)$. For $A \in \mathcal{B}\Big(H(D)\Big)$ and $\hat{\omega} \in \Omega$, define

$$P_{N,n,h}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : L_n(\lambda, \alpha, s + ikh) \in A \Big\}.$$

Let the function $u_n : \Omega \to H(D)$ be the same as in Lemma 2.7, i.e.,

$$u_n(\omega) = L_n(\lambda, \alpha, s, \omega), \quad \omega \in \Omega.$$

Lemma 3.2. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n,h}$ converges weakly to the measure $V_n = m_H u_n^{-1}$ as $N \to \infty$.*

Proof. We have seen in Chapter 2 that the function u_n is continuous. Moreover, by the definitions of $L_n(\lambda, \alpha, s)$ and u_n, we have that

$$u_n\Big(\Big(m+\alpha\Big)^{-ikh} : m \in \mathbb{N}_0\Big) = L_n(\lambda, \alpha, s + ikh).$$

Therefore, we find that, for all $A \in \mathcal{B}(H(D))$,

$$P_{N,n,h}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : L_n(\lambda, \alpha, s + ikh) \in A \Big\} =$$
$$\frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \Big((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\Big) \in u_n^{-1} A \Big\} = Q_{N,h}(u_n^{-1} A),$$

where $Q_{N,h}$ is from Lemma 3.1. Thus, the equality $P_{N,n,h} = Q_{N,h} u_n^{-1}$ holds.

This, Lemmas 3.1 and 2.6 show that $P_{N,n,h}$ converges weakly to the measures $m_H u_n^{-1} \stackrel{\text{def}}{=} V_n$. $\square$

Next, we will approximate the function $L(\lambda, \alpha, s)$ by $L_n(\lambda, \alpha, s)$ in discrete sense. For this, we will use a discrete mean square estimate for the function $L(\lambda, \alpha, s)$. To obtain this estimate, we will apply the following Gallagher lemma connecting continuous and discrete mean squares of some functions.

Lemma 3.3. *Let $T_0,\quad T \geqslant \delta > 0$ be real numbers, and $\mathcal{T}$ be a finite non-empty set lying in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1.$$

Suppose that $S(x)$ is a complex-valued continuous function on $[T_0, T_0 + T]$ having a continuous derivative in $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta} \int\limits_{T_0}^{T_0+T} |S(t)|^2 dt + \Big(\int\limits_{T_0}^{T_0+T} |S(t)|^2 dt \int\limits_{T_0}^{T_0+T} |S'(t)|^2 dt \Big)^{\frac{1}{2}}.$$

Proof of the lemma can be found in [44, Lemma 1.4].

Lemma 3.4. *Suppose that $\frac{1}{2} < \sigma < 1$. Then, for all $\sigma, \lambda, \alpha, h > 0$ and $t \in \mathbb{R}$,*

$$\sum_{k=0}^{N} |L(\lambda, \alpha, \sigma + ikh + it)|^2 \ll N(1 + |t|),$$

where implied constant in $\ll$ depends on σ, λ, α and h.

Proof. For $\frac{1}{2} < \sigma < 1$, the estimates

$$\int_0^T |L(\lambda, \alpha, \sigma + i\tau)|^2 d\tau \ll_{\sigma,\lambda,\alpha} T$$

and

$$\int_0^T |L'(\lambda, \alpha, \sigma + i\tau)|^2 d\tau \ll_{\sigma,\lambda,\alpha} T$$

are valid. These estimates imply the bounds

$$\int_0^T |L(\lambda, \alpha, \sigma + i\tau + it)|^2 d\tau \ll_{\sigma,\lambda,\alpha} T(1 + |t|)$$

and

$$\int_0^T |L'(\lambda,\alpha,\sigma+i\tau+it)|^2 d\tau \ll_{\sigma,\lambda,\alpha} T(1+|t|).$$

Now, the application of Lemma 3.3 with $\delta = h$ and the above estimates, give

$$\sum_{k=0}^{N} |L(\lambda,\alpha,\sigma+ikh+it)|^2 \ll \frac{1}{h}\int\limits_0^{Nh} |L(\lambda,\alpha,\sigma+i\tau+it)|^2 d\tau +$$

$$\Big(\int\limits_0^{Nh} |L(\lambda,\alpha,\sigma+i\tau+it)|^2 d\tau \int\limits_0^{Nh} |L'(\lambda,\alpha,\sigma+i\tau+it)|^2 d\tau\Big)^{\frac{1}{2}} \ll N(1+|t|)$$

with constant in $\ll$ depending on σ, λ, α and h. □

The next lemma consider a discrete approximation in the mean of the functions $L(\lambda,\alpha,s)$ by $L_n(\lambda,\alpha,s)$. The lemma is a discrete version of Lemma 2.8.

Lemma 3.5. *For all λ, α and $h>0$, the equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho\Big(L(\lambda,\alpha,s+ikh), L_n(\lambda,\alpha,s+ikh)\Big)=0$$

holds.

Proof. As in the continuous case (Lemma 2.8), it suffices to show that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-L_n(\lambda,\alpha,s+ikh)\Big|=0 \quad (3.5)$$

for every compact subset $K\subset D$.
Thus, let $K \subset D$ be an arbitrary compact subset. We fix $\varepsilon>0$ such that $\frac{1}{2}+2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for all points $s=\sigma+iv\in K$, and take

$$\theta = \sigma-\varepsilon-\frac{1}{2} > 0.$$

Then, using (2.10), we obtain that, for all $s=\sigma+iv\in K$,

$$\Big|L_n(\lambda,\alpha,s+ikh)-L(\lambda,\alpha,s+ikh)\Big| \leqslant$$

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big|L(\lambda,\alpha,s+ikh-\theta+it)\Big|\frac{|l_n(\alpha,-\theta+it)|}{|-\theta+it|}+\Big|R_n(s+ikh)\Big|$$

in the notation used in the proof of Lemma 2.8. This, after a shift $t+v\to t$, yields the inequality

$$\Big|L_n(\lambda,\alpha,s+ikh)-L(\lambda,\alpha,s+ikh)\Big| \leqslant$$

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big|L(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+kh))\Big|\frac{|l_n(\alpha,\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}dt+\Big|R_n s+ikh\Big|.$$

Hence, we find that

$$\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-L+n(\lambda,\alpha,s+ikh)\Big| \leqslant S_1+S_2, \quad (3.6)$$

where

$$S_1=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big|\frac{1}{N+1}\sum_{k=0}^{N}\Big(L\Big(\lambda,\alpha,\frac{1}{2}+\varepsilon+i(t+kh)\Big)\Big)\Big|$$

$$\sup_{s\in K}\Big|\frac{|l_n(\alpha,\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|}dt$$

and

$$S_2=\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\Big|R_n(s+ikh)\Big|.$$

In the definition of the function $l_n(\alpha,s)$, the gamma-function $\Gamma(s)$ occurs. Therefore, applying the estimate

$$\Gamma(\sigma+it)\ll\exp\Big\{-c|t|\Big\},\quad c>0,$$

we find, as in Chapter 2, that for $s\in K$,

$$\frac{l_n(\alpha,\frac{1}{2}+\varepsilon-s+it)}{\frac{1}{2}+\varepsilon-s+it}\ll_K (n+\alpha)^{-\varepsilon}\exp\{-c|t|\}.$$

Now, this and Lemma 3.4 show that

$$S_1\ll_K (n+\alpha)^{-\varepsilon}\int\limits_{-\infty}^{\infty}(1+|t|)\exp\{-c|t|\}dt\ll_K (n+\alpha)^{-\varepsilon}. \quad (3.7)$$

Moreover, for $s \in K$,

$$R_n(s+ikh) \ll (n+\alpha)^{1-\sigma} \exp\Big\{-c|kh-v|\Big\} \ll_K$$
$$(n+\alpha)^{1-\sigma}\exp\{-ckh\}.$$

Therefore,

$$S_2 \ll_K (n+\alpha)^{\frac{1}{2}-2\varepsilon}\frac{1}{N}\Big(\sum_{0\leqslant k\leqslant \log N} + \sum_{k>\log N}\Big)\exp\{-ckh\} \ll_{K,h}$$
$$(n+\alpha)^{\frac{1}{2}-2\varepsilon}\Big(\frac{\log N}{N}+\frac{1}{N}\exp\{-ch\log N\} \ll_{k,h} (n+\alpha)^{\frac{1}{2}-2\varepsilon}\frac{\log N}{N}.$$

Thus, by (3.6) and (3.7),

$$\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh), L_n(\lambda,\alpha,s+ikh)\Big| \ll_{K,h}$$
$$(n+\alpha)^{-\varepsilon}+(n+\alpha)^{\frac{1}{2}-2\varepsilon}\frac{\log N}{N}.$$

From this, we obtain the equality (3.5). The lemma is proved. $\square$

Now, we are in position to prove Theorem 3.3.

Proof of Theorem 3.3. We will prove that the measure $P_{N,h}$, as $N \to \infty$, converges weakly to the limit measure P of the measure V_n (V_n is the limit measure in Lemma 3.2) as $n \to \infty$. We have seen in the proof of Lemma 2.15 that the sequence of probability measures $\Big\{V_n : n \in \mathbb{N}\Big\}$ is relatively compact. For this, the linear independence over $\mathbb{Q}$ for the set $L(\alpha)$ was applied. However, the linear independence of the set $L(\alpha, h, \pi)$ implies that for the set $L(\alpha)$. Hence, the set $\Big\{V_n : n \in \mathbb{N}\Big\}$ remains also relatively compact under hypothesis of the theorem. Therefore, there exists a subsequence. $\Big\{V_{n_r}\Big\} \subset \Big\{V_n\Big\}$ such that $\Big\{V_{n_r}\Big\}$ converges weakly to a certain probability measure P on $\Big(H(D), \mathcal{B}(H(D))\Big)$ as $r \to \infty$. The latter fact can be written also as

$$Y_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P, \tag{3.8}$$

where Y_n is the $H(D)$-valued random element having the distribution V_n.

Now, let $\xi_{N,h}$ be a random variable defined on a certain probability space with the measure μ, and having the distribution

$$\mu(\xi_N = kh) = \frac{1}{N+1}, \quad k = 0, 1, ..., N.$$

Define

$$X_{N,n,h} = X_{N,n,h}(s) = L_n(\lambda, \alpha, s + i\xi_{N,h})$$

Then, in view of Lemma 3.2, we have that

$$X_{N,n,h} \xrightarrow[N\to\infty]{\mathcal{D}} Y_n. \tag{3.9}$$

Define one more $H(D)$-valued random element

$$Z_{N,h} = Z_{N,h}(s) = L(\lambda, \alpha, s + i\xi_{N,h}).$$

Then, Lemma 3.5 and the definition of $\xi_{N,h}$ imply that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\Big(\varrho\Big(X_{N,n,h}, Z_{N,h}\Big) \geqslant \varepsilon\Big) =$$
$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N :$$
$$\varrho\Big(L(\lambda, \alpha, s + ikh), L_n(\lambda, \alpha, s + ikh)\Big) \geqslant \varepsilon\Big\} \leqslant$$
$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \Big|\varrho\Big(L(\lambda, \alpha, s + ikh), L_n(\lambda, \alpha, s + ikh)\Big)\Big| = 0.$$

This, (3.8) and (3.9) show that all conditions of Lemma 2.14 are satisfied. Therefore, we obtain that

$$Z_{N,h} \xrightarrow[N\to\infty]{\mathcal{D}} P. \tag{3.10}$$

The latter relation is equivalent to the weak convergence of $P_{N,h}$ to P as $N \to \infty$. Moreover, the relation (3.10) shows that the measure P is independent of the choice of the subsequence $\Big\{V_{n_r}\Big\}$. Since the sequence $\Big\{V_n\Big\}$ is relatively compact, we obtain from this that

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

Thus, we have that the measure $P_{N,h}$, as $N \to \infty$, converges weakly to the limit measure P of V_n, as $n \to \infty$. This observation allows us to identify the limit measure P. Actually, in the proof of Lemma 2.15 it was obtained that the measure P coincides with P_L, where P_L is the distribution of the random element

$$L(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}, \quad s \in D.$$

The theorem is proved. $\square$

3.3 Proofs of Theorem 3.1 and 3.2

Proofs of Theorems 3.1 and 3.2 are based on Theorem 3.3 and Lemma 2.21.

Proof of Theorem 3.1. By Theorem 2.4, the support of the measure P_L is the whole of $H(D)$. For the proof of this results, the linear independence of the set $L(\alpha)$ is applied. In the case of theorem 3.1, the set $L(\lambda, \alpha, \pi)$ is linearly independent over $\mathbb{Q}$. Clearly, the linear independence of the set $L(\alpha, h, \pi)$ implies that of $L(\alpha)$. Therefore, under hypothesis that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, the support of the measure P_L also is the whole of $H(D)$.

In view of Lemma 2.21, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.11}$$

Define the set

$$G_\varepsilon = \{g \in H(D) : \sup_{s\in K} |g(s) - p(s)|\} < \frac{\varepsilon}{2}.$$

We have that G_ε is an open set of the space $H(D)$. Therefore, Theorem 3.3 and Lemma 2.22 imply the inequality

$$\liminf_{N\to\infty} P_{N,h}(G_\varepsilon) \geqslant P_L(G_\varepsilon). \tag{3.12}$$

Clearly, the polynomial $p(s)$ is an element of $H(D)$. Therefore, by the above remark, $p(s)$ belongs to the support of the measure P_L. Hence, we have the inequality

$$P_L(G_\varepsilon) > 0. \tag{3.13}$$

This and (3.12) show that

$$\liminf_{N\to\infty} P_{N,h}(G_\varepsilon) > 0,$$

and, by the definitions of $P_{N,h}$ and G, we obtain

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} \left|L(\lambda, \alpha, s+ikh) - p(s)\right| < \frac{\varepsilon}{2}\right\} > 0. \tag{3.14}$$

Now, we will replace the polynomial $p(s)$ by $f(s)$ in (3.14). Let $k \in \mathbb{N}_0$ satisfy the inequality

$$\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-p(s)\Big|<\frac{\varepsilon}{2}.$$

Then, for these k, we find using (3.11) that

$$\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-f(s)\Big|\leqslant$$
$$\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-p(s)\Big|+\sup_{s\in K}\Big|f(s)-p(s)\Big|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence,

$$\Big\{0\leqslant k\leqslant N:\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-p(s)\Big|<\frac{\varepsilon}{2}\Big\}\subset$$
$$\Big\{0\leqslant k\leqslant N:\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-p(s)\Big|<\varepsilon\Big\}.$$

Therefore,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-p(s)\Big|<\frac{\varepsilon}{2}\Big\}\leqslant$$
$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-f(s)\Big|<\varepsilon\Big\}.$$

The latter inequality together with (3.13) show that

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}\Big|L(\lambda,\alpha,s+ikh)-f(s)\Big|<\varepsilon\Big\}>0.$$

The theorem is proved. □

Proof of Theorem 3.2. We preserve the notation of the set G_ε from the proof of Theorem 3.1, and, additionally, define a new set

$$\hat{G}_\varepsilon=\Big\{g\in H(D):\sup_{s\in K}|g(s)-f(s)|<\varepsilon\Big\}.$$

Then we have that the boundary $\partial\hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ belongs the set

$$\Big\{g\in H(D):\sup_{s\in K}|g(s)-f(s)|=\varepsilon\Big\}.$$

Hence, it follows that $\partial\hat{G}_{\varepsilon_1}\bigcap\partial\hat{G}_{\varepsilon_2}=\varnothing$ for $\varepsilon_1\neq\varepsilon_2$, $\varepsilon_1,\varepsilon_2>0$. From this, we obtain that the set $\hat{G}_\varepsilon$ is a continuity set of the measure P_L ($P_L(\partial\hat{G}_\varepsilon)=0$) for all but at most countably many $\varepsilon>0$. Therefore, using of Theorem 3.3 and Lemma

2.16 implies the existence of the limit

$$\lim_{N\to\infty} P_{N,h}(\hat{G}_\varepsilon) = P_L(\hat{G}_\varepsilon) \tag{3.15}$$

for all but at most countably many $\varepsilon > 0$. It remains to show the positivity of $P_L(\hat{G}_\varepsilon)$.

We take $g \in H(D)$ such that

$$\sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2},$$

where $p(s)$ is the polynomial from (3.11). Then, in view of (3.11), for these g,

$$\sup_{s\in K} |g(s) - f(s)| \leqslant \sup_{s\in K} |g(s) - p(s)| + \sup_{s\in K} |g(s) - f(s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Hence, in view of (3.13), $P_L(\hat{G}_\varepsilon) > 0$. This and (3.15) give that the limit

$$\lim_{N\to\infty} P_{N,h}(\hat{G}_\varepsilon) > 0$$

exists for all but at most countably many $\varepsilon > 0$. Using the definitions of $P_{N,h}$ and $\hat{G}_\varepsilon$ completes the proof of Theorem 3.2. $\square$

The results of Chapter 3 are published in [33].

Chapter 4

Joint continuous universality theorems for Lerch zeta-functions

For $j = 1, ..., r$, let α_j, $0 < \alpha_j \leqslant 1$, and λ_j, $0 < \lambda j \leqslant 1$, be fixed parameters, and let $L(\lambda_j, \alpha_j, s)$ be the corresponding Lerch zeta-function. This chapter is devoted to the simultaneous approximation of a given collection $(f_1(s), ..., f_r(s))$ of analytic functions by a collection of shifts $(L(\lambda_1, \alpha_1, s + i\tau), ..., L(\lambda_r, \alpha_r, s + i\tau)), \quad \tau \in \mathbb{R}$. The results of the chapter are multidimensional generalizations of the theorems obtained in Chapter 2.

4.1 Statements of the theorems

We recall that

$$L(\alpha_1, ..., \alpha_r) = \{(\log(m + \alpha_1) : m \in \mathbb{N}_0), ..., (\log(m + \alpha_r) : m \in \mathbb{N}_0)\}.$$

Thus, the set $L(\alpha_1, ..., \alpha_r)$ consists of all logarithms $\log(m + \alpha_j)$, $m \in \mathbb{N}_0, \quad j = 1, ..., r$.

Theorem 4.1. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| L(\lambda_j, \alpha_j, s + i\tau) - f_j(s) \right| < \varepsilon \right\} > 0.$$

Theorem 4.1 has the following modification.

Theorem 4.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$.*

For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then the limit

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left|L(\lambda_j, \alpha_j, s+i\tau) - f_j(s)\right| < \varepsilon\right\} > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 4.1 shows that the set of shifts $(L(\lambda_1, \alpha_1, s+i\tau), ..., L(\lambda_r, \alpha_r, s+i\tau))$ satisfying the inequality

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left|L(\lambda_j, \alpha_j, s+i\tau) - f_j(s)\right| < \varepsilon$$

has a positive lower density. Theorem 4.2 is stronger than theorem 4.1 because it shows that the above collection of shifts has a positive density, however, with an possible exception of „small“ set of values of positive ε.

Theorems 4.1 and 4.2 will be derived from a joint continuous limit theorem for probability measures in the multidimensional space of analytic functions.

4.2 A joint continuous limit theorem

In this section, we will prove a multidimensional generalization of Theorem 2.3. Let, as above, $H(D)$ be the space of analytic functions on D. Denote

$$H^r(D) = \underbrace{H(D) \times ... \times H(D)}_{r},$$

and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_T^r(A) = \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau) \in A\right\},$$

where

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, s) = (L(\lambda_1, \alpha_1, s), ..., L(\lambda_r, \alpha_r, s))$$

with $\underline{\lambda} = (\lambda_1, ..., \lambda_r)$ and $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$. For the statement of a limit theorem for P_T^r as $T \to \infty$, we need some definitions.
The torus Ω is the same as in Section 2.2. Define

$$\Omega^r = \Omega_1 \times ... \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, ..., r$. Since Ω is a compact topological group, by the Tikhonov theorem, we have that Ω^r is a compact topological group as well.

Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r exists, and this gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. Denote by $\omega_j(m)$ the m th component of an element $\omega_j \in \Omega_j$, $m \in \mathbb{N}_0$, $j = 1, ..., r$, and by $\omega = (\omega_1, ..., \omega_r)$ the elements of Ω^r. Now, on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $H^r(D)$-valued element

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega) = (L(\lambda_1, \alpha_1, s, \omega_1), ..., L(\lambda_r, \alpha_r, s, \omega_r)),$$

where

$$L(\lambda_j, \alpha_j, s, \omega_j m) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j m}{(m+\alpha)^s}, \quad j = 1, ..., r.$$

The main result of this section is the following functional limit theorem.

Theorem 4.3. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, P_T^r converges weakly to the distribution $P_{\underline{L}}^r$ of the random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega)$ as $T \to \infty$. Moreover, the support of the measure $P_{\underline{L}}^r$ is the whole of $H^r(D)$.*

We remind that

$$P_{\underline{L}}^r(A) = m_r^H \Big\{ \omega \in \Omega^r : \underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega) \in A \Big\}, \quad A \in \mathcal{B}(H^r(D)).$$

We divide the proof of Theorem 4.3 into lemmas.
We start with a limit theorem on Ω^r. For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_T^r(A) = \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0, T] :$$
$$(((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), ..., ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) \in A \Big\}.$$

Lemma 4.1. *Suppose, that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then Q_T^r converges weakly to the Haar measure m_H^r as $T \to \infty$.*

Proof. We consider the Fourier transform $g_{T,r}(\underline{k}_1, ..., \underline{k}_r)$, $\underline{k}_j = (k_{jm} : k_{jm} \in \mathbb{Z},\ \ m \in \mathbb{N}_0,\ \ j = 1, ..., r)$, of the measure Q_T^r. The dual group of Ω^r is isomorphic to

$$\bigoplus_{j=1}^{r} \bigoplus_{j=1}^{\infty} \mathbb{Z}_{jm},$$

where $\mathbb{Z}_{jm} = \mathbb{Z}$ for all $m \in \mathbb{N}_0$ and $j = 1, ..., r$. Therefore, the characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{m=0}^{\infty}{}' \omega^{k_{jm}}(m),$$

where the sign „ ' “ means that only a finite number of integers k_{mj} are distinct from zero. Hence, we have that

$$g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = \int_{\Omega^r} \Big(\prod_{j=1}^{r} {\prod_{m=0}^{\infty}}' \omega^{k_{jm}}(m) \Big) dQ_T^r.$$

Thus, the definition of Q_T^r shows that

$$\begin{gathered} g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = \\ \frac{1}{T} \int_0^T \Big(\prod_{j=1}^{r} {\prod_{m=0}^{\infty}}' (m+\alpha_j)^{-ik_{jm}\tau} \Big) d\tau = \\ \frac{1}{T} \int_0^T \exp\{-i\tau \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j)\} d\tau. \end{gathered} \tag{4.1}$$

Since the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$, we have that

$$\sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j) = 0$$

if and only if all $k_{jm} = 0$. Therefore, in view of (4.1),

$$g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = 1 \tag{4.2}$$

for $(\underline{k}_1, ..., \underline{k}_r) = (\underline{0}, ..., \underline{0})$. If $(\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0})$, then integrating in (4.1) gives

$$g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = \frac{1 - \exp\Big\{ -iT \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j) \Big\}}{iT \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j)}.$$

This and (4.2) show that

$$\lim_{T\to\infty} g_{T,r}(\underline{k}_1, ..., \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, ..., \underline{k}_r) = (\underline{0}, ..., \underline{0}), \\ 0 & \text{if } (\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r, Lemma 2.2 proves the lemma.

Lemma 4.1 implies a joint limit theorem in the space $H^r(D)$ for absolutely convergent Dirichlet series. For a fixed $\theta > \frac{1}{2}$, and $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, define

$$v_n(m, \alpha_j) = \exp\Big\{ -\Big(\frac{m+\alpha_j}{n+\alpha_j}\Big)^{\theta} \Big\}, \quad j = 1, ..., r,$$

and

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s) = \Big(L_n(\lambda_1, \alpha_1, s), ..., L_n(\lambda_r, \alpha_r, s)\Big),$$

where

$$L_n(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda jm} v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, ..., r.$$

Moreover, for $\omega = (\omega_1, ..., \omega_r) \in \Omega^r$, we put

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega) = \Big(L_n(\lambda_1, \alpha_1, s, \omega_1), ..., L_n(\lambda_r, \alpha_r, s, \omega_r)\Big),$$

where

$$L_n(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda_j m} v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, ..., r.$$

By Lemma 2.5, we have that the series for $L_n(\lambda_j, \alpha_j, s)$ and $L_n(\lambda_j, \alpha_j, s, \omega_j)$ are absolutely convergent for $\sigma > \frac{1}{2}$.
Now, for $A \in \mathcal{B}(H^r(D))$, define

$$P^r_{T,n}(A) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T] : \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\tau) \in A\},$$

and, for a fixed $\hat{\omega} = (\hat{\omega_1}, ..., \hat{\omega_r})$,

$$P^r_{T,n,\hat{\omega}}(A) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T] : \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\tau, \hat{\omega}) \in A\}.$$

□

Lemma 4.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure V^r_n such that both the measures $P^r_{T,n}$ and $P_{T,n,\hat{\omega}}$ converge weakly to V^r_n as $T \to \infty$.*

Proof. We apply the same arguments as in the proof of Lemma 2.7. Define the function $u^r_n : \Omega^r \to H^r(D)$ by the formula

$$u^r_n(\omega) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega), \quad \omega \in \Omega^r.$$

Since the series $L_n(\lambda_j, \alpha_j, s, \omega_j), \quad j = 1, ..., r$, are absolutely convergent for $\sigma > \frac{1}{2}$, we have that the function u^r_n is continuous. Moreover, by the definitions of $P_{T,n}^r$ and Q_T^r,

$$P_{T,n}^r(A) = \frac{1}{T}\operatorname{meas}\{\tau \in [0,T] :$$
$$(((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), ..., ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) \in (u_n^r)^{-1}A\} =$$
$$Q_T^r((u_n^r)^{-1}A)$$

for every $A \in \mathcal{B}(H^r(D))$. Thus, the equality $P_{T,n}^r = Q_T^r(u_n^r)^{-1}$ is true. This, the continuity of the function u_n^r, Lemmas 4.1 and 2.6 show that $P_{T,n}^r$ converges weakly to $V_n^r \stackrel{def}{=} m_H^r(u_n^r)^{-1}$ as $T \to \infty$. In the case of $P_{T,n,\hat{\omega}}^r$, we apply the same arguments as for $P_{T,n}^r$. Define the function $\hat{u}_n^r : \Omega^r \to H^r(D)$ by the formula

$$\hat{u}_n^r(\omega) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega\hat{\omega}), \quad \omega \in \Omega^r,$$

which is continuous as well. Therefore, similarly as above, we obtain that $P_{T,n,\hat{\omega}}^r$ converges weakly to the measure $\hat{V}_n^r \stackrel{def}{=} m_H^r(\hat{u}_n^r)^{-1}$ as $T \to \infty$. It remains to prove that $V_n^r = \hat{V}_n^r$. We have, by the definitions of u_n^r and $\hat{u}_n^r$, that, for all $\omega \in \Omega^r$, $\hat{u}_n^r(\omega) = u_n^r(u(\omega))$, where the function $u : \Omega^r \to \Omega^r$ is given by $u(\omega) = \omega\hat{\omega}$. Since the Haar measure m_H^r is invariant with respect to translations by point from Ω^r, we find that

$$\hat{V}_n^r = m_H^r(\hat{u}_n^r)^{-1} = m_H^r(\hat{u}_n^r u)^{-1} = (m_H^r u^{-1})(u_n^r)^{-1} = m_H^r(u_n^r)^{-1} = V_n^r,$$

i.e., $P_{T,n,\hat{\omega}}^r$ as $T \to \infty$, also converges weakly to the same probability measure V_n^r as $P_{T,n}^r$. Thus, we have that both the measures $P_{T,n}^r$ and $P_{T,n,\hat{\omega}}^r$ converges weakly to the measure $V_n^r \stackrel{def}{=} m_H^r u_n^{-1}$ as $T \to \infty$. □

The next step of the proof of Theorem 4.3 includes the approximation of $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ by $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$, and of $\underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega)$ by $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega)$. Let ϱ be the metric in $H(D)$ which is used in Lemma 2.8. For $\underline{g}_1 = (g_{11}, ..., g_{1r})$, $\underline{g}_2 = (g_{21}, ..., g_{2r}) \in H^r(D)$, define

$$\underline{\varrho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \varrho(g_{1j}, g_{2j}).$$

Then we have that $\underline{\varrho}$ is a metric in the space $H^r(D)$ inducing its product topology.

Lemma 4.3. *For all $\underline{\lambda}$ and $\underline{\alpha}$, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\varrho}(\underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s+i\tau))d\tau = 0$$

holds.

Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, for all λ and almost all $\omega \in \Omega^r$, the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\varrho}(\underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau, \omega), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s+i\tau, \omega))d\tau = 0$$

holds.

Proof. From the definition of the metric $\underline{\varrho}$, it follows that the equalities of the lemma are implied by the equalities

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho(L(\lambda_j, \alpha_j, s+i\tau), L_n(\lambda_j, \alpha_j, s+i\tau))d\tau = 0, \quad j = 1, ..., r,$$

and, for almost all ω_j,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho(L(\lambda_j, \alpha_j, s+i\tau, \omega_j), L_n(\lambda_j, \alpha_j, s+i\tau, \omega_j))d\tau = 0, \tag{4.3}$$

where $j = 1, ..., r$. However, the first equalities are contained in Lemma 2.8, while the second equalities follows from Lemma 2.12. Actually, the linear independence of the set $L(\alpha_1, ..., \alpha_r)$ over $\mathbb{Q}$ implies that of the sets $L(\alpha_j), \quad j = 1, ..., r$. Moreover, the Haar measure m_H^r is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j)), \quad j = 1, ..., r$. Thus, if (4.3) are true for $A_j \subset \Omega_j$, then we have that $m_{jH}(A_j) = 1, \quad j = 1, ..., r$. Hence, for $A = A_1 \times ... \times A_r$, it follows that

$$m_H^r(A) = m_{1H}(A_1) \times ... \times m_{rH}(A_r) = 1.$$

The lemma is proved. □

For $A \in \mathcal{B}(H^r(D))$ and $\omega \in \Omega^r$ satisfying the second part of Lemma 4.3, define

$$P_{T,\omega}^r(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau, \omega) \in A\}.$$

Then the following statement is true.

Lemma 4.4. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, on $\Big(H^r(D), \mathcal{B}(H^r(D))\Big)$, then exists a probability measure P^r such that both the measure P_T^r and $P_{T,\omega}^r$ converge weakly to P^r as $T \to \infty$.*

Proof. We start with the measure P_T^r. Let, as in the proof of Lemma 2.15, ξ be a random variable uniformly distributed in the interval $[0, 1]$, and defined on a certain probability space with the measure ν. Define the $H^r(D)$-valued random element

$$\underline{X}_{T,n}^r = \underline{X}_{T,n}^r(s) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\xi T).$$

Then, in view of Lemma 4.2, we have that

$$\underline{X}_{T,n}^r \xrightarrow[T\to\infty]{\mathcal{D}} Y_n^r, \tag{4.4}$$

where $Y_n^r = Y_n^r(s)$ is the $H^r(D)$-valued random element having the distribution V_n^r(V_n^r is the limit measure in Lemma 4.2).
Now, we will prove that the family of probability measures $\{V_n^r : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K^r = K^r(\varepsilon) \subset H^r(D)$ such that

$$V_n^r(K^r) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. For this, we will apply the properties of the marginal measures

$$V_{j,n}^r(A) = V_n^r(\underbrace{H(D) \times ... \times H(D)}_{j-1} \times A \times H(D) \times ... \times H(D)), \quad A \in \mathcal{B}(H(D)),$$

where $j = 1, ..., r$, of the measure V_n^r. Under hypotheses of the lemma (the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$), the sets $L(\alpha_1), ..., L(\alpha_r)$ are linearly independent over $\mathbb{Q}$. Therefore, by the proof of Lemma 2.15, $V_{j,n}^r$ converges weakly to a certain probability measure V_j^r as $n \to \infty$, $\quad j = 1, ..., r$. Hence, the family $\{V_{j,n}^r : n \in \mathbb{N}\}$ is relatively compact, $j = 1, ..., r$. The space $H(D)$ is complete and separable. Therefore, by the second part of Lemma 2.13 (the inverse Prokhorov theorem), the family $\{V_{j,n}^r : n \in \mathbb{N}\}$ is tight, $j = 1, ..., r$. Hence, for every $\varepsilon > 0$, there exists a compact set $K_j = K_j(\varepsilon) \subset H(D)$ such that

$$V_{j,n}^r(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, ..., r, \tag{4.5}$$

for all $n \in \mathbb{N}$. Let $K^r = K_1 \times ... \times K_r$. Then the set K^r is compact in the space

$H(D)$, and, in view of (4.5),

$$V_n^r(H^r(D)\setminus K^r) = V_n^r(\cup_{j=1}^r H(D)\setminus K_j)) \leqslant$$
$$\sum_{j=1}^r V_n^r(\underbrace{H(D)\times ...\times H(D)}_{j-1}\times A\times H(D)\times ...\times H(D)) =$$
$$\sum_{j=1}^r V_{j,n}^r(H(D)\setminus K_j) < \sum_{j=1}^r (1-(1-\frac{\varepsilon}{r})) = \frac{\varepsilon}{r}\sum_{j=1}^r = \varepsilon$$

for all $n\in\mathbb{N}$. Thus, the family of probability measures $\{V_n^r : n\in\mathbb{N}\}$ is tight. By the first part of Lemma 2.13 (the Prokhorov theorem), the family of probability measures $\{V_n^r : n\in\mathbb{N}\}$ is relatively compact. Therefore, every subsequence of $\{V_n^r\}$ contains a subsequence $\{V_{n_k}^r\}$ weakly convergent to a certain probability measure P^r on $(H^r(D),\mathcal{B}(H^r(D)))$ as $k\to\infty$. In other words, we have the relation

$$\underline{Y}_{n_k}^r \xrightarrow[k\to\infty]{\mathcal{D}} P^r. \tag{4.6}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}_T^r = \underline{X}_T^r(s) = \underline{L}(\underline{\lambda},\underline{\alpha},s+i\xi T).$$

Then, by the first part of Lemma 4.3, we obtain that, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\nu\Big(\underline{\varrho}(\underline{X}_{T,n}^r,\underline{X}_T^r)\geqslant\varepsilon\Big) =$$
$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T] : \underline{\varrho}(\underline{L}(\underline{\lambda},\underline{\alpha},s+i\tau),\underline{L}_n(\underline{\lambda},\underline{\alpha},s+i\tau))\geqslant\varepsilon\Big\}\leqslant$$
$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T\varepsilon}\int_0^T\underline{\varrho}(\underline{L}(\underline{\lambda},\underline{\alpha},s+i\tau),\underline{L}_n(\underline{\lambda},\underline{\alpha},s+i\tau))d\tau = 0.$$

This equality, and relations (4.4) and (4.6) show that all conditions of Lemma 2.14 are satisfied. Therefore, we have that

$$\underline{X}_T^r \xrightarrow[T\to\infty]{\mathcal{D}} P^r, \tag{4.7}$$

or, in other words, P_T^r converges weakly to the measure P as $T\to\infty$. Moreover, the relation (4.7) shows that the limit measure P does not depend of the subsequence $\{V_{n_k}^r\}$. This and the relative compactness of the family $\{V_n^r : n\in\mathbb{N}\}$ imply the relation

$$\underline{Y}_n^r \xrightarrow[n\to\infty]{\mathcal{D}} P^r. \tag{4.8}$$

It remains to prove that $P_{T,\omega}^r$ also converges weakly to P^r as $T\to\infty$. For this purpose, we define two $H^r(D)$-valued random elements

$$\underline{X}^r_{T,n,\omega} = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\xi T, \omega)$$

and

$$\underline{X}^r_{T,\omega} = \underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\xi T, \omega).$$

Then Lemma 4.2, the second part of Lemma 4.3, relation (4.8) and repeating of the above arguments for $\underline{X}^r_{T,n,\omega}$ and $\underline{X}^r_{T,\omega}$ show that $P^r_{T,\omega}$ also converges weakly to the measure P^r as $T \to \infty$. The lemma is proved. $\square$

For the identification of the limit measure P^r in the previous lemma, we will apply Lemma 2.10 (the Birckhoff-Khintchine ergodic theorem). For brevity, let, for $\tau \in \mathbb{R}$,

$$a^r_\tau = (((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), ..., ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0)).$$

Define the family of transformations $\{\varphi^r_\tau : \tau \in \mathbb{R}\}$ of Ω^r by

$$\varphi^r_\tau(\omega) = a^r_\tau \omega.$$

Then the family $\{\varphi^r_\tau : \tau \in \mathbb{R}\}$ is a group. Obviously, the transformations φ^r_τ are continuous, hence, they are measurable. Moreover, in virtue of the invariance of the Haar measure m^r_H, the transformations φ^r_τ are measure preserving. Thus, on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m^r_H)$, we have the group $\{\varphi^r_\tau : \tau \in \mathbb{R}\}$ of measurable measure preserving transformations.

Lemma 4.5. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the transformation group $\{\varphi^r_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof. The lemma is proved in [26], Lemma 10, by the Fourier transform method. We only observe that the linear independence of the set $L(\alpha_1, ..., \alpha_r)$ is used to show that if χ is a non-trivial character of the group Ω^r, then, for $(\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0})$,

$$\sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m + \alpha) \neq 0,$$

hence, there exists $\tau_0 \neq 0$ such that

$$\chi(a^r_{\tau_0}) = \exp\{-i\tau_0 \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m + \alpha)\} \neq 1.$$

Here the notation of Lemma 4.1 is used. $\square$

Proof of Theorem 4.3. We start with the identification of the measure P^r in Lemma 4.4. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the random variable

$$\eta(\omega) = \begin{cases} 1 & \text{if } \underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega) \in A, \\ 0 & \text{if } \underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega) \notin A, \end{cases}$$

where A is a fixed continuity set of the measure P^r. Then using of Lemmas 4.4 and 2.16 yields the equality

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau, \omega) \in A\right\} = P^r(A). \tag{4.9}$$

Lemma 4.5 implies the ergodicity of the random process $\eta(\varphi_\tau^r(\omega))$. Therefore, by Lemma 2.10,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \eta(\varphi_\tau^r(\omega)) d\tau = \mathbb{E}\eta \tag{4.10}$$

for almost all $\omega \in \Omega^r$. From the definition of the random variable η, it follows that

$$\mathbb{E}\eta = \int_{\Omega^r} \eta dm_H^r = m_H^r = \{\omega \in \Omega^r : \underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega) \in A\} = P_{\underline{L}}^r(A). \tag{4.11}$$

Moreover, by the definitions of η and φ_τ^r, we find that

$$\frac{1}{T} \int_0^T \eta(\varphi_\tau^r(\omega)) d\tau = \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau, \omega) \in A\right\}.$$

Therefore, (4.10) and (4.11) imply the equality

$$\lim_{T\to\infty} \operatorname{meas} \frac{1}{T}\left\{\tau \in [0,T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s+i\tau, \omega) \in A\right\} = P_{\underline{L}}^r(A).$$

The latter equality together with (4.9) shows that $P^r(A) = P_{\underline{L}}^r(A)$. Since A is an arbitrary continuity set of the measure P^r, we have that $P^r(A) = P_{\underline{L}}^r(A)$ for all continuity sets A of P^r. However, all continuity sets constitute the determining class. Hence, $P^r(A) = P_{\underline{L}}^r(A)$ for all $A \in \mathcal{B}(H^r(D))$, i.e., $P^r = P_{\underline{L}}^r$. The first part of the theorem is proved.
It remains to find the support of the measure $P_{\underline{L}}^r$. The space $H(D)$ is separable. Therefore, it is known [6] that

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times ... \times \mathcal{B}(H(D))}_{r}.$$

This shows that it is sufficient to consider the measure $P_{\underline{L}}^r$ on the sets of the form

$$A = A_1 \times ... \times A_r$$

with $A_1, ..., A_r \in \mathcal{B}(H(D))$. Moreover, the Haar measure m_H^r is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j)), \quad j = 1, ..., r$. By the definition of the measure P_L^r, we have that

$$P_L^r(A) = m_H^r\{\omega \in \Omega^r : \underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega) \in A\} = \tag{4.12}$$
$$\prod_{j=1}^{r} m_{jH}\{\omega_j \in \Omega_j : L(\lambda_j, \alpha_j, s, \omega_j) \in A_j\}.$$

Since the sets $L(\alpha_1), ..., L(\alpha_r)$ are linearly independent over $\mathbb{Q}$, by Theorem 2.4, the support of the measure

$$m_{jH}\{\omega_j \in \Omega_j : L(\lambda_j, \alpha_j, s, \omega_j) \in A_j\}, \quad j = 1, ..., r,$$

is the whole of $H(D)$. Obviously, $P_L^r(H^r(D)) = 1$. Moreover, if $A_j \in \mathcal{B}(H(D))$ and $A_j \neq H(D)$ for some $j = 1, ..., r$, then we have that

$$m_{jH}\{\omega_j \in \Omega_j : L(\lambda_j, \alpha_j, s, \omega_j) \in A_j\} < 1.$$

Therefore, in view of (4.12), we find that $P_{\underline{P}}^r(A) < 1$. Therefore, we obtain that $A_j = H(D)$ for all $j = 1, ..., r$, and the support of $P_{\underline{L}}^r$ is the whole of $H^r(D)$. $\square$

4.3 Proofs of joint universality theorems

Theorem 4.1 and 4.2 are consequences of Theorem 4.3, Lemma 2.21(the Mergelyan theorem), and properties of the weak convergence of probability measures.

Proof of Theorem 4.1. By Lemma 2.21, there exist polynomials $p_1(s), ..., p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{4.13}$$

Define the set

$$G_\varepsilon^r = \Big\{(g_1, ..., g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\varepsilon}{2}\Big\}.$$

Then, by the second part of Theorem 4.3, G_ε^r is an open neighbourhood of an element $(p_1(s), ..., p_r(s))$ of the support of the measure $P_{\underline{L}}^r$. Thus, by the properties of the support,

$$P_{\underline{L}}^r(G_\varepsilon^r) > 0. \tag{4.14}$$

By the first part of Theorem 4.3 and Lemma 2.22,

$$\liminf_{T\to\infty} P_T^r(G_\varepsilon^r) \geqslant P_{\underline{L}}^r(G_\varepsilon^r).$$

Therefore, the definitions of P_T^r and G_ε^r together with (4.14) give the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left|L(\lambda_j, \alpha_j, s+i\tau) - p_j(s)\right| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (4.13) gives the assertion of the theorem. □

Proof of Theorem 4.2. Define the set

$$\hat{G}_\varepsilon^r = \left\{(g_1, ..., g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\right\}.$$

The set $\hat{G}_\varepsilon^r$ is open in the space $H^r(D)$. Moreover, its boundary $\partial\hat{G}_\varepsilon^r$ lies in the set

$$\left\{(g_1, ..., g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon\right\}.$$

Therefore, $\partial\hat{G}_{\varepsilon_1}^r \cap \partial\hat{G}_{\varepsilon_2}^r = \varnothing$ for positive $\varepsilon_1 \neq \varepsilon_2$. Hence, we have that $P_{\underline{L}}^r(\partial\hat{G}_\varepsilon^r) > 0$ for at most countably many $\varepsilon > 0$. Therefore, by the first part of Theorem 4.3 and Lemma 2.16, we have that

$$\lim_{T\to\infty} P_T^r(\hat{G}_\varepsilon^r) = P_{\underline{L}}^r(\hat{G}_\varepsilon^r) \tag{4.15}$$

for all but at most countably many $\varepsilon > 0$. It remains to show that $P_{\underline{L}}^r(\hat{G}_\varepsilon^r) > 0$. Suppose that $(g_1, ..., g_r) \in G_\varepsilon^r$, where G_ε^r is the set defined in the proof of Theorem 4.1. Then, using (4.13), we find that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| \leqslant$$
$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - p_j(s)| + \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| <$$
$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This show that $(g_1, ..., g_r) \in \hat{G}_\varepsilon^r$. Hence, $G_\varepsilon^r \subset \hat{G}_\varepsilon^r$. Therefore, by monotonicity of P_L^r and (4.14), the inequality $P_{\underline{L}}^r(\hat{G}_\varepsilon^r) > 0$ is true. This together with (4.15) proves the theorem. □

Chapter 5

Joint discrete universality theorems for Lerch zeta-functions

Let, as in Chapter 4, $L(\lambda_1, \alpha_1, s), ..., L(\lambda_r, \alpha_r, s)$ be the Lerch zeta-functions. In this chapter, we will prove discrete versions of Theorems 4.1 and 4.2 that are joint generalizations of theorems obtained in Chapter 3.

5.1 Statements of the theorems

For $h > 0$, define the set

$$L(\alpha_1, ..., \alpha_r; h, \pi) = \{(\log(m+\alpha_1) : m \in \mathbb{N}_0), ..., (\log(m+\alpha_r) : m \in \mathbb{N}_0), \frac{2\pi}{h}\}.$$

Thus, all logarithms $\log(m + \alpha_j), \quad m \in \mathbb{N}_0, \quad j = 1, ..., r$, and the element $\dfrac{2\pi}{h}$ form the set
$L(\alpha_1, ..., \alpha_r; h, \pi)$, its elements are not necessarily different.

Theorem 5.1. *Suppose that the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, and $0 < \lambda_j \leqslant 1$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \Big| L(\lambda_j, \alpha_j, s+ikh) - f_j(s) \Big| < \varepsilon \Big\} > 0.$$

In the next theorem, „liminf“ is replaced by „lim“.

Theorem 5.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ is linearly independent over*

$\mathbb{Q}$. *For* $j = 1, ..., r$, *let* $K_j \in \mathcal{K}$, $f_j(s) \in H(K_j)$, *and* $0 < \lambda_j \leqslant 1$. *Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \Big| L(\lambda_j, \alpha_j, s+ikh) - f_j(s) \Big| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Thus, as in Chapter 4, we see that Theorem 5.2, is stronger that Theorem 5.1 because in Theorem 5.2, the set of shifts $(L(\lambda_1, \alpha_1, s+ikh), ..., L(\lambda_r, \alpha_r, s+ikh))$ approximating a given collection $(f_1(s), ..., f_r(s))$ of analytic functions has a positive density for „almost all“ $\varepsilon > 0$, while, in Theorem 5.1, this set has a positive lower density, however, for all $\varepsilon > 0$.
The proofs of Theorems 5.1 and 5.2 are based on joint statistical properties of Lerch zeta-functions, more precisely, on joint discrete limit theorems of weakly convergent probability measures in the r-dimensional space of analytic functions.

5.2 A joint discrete limit theorem

This section is devoted to a multidimensional generalization of Theorem 3.3.
For $A \in \mathcal{B}(H^r(D))$, define

$$P_N^r(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \underline{L}(\underline{\lambda}, \underline{\alpha}, s+ikh) \in A \Big\},$$

where $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ is the same as in Section 4.2. Also, we preserve the notation of Section 4.2 for the r-dimensional torus Ω^r and the $H^r(D)$-valued random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega)$.

Theorem 5.3. *Suppose that the set* $L(\alpha_1, ...\alpha_r; h, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Then* P_N^r *converges weakly to the distributions* $P_{\underline{L}}^r$ *of the random element* $\underline{L}(\underline{\lambda}, \underline{\alpha}, s, \omega)$ *as* $N \to \infty$.

We begin the proof of Theorem 5.3 with a discrete limit theorem on the torus Ω^r. For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_N^r(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \\ ((m+\alpha_1)^{-ikh} : m \in \mathbb{N}_0), ..., (m+\alpha_r)^{-ikh} : m \in \mathbb{N}_0)) \in A \Big\}.$$

In the next lemma, the linear independence of the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ plays the crucial role.

Lemma 5.1. *Suppose that the set* $L(\alpha_1, ..., \alpha_r; h, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Then* Q_N^r *converges weakly to the Haar measure* m_H^r *as* $N \to \infty$.

Proof. Let $g_{N,r}(\underline{k}_1, ..., \underline{k}_r)$, $\quad \underline{k}_j = (k_{jm} : k_{jm} \in \mathbb{Z}, m \in \mathbb{N}_0, j = 1, ..., r)$, be the Fourier transform of the measure Q_N^r. Then, as in Section 4.2, we have that

$$g_{N,r}\Big(\underline{k}_1, ..., \underline{k}_r\Big) = \int_{\Omega^r} \Big(\prod_{j=1}^{r} {\prod_{m=0}^{\infty}}' \omega_j^{k_{jm}}(m) \Big) dQ_N^r.$$

Therefore, by the definition of Q_N^r,

$$\begin{gathered} g_{N,r}(\underline{k}_1, ..., \underline{k}_r) = \\ \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} {\prod_{m=0}^{\infty}}' (m + \alpha_j)^{-ikhk_{jm}} = \\ \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ - ikh \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m + \alpha_j) \Big\}. \end{gathered} \tag{5.1}$$

Obviously,

$$g_{N,r}(\underline{0}, ..., \underline{0}) = 1. \tag{5.2}$$

Since the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$, we have that

$$\exp\Big\{ - ih \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m + \alpha_j) \Big\} \neq 1$$

for $(\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0})$. Actually, if the latter inequality is not true, then

$$\exp\Big\{ - ih \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m + \alpha_j) \Big\} = e^{2\pi i l}$$

with some $l \in \mathbb{R}$. Hence,

$$\sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m + \alpha_j) - \frac{2\pi l_1}{h} = 0$$

with some $l_1 \in \mathbb{Z}$, and this contradicts the linear independence of the set $L(\alpha_1, ..., \alpha_r; h, \pi)$. Thus, in the case $(\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0})$, we find by (5.1) using the formula for the sum of the geometrical progression that

$$g_N^r(\underline{k}_1, ..., \underline{k}_r) = \frac{1 - \exp\Big\{ - (N+1)ih \sum_{j=1}^{r} \sum_{m=0}^{\infty}{}' k_{jm} \log(m + \alpha_j)\Big\}}{\Big(N+1\Big)\Big(1 - \exp\Big\{ - ih \sum_{j=1}^{r} \sum_{m=0}^{\infty}{}' k_{jm} \log(m + \alpha_j)\Big\}\Big)}.$$

This together with (5.2) shows that

$$\lim_{N\to\infty} g_{N,r}(\underline{k}_1, ..., \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, ..., \underline{k}_r) = (\underline{0}, ..., \underline{0}), \\ 0 & \text{if } (\underline{k}_1, ..., \underline{k}_r) \neq (\underline{0}, ..., \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r, the lemma follows by Lemma 2.2. □

Lemma 5.1 leads to a joint discrete limit theorem for absolutely convergent Dirichlet series. We preserve the notation of Section 4.2 for $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$ and $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega)$. For $A \in \mathcal{B}(H^r(D))$, define

$$P_{N,n}^r(A) = \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + ikh) \in A\Big\},$$

Moreover, define the function $u_n^r : \Omega^r \to H^r(D)$ by the formula

$$u_n^r(\omega) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega), \quad \omega \in \Omega^r.$$

We have seen in Section 4.2 that the function u_n^r is continuous. Moreover, by the definitions of $P_{N,r}^r$ and Q_N^r, for all $A \in \mathbb{B}(H^r(D))$, the equality

$$P_{N,n}^r(A) = \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N :$$
$$\Big((m+\alpha_1)^{-ikh} : m \in \mathbb{N}_0\Big), ..., \Big((m+\alpha_r)^{-ikh} : m \in \mathbb{N}_0\Big) \in (u_n^r)^{-1}A\Big\} =$$
$$Q_N^r(u_n^r)^{-1}A).$$

holds. Thus, we obtain that $P_{N,n}^r = Q_N^r(u_n^r)^{-1}$. This equality, the continuity of u_n^r, and Lemmas 5.1 and 2.6 prove the following lemma.

Lemma 5.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n}^r$ converges weakly to the measure $V_n^r \stackrel{def}{=} m_H^r u_n^{-1}$ as $N \to \infty$.*

The next lemma is devoted to the approximation in the mean of $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ by $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$.

Lemma 5.3. *For all $\underline{\lambda}$, $\underline{\alpha}$ and $h > 0$, the equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\varrho}(\underline{L}(\underline{\lambda}, \underline{\alpha}, s+ikh), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s+ikh)) = 0$$

holds.

Proof. The definition of the metric $\underline{\varrho}$ shows that the equality of the lemma follows from the equalities

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\varrho}(L(\lambda_j, \alpha_j, s+ikh), L_n(\lambda_j, \alpha_j, s+ikh)) = 0,$$

$j = 1, ..., r$. However, these equalities are true in view of Lemma 3.5. This proves the lemma.

□

Now, we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. We observe that the linear independence over $\mathbb{Q}$ of the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ implies that of the set $L(\alpha_1, ..., \alpha_r)$. Therefore, by the proof of Lemma 4.4, we have that the family of probability measures $\{V_n^r : n \in \mathbb{N}\}$ (V_n^r is the limit measure in Lemma 5.2) is tight. Hence, in virtue of Lemma 2.13, the family $\{V_n : n \in \mathbb{N}\}$ is relatively compact. Therefore, there exists a subsequence $\{V_{n_k}^r\} \subset \{V_n^r\}$ such that $V_{n_k}^r$ converges weakly to a certain probability measure P_r on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. Hence, denoting by $Y_n^r = Y_n^r(s)$ the $H^r(D)$-valued random element having the distribution V_n^r, we obtain that

$$Y_{n_k}^r \xrightarrow[k\to\infty]{\mathcal{D}} P^r. \tag{5.3}$$

Now, on a certain probability space with measure ν, define a random variable ξ_N by the formula

$$\nu(\xi_N = kh) = \frac{1}{N+1}, \quad k = 0, 1, ..., N.$$

Next, define the $H^r(D)$-valued random element

$$\underline{X}_{N,n}^r = \underline{X}_{N,n}^r(s) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\xi_N).$$

Then, by Lemma 5.2,

$$\underline{X}_{N,n}^r \xrightarrow[N\to\infty]{\mathcal{D}} \underline{Y}_n^r. \tag{5.4}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}_N^r = \underline{X}_N^r(s) = \underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\xi_N).$$

Then, in view of Lemma 5.3, we have that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \nu(\underline{\varrho}(\underline{X}_{N,n}^r, X_N^r) \geqslant \varepsilon) = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\{0 \leqslant k \leqslant N :$$
$$\underline{\varrho}(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + ikh), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + ikh)) \geqslant \varepsilon\} \leqslant$$
$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \underline{\varrho}(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + ikh), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + ikh)) = 0.$$

This equality together with relations (5.3) and (5.4) shows that all hypotheses of Lemma 2.14 are satisfied. Therefore, we obtain the relation

$$\underline{X}_N^r \xrightarrow[N\to\infty]{\mathcal{D}} P^r. \tag{5.5}$$

Thus, we have that P_N^r converges weakly to P^r as $N \to \infty$. Moreover, the relation (5.5) shows that the measure P^r is independent of the choice of the subsequence $\{V_{n_k}^r\}$. Since the sequence $\{V_n^r\}$ is relatively compact, hence we obtain that

$$Y_n^r \xrightarrow[n\to\infty]{\mathcal{D}} P^r.$$

This means that V_n^r converges weakly to P^r as $n \to \infty$. The latter remark allows to identify easily the measure P^r. Actually, in Section 4.2, it was obtained that, under hypothesis that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$,

$$\frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\tau) \in A\Big\}, \quad A \in \mathcal{B}(H^r(D)),$$

also converges weakly to the limit measure P^r of V_n^r as $n \to \infty$, and that P^r coincides $P_{\underline{L}}^r$. Clearly, the linear independence of the set $L(\alpha_1, ..., \alpha_r; h, \pi)$ implies that of the set $L(\alpha_1, ..., \alpha_r)$. Therefore, P_N^r also converges weakly to $P_{\underline{L}}^r$ which is the limit measure of V_n^r. The theorem is proved. □

5.3 Proofs of universality

The proofs of Theorems 5.1 and 5.2 are analogical to those of Theorems 4.1 and 4.2.

Proof of Theorem 5.1. Using Lemma 2.21, we find polynomials $p_1(s), ..., p_r(s)$

such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-p_j(s)|<\frac{\varepsilon}{2}. \tag{5.6}$$

Now, define the set

$$G_\varepsilon^r=\Big\{(g_1,...,g_r)\in H^r(D):\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s)-p_j(s)|<\frac{\varepsilon}{2}\Big\}.$$

By Theorem 4.3, the support of the measure $P^r_{\underline{L}}$ is the whole of $H^r(D)$. Therefore, the set G_ε^r is an open neighbourhood of the element $(p_1(s),...,p_r(s))$ of the support of the measure $P^r_{\underline{L}}$. Thus,

$$P^r_{\underline{L}}(G_\varepsilon^r)>0. \tag{5.7}$$

Therefore, by Theorem 5.3 and Lemma 2.22,

$$\liminf_{N\to\infty}P_N^r(G_\varepsilon^r)\geqslant P^r_{\underline{L}}(G_\varepsilon^r)>0.$$

Hence, by the definitions of P_N^r and G_ε^r, we find that

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N: \tag{5.8}$$
$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-p_j(s)|<\frac{\varepsilon}{2}\Big\}>0.$$

Suppose that $k\in\mathbb{N}_0$ satisfies the inequality

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-p_j(s)|<\frac{\varepsilon}{2}.$$

Then, taking into account (5.6), for those k we find that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-f_j(s)|\leqslant$$
$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-p_j(s)|+\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s)-p_j(s)|<$$
$$\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This shows that

$$\{0\leqslant k\leqslant N:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-p_j(s)|<\frac{\varepsilon}{2}\}\subset$$
$$\{0\leqslant k\leqslant N:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-f_j(s)|<\varepsilon\}.$$

Therefore, (5.8) implies the inequality

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |L(\lambda_j, \alpha_j, s+ikh) - f_j(s)| < \varepsilon\Big\} > 0.$$

The theorem is proved. $\square$

Proof of Theorem 5.2. Consider the set

$$\hat{G}^r_\varepsilon = \{(g_1, ..., g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\}.$$

Clearly, $\hat{G}^r_\varepsilon$ is an open set in $H^r(D)$, thus $\hat{G}_\varepsilon \in \mathcal{B}(H^r(D))$. Since the boundary $\partial\hat{G}^r$ belongs to the set

$$\{(g_1, ..., g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon\},$$

we have that the boundaries $\partial\hat{G}^r_{\varepsilon_1}$ and $\partial\hat{G}^r_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Thus, $P^r_{\underline{L}}(\partial\hat{G}^r_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. Therefore, the set $\hat{G}^r_\varepsilon$ is a continuity set of the measure $P^r_{\underline{L}}$ for all but at most countably many $\varepsilon > 0$. Hence, in virtue of Theorem 5.3 and Lemma 2.16, Hence, by the definitions of P^r_N and G^r_ε, we find that

$$\lim_{N\to\infty} P^r_N(\hat{G}^r_\varepsilon) = P^r_{\underline{L}}(\hat{G}^r_\varepsilon) \tag{5.9}$$

for all but at most countably many $\varepsilon > 0$. It is not difficult to see that $G^r_\varepsilon \subset \hat{G}^r_\varepsilon$, where G^r_ε was used in the proof of Theorem 5.1. Actually, if $(g_1, ..., g_r) \in G^r_\varepsilon$, then, in virtue of (5.6), we obtain that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |g_j(s) - f_j(s)| \leqslant$$
$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |g_j(s) - p_j(s)| + \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(g_1, ..., g_r) \in \hat{G}^r_\varepsilon$. Now, the inclusion $G^r_\varepsilon \subset \hat{G}^r_\varepsilon$ and (5.7) show that $P^r_{\underline{L}}(\hat{G}_\varepsilon) > 0$. Therefore, (5.9) implies that the limit

$$\lim_{N\to\infty} P^r_N(\hat{G}^r_\varepsilon) > 0$$

exists for all but at most countably $\varepsilon > 0$. Taking into account the definitions of

P_N^r and $\hat{G}_\varepsilon^r$, hence we obtain that the limit

$$\lim_{T\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \Big|L(\lambda_j, \alpha_j, s+ikh) - f_j(s)\Big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The theorem is proved. $\square$

Chapter 6

Functional independence of the Lerch zeta-function

In this chapter, we will prove one of corollaries of the universality-theorems on the functional independence of the Lerch zeta-function.

6.1 Denseness lemmas

Define the mapping $u : \mathbb{R} \to \mathbb{C}^k$ by the formula

$$u(t) = (L(\lambda, \alpha, \sigma + it), L'(\lambda, \alpha, \sigma + it), ..., L^{(k-1)}(\lambda, \alpha, \sigma + it)),$$

where σ, $\frac{1}{2} < \sigma < 1$, is a fixed number.

Lemma 6.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. Then the image of u is everywhere dense in $\mathbb{C}^k$.*

Proof. We fix $\varepsilon > 0$, and take an arbitrary point

$$\underline{a} = (a_0, a_1, ..., a_{k-1}) \in \mathbb{C}^k.$$

We have to show that there exists $t \in \mathbb{R}$ such that

$$|u(t) - \underline{a}|_{\mathbb{C}^k} < \varepsilon,$$

where $|.|_{\mathbb{C}^k}$ is the distance in $\mathbb{C}^k$. For this, it suffices to obtain that

$$|L^{(j)}(\lambda, \alpha, \sigma + it) - a_j| < \frac{\varepsilon}{k}$$

for all $j = 0, 1, ..., k-1$. Consider the polynomial

$$p_k(s) = \sum_{l=0}^{k-1} \frac{a_l s^l}{l!}.$$

Clearly,

$$p_l^{(j)}(0) = a_j, \quad j = 0, ..., k-1. \tag{6.1}$$

We take $\hat{\sigma}$, $\frac{1}{2} < \hat{\sigma} < 1$. Let $K \in \mathcal{K}$ be such that $\hat{\sigma}$ is an interior point of K. Denote by δ the distance of $\hat{\sigma}$ from the boundary of K. In virtue of Theorem 2.1, there exists a sequence $\{\tau_n\} \subset \mathbb{R}$, $\lim_{n\to\infty} \tau_n$, such that

$$\sup_{s\in K} |L(\lambda, \alpha, s + i\tau_n) - p(s - \hat{\sigma})| < \frac{\varepsilon\delta^{k-1}}{2^{k-1}(k-1)!k}$$

Therefore, the application of the Cauchy integral formula and (6.1) shows that, for $j = 0, 1, ..., k-1$,

$$|L^j(\lambda, \alpha, \hat{\sigma} + i\tau_n) - a_j| = \frac{j!}{2\pi}\Big| \int_{|z-\hat{\sigma}|=\frac{\delta}{2}} \frac{L(\lambda, \alpha, z + i\tau_n) - p_k(z - \hat{\sigma})}{(z - \hat{\sigma}^{j+1})} dz \Big| < \frac{\varepsilon}{k}$$

and the lemma is proved. □

The next lemma is a multidimensional analogue of Lemma 6.1. Define the mapping $u : \mathbb{R} \to \mathbb{C}^{k_1+...+k_r}$ by the formula

$$\begin{aligned} u_r(t) = (&L(\lambda_1, \alpha_1, \sigma + it), L'(\lambda_1, \alpha_1, \sigma + it), ..., L^{(k_1-1)}(\lambda_1, \alpha_1, \sigma + it), ..., \\ &L(\lambda_r, \alpha_r, \sigma + it), L'(\lambda_r, \alpha_r, \sigma + it), ..., L^{(k_r-1)}(\lambda_r, \alpha_r, \sigma + it)), \end{aligned}$$

where σ, $\frac{1}{2} < \sigma < 1$, is a fixed number.

Lemma 6.2. *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda_j \leqslant 1$, $j = 1, ..., r$. Then the image of u_r is everywhere dense in $\mathbb{C}^{k_1+...+k_r}$.*

Proof. We fix $\varepsilon > 0$ and

$$\underline{a}_r = (a_{10}, a_{11}, ..., a_{1,k_1-1}, ..., a_{r0}, a_{r1}, ..., a_{r,k_r-1}) \in \mathbb{C}^{k_1+...+k_r}.$$

We will prove that there exists a sequence $\{\tau_n\} \subset \mathbb{R}$, $\lim_{n\to\infty} \tau_k = +\infty$, such that

$$|u_r(\tau_n - \underline{a}_r)|_{\mathbb{C}^{k_1+...+k_r}} < \varepsilon.$$

For this, we will show that there exists a sequence $\{\tau_n\}$ such that

$$\left|L^{(l_j)}(\lambda_j, \alpha_j, \sigma + i\tau_n) - a_j l_j\right| < \frac{\varepsilon}{k_1 + ... + k_r}$$

for $j = 1, ..., r$, $l_j = 0, 1, ..., k_j - 1$. Define the polynomials

$$p_{k_j}(s) = \sum_{l=0}^{k_j - 1} \frac{a_{jl} s^l}{l!}, \quad j = 1, ..., r.$$

Then we have that

$$p_{k_j}^{(l)}(0) = a_{jl}, \quad j = 1, ..., r, \quad l = 0, 1, ..., k_j - 1.$$

We fix $\hat{\sigma}$, $\frac{1}{2} < \hat{\sigma} < 1$, and take a compact set $K \in \mathcal{K}$ such that $\hat{\sigma}$ is an interior point of K. Let δ be the distance of $\hat{\sigma}$ from the boundary of K. Then Theorem 4.1 implies the existence of $\tau_n \to \infty$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K} \left|L(\lambda_j, \alpha_j, s + i\tau_n) - p_{l_j}(s - \sigma_0)\right| < \frac{\varepsilon \delta^{k-1}}{2^{k-1}(k-1)!(k_1 + ... + k_r)},$$

where $k = \max\limits_{1 \leqslant j \leqslant r} k_j$. Then, by the Cauchy integral formula,

$$\left|L^{(l_j)}(\lambda_j, \alpha_j, \hat{\sigma} + i\tau_n) - a_{jl}\right| =$$

$$\frac{l_j!}{2\pi}\left| \int\limits_{|z - \hat{\sigma}| = \frac{\delta}{2}} \frac{L(\lambda_j, \alpha_j, s + i\tau_n) - p_j(z - \hat{\sigma})}{(z - \hat{\sigma})^{l_j + 1}} dz \right| <$$

$$\frac{\varepsilon}{k_1 + ... + k_r.}$$

The lemma is proved. $\square$

6.2 Theorems on the functional independence

In this section, we will apply Lemmas 6.1 and 6.2 for the proof of the following theorems.

Theorem 6.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $0 < \lambda \leqslant 1$. For $j = 0, ..., n$, let $V_j : \mathbb{C}^k \to \mathbb{C}$ be a continuous function , and let*

$$\sum_{j=0}^{n} s^j V_j\Big(L(\lambda, \alpha, s), L'(\lambda, \alpha, s), ..., L^{(k-1)}(\lambda, \alpha, s)\Big) = 0$$

identically for s. Then $V_j \equiv 0$ *for* $j = 0, ..., n$.

Proof. Let $V : \mathbb{C}^k \to \mathbb{C}$ be a continuous function such that

$$V\Big(L(\lambda, \alpha, \sigma - it), L'(\lambda, \alpha, \sigma + it), ..., L^{(k-1)}(\lambda, \alpha, \sigma + it)\Big) \equiv 0. \qquad (6.2)$$

Then $V \equiv 0$. Actually, this follows easily from the continuity of V and Lemma 6.1. Suppose, on the contrary, that $V \not\equiv 0$. Then these exists $(s_0, s_1, ..., s_{k-1}) \in \mathbb{C}^k$ such that $V(s_0, s_1, ..., s_{k-1}) \neq 0$. By the continuity of V, these exists a bounded region $G \subset \mathbb{C}^k$ containing $(s_0, s_1, ..., s_{k-1})$ such that

$$|V(\underline{a})| > C > 0$$

for all points $\underline{a} \in G$, and in view of Lemma 6.1, we obtain the contradiction to (6.2).

Let $l \leqslant n$ be the greatest number such that

$$\sup_{\underline{a} \in G} |V_l(\underline{a})| \neq 0.$$

If $l = 0$, then the theorem follows by the above remark on the function V. If $l > 0$, then there exists a region $G_1 \subset G$ such that

$$\inf_{\underline{a} \in G_1} |V_l(\underline{a})| > C_1 > 0. \qquad (6.3)$$

By the proof of Lemma 6.1, we can find a sequence $\{t_m\}$, $\lim_{m\to\infty} t_m = +\infty$, such that

$$\Big(L(\lambda, \alpha, \sigma + it_m), L'(\lambda, \alpha, \sigma + it_m), ..., L^{(k-1)}(\lambda, \alpha, \sigma + it_m)\Big) \in G_1.$$

This together with (6.3) shows that

$$\lim_{n\to\infty} (\sigma + i_{tm})^l \Big| V_l(L(\lambda, \alpha, \sigma + it_m),$$
$$L'(\lambda, \alpha, \sigma + it_m), ..., L^{(k-1)}(\lambda, \alpha, \sigma + it_m))\Big| = +\infty.$$

This gives contradiction to the equality of the theorem. $\square$

Theorem 6.2. *Suppose that the set* $L(\alpha_1, ..., \alpha_r)$ *is linearly independent over* $\mathbb{Q}$*, and* $0 < \lambda_j \leqslant 1$. *Let* $V_j : \mathbb{C}^{k_1+...+k_r} \to \mathbb{C}$*,* $j = 0, ..., n$*, be a continuous function,* *and let the equality*

$$\sum_{j=0}^{n} s^j V_j\Big(L(\lambda_1, \alpha_1, s), L'(\lambda_1, \alpha_1, s), ..., L^{(k_1-1)}(\lambda_1, \alpha_1, s), ...,$$
$$L(\lambda_r, \alpha_r, s), L'(\lambda_r, \alpha_r, s), ..., L^{(k_r-1)}(\lambda_r, \alpha_r, s)\Big) = 0$$

holds identically for s. Then $V_j \equiv 0$ for $j = 0, ..., n$.

Proof. We apply similar arguments to those used in the proof of Theorem 1.
Let $V : \mathbb{C}^{n_1+...+n_r} \to \mathbb{C}$ be a continuous function. We will prove that if the equality

$$V\Big(L(\lambda_1, \alpha_1, s), L'(\lambda_1, \alpha_1, s), ..., L^{(k_1-1)}(\lambda_1, \alpha_1, s), ..., \tag{6.4}$$
$$L(\lambda_r, \alpha_r, s), L'(\lambda_r, \alpha_r, s), ..., L^{(k_r-1)}(\lambda_r, \alpha_r, s)\Big) = 0$$

is satisfied for all s, then $V \equiv 0$. On the contrary, suppose that these exists a point $\underline{a} \in \mathbb{C}^{k_1+...k_r}$ such that $V(\underline{a}) \neq 0$. From the continuity of V, there exists a bounded region $G \subset \mathbb{C}^k$ containing the point $\underline{a}$ such that

$$|V(\underline{b})| > C < 0 \tag{6.5}$$

for all points $\underline{b} \in G$. Then, in view of Lemma 6.2, these exists $t \in \mathbb{R}$ such that, for fixed σ, $\frac{1}{2} < \sigma < 1$,

$$\Big(L(\lambda_1, \alpha_1, \sigma + it), L'(\lambda_1, \alpha_1, \sigma + it)), ..., L^{(k_1-1)}(\lambda_1, \alpha_1, \sigma + it)), ...,$$
$$L(\lambda_r, \alpha_r, \sigma + it)), L'(\lambda_r, \alpha_r, \sigma + it)), ..., L^{(k_r-1)}(\lambda_r, \alpha_r, \sigma + it)\Big) \in G$$

and this together with (6.5) contradicts (6.4).

Without loss of generality, we suppose that $V_0 \not\equiv 0$. Then, by the above remark, there exists a bounded region $G \subset \mathbb{C}^{k_1+...+k_r}$ such that

$$|V_0(\underline{b})| > C > 0$$

for all $\underline{b} \in G$. Let j_0 be the greatest non-negative integer $\leqslant n$ such that

$$\sup_{\underline{b} \in G} |V_{j_0}(\underline{b})| \neq 0.$$

If $j_0 = 0$, then, by the above remark, the theorem is proved. Therefore, suppose that $j_0 > 0$. Then we find a region $\hat{G} \subset G$ such that

$$\inf_{\underline{b} \in \hat{G}} |V_{j_0}(\underline{b})| > C < 0. \tag{6.6}$$

However, by Lemma 6.2, there exists a sequence $\{t_m\} \subset \mathbb{R}$, $\lim_{m \to \infty} t_n = +\infty$ such that

$$\Big(L(\lambda_1, \alpha_1, \sigma + it_m), L'(\lambda_1, \alpha_1, \sigma + it_m)), ..., L^{(k_1-1)}(\lambda_1, \alpha_1, \sigma + it_m)), ...,$$
$$L(\lambda_r, \alpha_r, \sigma + it_m)), L'(\lambda_r, \alpha_r, \sigma + it_m)), ..., L^{(k_r-1)}(\lambda_r, \alpha_r, \sigma + it_m)\Big) \in \hat{G}$$

for fixed σ, $\frac{1}{2} < \sigma < 1$. From this and (6.6), we obtain that

$$|\sigma + it_m|^{j_0} \Big| V_{j_0}\Big(L(\lambda_1, \alpha_1, \sigma + it_m), L'(\lambda_1, \alpha_1, \sigma + it_m)), ...,$$
$$L^{(k_1-1)}(\lambda_1, \alpha_1, \sigma + it_m)), ..., L(\lambda_r, \alpha_r, \sigma + it_m)), L'(\lambda_r, \alpha_r, \sigma + it_m)), ...,$$
$$L^{(k_r-1)}(\lambda_r, \alpha_r, \sigma + it_m)\Big)\Big| \to +\infty$$

as $m \to \infty$. This contradicts the hypothesis of the theorem. The theorem is proved. $\square$

Conclusions

1. The Lerch zeta-function $L(\lambda, \alpha, s)$ with parameter α such that the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$ has a continuous universality property on the approximation of analytic functions by shifts $L(\lambda, \alpha, s+i\tau)$.

2. The Lerch zeta-function $L(\lambda, \alpha, s)$ with parameter α such that the set $\{(\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h}\}$ is linearly independent over $\mathbb{Q}$, for all λ, $0 < \lambda \leqslant 1$, has a discrete universality property on the approximation of analytic functions by shifts $L(\lambda, \alpha, s + ikh)$.

3. The Lerch zeta-functions $L(\lambda_1, \alpha_1, s), ..., L(\lambda_r, \alpha_r, s)$ with parameters $\alpha_1, ..., \alpha_r$ such that the set $\{\log(m + \alpha_j) : m \in \mathbb{N}_0, \quad k = 1, ..., r\}$ is linearly independent over $\mathbb{Q}$ for all $0 < \lambda_j \leqslant 1$, have a joint continuous universality property on the approximation of collections of analytic functions by shifts $(L(\lambda_1, \alpha_1, s + i\tau), ..., L(\lambda_r, \alpha_r, s + i\tau))$

4. The Lerch zeta-functions $L(\lambda_1, \alpha_1, s), ..., L(\lambda_r, \alpha_r, s)$ with parameters $\alpha_1, ..., \alpha_r$ such that the set $\{(\log(m+\alpha_1) : m \in \mathbb{N}_0), ..., (\log(m+\alpha_r) : m \in \mathbb{N}_0), \dfrac{2\pi}{h}\}$ is linearly independent over $\mathbb{Q}$, for all $0 < \lambda_j \leqslant 1$, have a joint discrete universality property on the simultaneous approximation of collections of analytic functions by shifts $(L(\lambda_1, \alpha_1, s + ikh), ..., L(\lambda_r, \alpha_r, s + ikh))$.

5. The Lerch zeta-function $L(\lambda, \alpha, s)$ with parameter α such that the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$ is functionally independent.

6. The Lerch zeta-functions $L(\lambda_1, \alpha_1, s), ..., L(\lambda_r, \alpha_r, s)$ with parameters $\alpha_1, ..., \alpha_r$ such that the set $\{(\log(m + \alpha_1) : m \in \mathbb{N}_0), ..., (\log(m + \alpha_r) : m \in \mathbb{N}_0)\}$ is linearly independent over $\mathbb{Q}$ are functionally independent.

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Notation

k, l, m, n, r	integer integers
p	prime number
$\mathbb{Z}$	set of all integer numbers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$s = \sigma + it, \sigma, t \in \mathbb{R}, i = \sqrt{-1}$	complex number
$\zeta(s)$	Rieman zeta-function defined for $\sigma > 1$, by the series $\zeta(s) = \sum\limits_{m=0}^{\infty} \frac{1}{m^s}$, and by analytic continuation elsewhere
$L(s, \chi)$	Dirichle L-function defined, for $\sigma > 1$, by the series $L(s, \chi) = \sum\limits_{m=0}^{\infty} \frac{\chi(m)}{m^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha)$	Hurwitz zeta-function defined, for $\sigma > 1$, by the series $\zeta(s, \alpha) = \sum\limits_{m=0}^{\infty} \dfrac{1}{(m+\alpha)^s}$, and by analytic continuation elsewhere
$L(\lambda, \alpha, s)$	Lerch zeta-function defined, for $\sigma > 1$, by the series $L(\lambda, \alpha, s) = \sum\limits_{m=0}^{\infty} \dfrac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$, and by analytic continuation elsewhere
$\mathbb{E}X$	expectation of a random element X
$X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$	convergence in distribution
$\operatorname{meas} A$	Lebesgue measure of a set A
$\#A$	cardinality of a set A
$f(x) \ll_\lambda g(x), g(x) > 0, x \in X$	there exists a constant $C(\lambda) > 0$ such that, for all $x \in X$, $\lvert f(x)\rvert \leqslant Cg(x)$

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