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A transformation formula related to Dirichlet L-functions with principal character

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Abstract. We prove a transformation formula for the function for the exponential sum involving the divisor function. This formula can be applied to obtain meromorphic continuation for the Mellin transform of the square of Dirichlet L-function with principal character.

Keywords: Dirichlet L-function, Estermann zeta-function, Lerch zeta-function.

Introduction

Let $s = \sigma + it$ be a complex variable, χ_0 be the principal character modulo q, $q > 1$, and let

$$L(s, \chi_0) = \sum_{m=1}^{\infty} \frac{\chi_0(m)}{m^s}, \quad \sigma > 1$$

denote the corresponding Dirichlet L-function. It is well known that the function $L(s, \chi_0)$ has meromorphic continuation to the whole complex plane with unique simple pole at $s = 1$ with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right) = \frac{\varphi(q)}{q},$$

where p denotes a prime number, and $\varphi(q)$ is the Euler totient function.

For the investigation of the mean square

$$\int_0^T \left| L\left(\frac{1}{2} + it, \chi_0\right)\right|^2 dt,$$

the modified Mellin transform

$$\mathcal{Z}_1(s, \chi_0) = \int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi_0\right)\right|^2 x^{-s}\, dx, \quad \sigma > 1,$$

is needed. Meromorphic continuation of $\mathcal{Z}_1(s, \chi_0)$ requires a certain transformation formula. Similar transformation formulae are also used in the case of the Riemann zeta-function [6, 5]. Let

$$d(m) = \sum_{d|m} 1$$

denote the divisor function, γ be the Euler constant and

$$\Phi(z)=\sum_{m=1}^{\infty} d(m)e^{-mz}-\frac{\gamma-\log z}{z}.$$

Then in [6] and [5], the formulae for $\Phi(z^{-1})$ where obtained. Let $\frac{1}{2}<\varrho<1$,

$$\sigma_a=\sum_{d|m} d^a,$$

and

$$\Phi_\varrho(z)=\sum_{m=1}^{\infty}\frac{\sigma_{2\varrho-1}(m)}{m^{2\varrho-1}}e^{-zm}-\Gamma(2-2\rho)\zeta(2-2\rho)z^{2\rho-2}-\zeta(2\rho)z^{-1},$$

where $\Gamma(z)$ denotes the Euler gamma-function, and $\zeta(s)$ is the Riemann zeta-function. Then in [4], a formula for $\Phi_\varrho(z^{-1})$ has been obtained.

Let k and l be coprime positive integers, $z\neq 0$ and

$$\Phi\left(z,\frac{k}{l}\right)=\sum_{m=1}^{\infty} d(m)e^{2\pi i\frac{km}{l}}e^{-mz}-\frac{\gamma-2\log l-\log z}{lz}.$$

The aim of this note is to obtain a formula for $\Phi(z^{-1},\frac{k}{l})$.

Let c_0' and c_0'' be the constant terms in [2] for $E(s,\frac{\overline{k}}{l},0)$ and $E(s,-\frac{\overline{k}}{l},0)$ respectively. Moreover, denote

$$\delta=\begin{cases}1 & \text{if } \operatorname{Im} z>0,\\ -1 & \text{if } \operatorname{Im} z<0,\end{cases}$$

and, for $1<b<2$, define

$$I(z)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{l}\right)^{1-2\omega}\Gamma(\omega)\left(\bigl(\sin(\pi\omega)\bigr)^{-1}E\left(\omega,\frac{\overline{k}}{l},0\right)\right.$$
$$\left.+\bigl(\cot(\pi\omega)+\delta i\bigr)E\left(\omega,-\frac{\overline{k}}{l},0\right)\right)z^{1-\omega}\,d\omega.$$

Theorem 1. *Supose that* $\operatorname{Re} z>0$ *and* $\operatorname{Im} z\neq 0$*. Then*

$$\Phi\left(z^{-1},\frac{k}{l}\right)=-\frac{2\pi i\delta z}{l}\sum_{m=1}^{\infty} d(m)e^{2\pi i\frac{km}{l}}e^{-\frac{4\pi^2 mz}{l^2}}+\frac{l}{2\pi^2}(c_0'-c_0'')+\frac{1}{4}+I(z).$$

1 Estermann zeta-function

Let $l>1$ and $(k,l)=1$. The Estermann zeta-function $E(s,\frac{k}{l},\alpha)$, for $\sigma>\max(1+\operatorname{Re}\alpha,1)$, is defined by the series

$$E\left(s,\frac{k}{l},\alpha\right)=\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)}{m^s}\exp\left\{2\pi i m\frac{k}{l}\right\}.$$

For $\lambda \in \mathbb{R}$ and $0 < \beta \leqslant 1$, denote by $L(\lambda, \beta, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i m\lambda}}{(m+\beta)^s}$. It is well known, see, for example [3], $L(\lambda, \beta, s)$, for $\lambda \notin \mathbb{Z}$, is analytically continuable to an entire function, while for $\lambda \in \mathbb{Z}$, the function $L(\lambda, \beta, s)$ becomes the Hurwitz zeta-function which is meromorphically continuable to the whole complex plane where it has a simple pole at $s = 1$ with residue 1.

It is not difficult to see that, for $\sigma > \max(\operatorname{Re}\alpha + 1, 1)$

$$E\left(s, \frac{k}{l}, \alpha\right) = l^{\alpha-s} \sum_{v=1}^{l} \exp\left\{2\pi i \frac{vk}{l}\right\} L\left(1, \frac{v}{l}, s-\alpha\right) L\left(\frac{v}{l}, 1, s\right). \tag{1}$$

This equality show that the function $E(s, \frac{k}{l}, \alpha)$ is analytic in the whole complex plane, except for two simple poles at $s = 1$ and $s = 1 + \alpha$ if $\alpha \neq 0$, and a double pole $s = 1$ if $\alpha = 0$.

Now let $\overline{k}$ is connected to k by the congruence $k\overline{k} \equiv 1 \pmod{l}$. Then equality (1) together with the functional equation for the Lerch zeta-function, see, for example [3], leads to the functional equation for $E(s, \frac{k}{l}, \alpha)$

$$\begin{aligned} E\left(s, \frac{k}{l}, \alpha\right) = \frac{1}{\pi}\left(\frac{2\pi}{l}\right)^{2s-1-\alpha} & \Gamma(1-s)\Gamma(1+\alpha-s) \\ \times \Bigg(& \cos\frac{\pi\alpha}{2} E\left(1+\alpha-s, \frac{\overline{k}}{l}, \alpha\right) \\ & - \cos\left(\pi s - \frac{\pi\alpha}{2}\right) E\left(1+\alpha-s, -\frac{\overline{k}}{l}, \alpha\right)\Bigg). \end{aligned} \tag{2}$$

The function $E(s, \frac{k}{l}, \alpha)$, for $\alpha = 0$, was introduced by T. Estermann in [1] for needs of the representation of numbers as a sum of two products. In [2], the extension for $\alpha \in [-1, 0]$ was given.

The series of definition of $\Phi(z, \frac{k}{l})$ contains the product $d(m)\exp\{2\pi i \frac{k}{l}\}$ which are coefficients of the Dirichlet series for $E(s, \frac{k}{l}, 0)$. Therefore, the function $E(s, \frac{k}{l}, 0)$ involved in the formula for $\Phi(z^{-1}, \frac{k}{l})$.

2 Proof of the transformation formula

We start with the well-known Mellin formula

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z) y^{-z}\, dz = e^{-y}, \quad c > 0.$$

The later formula together with definition gives the equality

$$\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz} = \frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \Gamma(\omega) E\left(\omega, \frac{k}{l}, 0\right) z^{-\omega}\, d\omega. \tag{3}$$

Now we move the line of integration in (3) to the left. Let $0 < a < 1$. Since, as it is noted in Section 1, the function $E(\omega, \frac{k}{l}, 0)$ has a double pole at $s = 1$, we obtain that

$$\begin{aligned}\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz} \\ = \frac{1}{2\pi i}\int_{a-\infty}^{a+\infty} \Gamma(\omega) E\left(\omega, \frac{k}{l}, 0\right) z^{-\omega}\, d\omega + \operatorname*{Res}_{z=1} \Gamma(\omega) E\left(\omega, \frac{k}{l}, 0\right) z^{-\omega}.\end{aligned} \tag{4}$$

Clearly,

$$\Gamma(\omega) = 1 - \gamma(\omega - 1) + \frac{\Gamma''(1)(\omega-1)^2}{2} + \cdots, \tag{5}$$

$$z^{-\omega} = z^{-1} e^{-(\omega-1)\log z} = z^{-1}\left(1 - (\omega-1)\log z + \frac{(\omega-1)^2 \log^2 z}{2} + \cdots\right).$$

Therefore, in view of (2),

$$\operatorname*{Res}_{z=1} \Gamma(\omega) E\left(\omega, \frac{k}{l}, 0\right) z^{-\omega} = \frac{\gamma - 2l - \log z}{2}.$$

This, (4) and the definition of $\Phi(z, \frac{k}{l})$ show that

$$\Phi\left(z, \frac{k}{l}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(\omega) E\left(\omega, \frac{k}{l}, 0\right) z^{-\omega}\, d\omega. \tag{6}$$

Hence,

$$\begin{aligned}\Phi\left(z^{-1}, \frac{k}{l}\right) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(\omega) E\left(\omega, \frac{k}{l}, 0\right) z^{\omega}\, d\omega \\ &= \frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a-+i\infty} \Gamma(1-\omega) E\left(1-\omega, \frac{k}{l}, 0\right) z^{1-\omega}\, d\omega.\end{aligned} \tag{7}$$

Applying the functional equation for the Estermann zeta-function, we find that

$$E\left(1-\omega, \frac{k}{l}, 0\right) = \frac{1}{\pi}\left(\frac{2\pi}{l}\right)^{2-2\omega} \Gamma^2(\omega)\left(E\left(\omega, \frac{\overline{k}}{l}, 0\right) + \cos(\pi\omega) E\left(\omega, -\frac{\overline{k}}{l}, 0\right)\right).$$

Moreover,

$$\Gamma(1-\omega) = \frac{\pi}{\Gamma(\omega)\sin \pi\omega}.$$

Thus, substituting in (7) gives

$$\begin{aligned}\Phi\Big(z^{-1},\frac{k}{l}\Big) &= \frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Big(\frac{2\pi}{l}\Big)^{1-2\omega}\Gamma(\omega)(\sin\pi\omega)^{-1}E\Big(\omega,\frac{\overline{k}}{l},0\Big)z^{1-\omega}\,d\omega\\
&\quad+\frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Big(\frac{2\pi}{l}\Big)^{1-2\omega}\Gamma(\omega)\cot(\pi\omega)E\Big(\omega,-\frac{\overline{k}}{l},0\Big)z^{1-\omega}\,d\omega\\
&= -\frac{\delta i2\pi z}{l}\Big(\frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Gamma(\omega)E\Big(\omega,-\frac{\overline{k}}{l},0\Big)\Big(\frac{4\pi^2 z}{l^2}\Big)^{-\omega}d\omega\Big)\\
&\quad+\frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Big(\frac{2\pi}{l}\Big)^{1-2\omega}\Gamma(\omega)\Big(\big(\sin(\pi\omega)\big)^{-1}E\Big(\omega,\frac{\overline{k}}{l},0\Big)\\
&\quad+\big(\cot(\pi\omega)+\delta i\big)E\Big(\omega,-\frac{\overline{k}}{l},0\Big)\Big)z^{1-\omega}\,d\omega\\
&= -\frac{2\pi i\delta z}{l}\Phi\Big(\frac{4\pi^2 z}{l^2},-\frac{\overline{k}}{l}\Big)+\frac{1}{2\pi i}\int_{1-a-i\infty}^{1-a+i\infty}\Big(\frac{2\pi}{l}\Big)^{1-2\omega}\Gamma(\omega)\\
&\quad\times\Big(\big(\sin(\pi\omega)\big)^{-1}E\Big(\omega,\frac{\overline{k}}{l},0\Big)+\big(\cot(\pi\omega)+\delta i\big)E\Big(\omega,-\frac{\overline{k}}{l},0\Big)\Big)z^{1-\omega}\,d\omega
\end{aligned}\tag{8}$$

in view of (5), since $0<1-a<1$.

It remains to transform the integral I in (8). For this, we move the line of integration to the right. Let $1<b<2$. Then we have that

$$\begin{aligned}I=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\Big(\frac{2\pi}{l}\Big)^{1-2\omega}\Gamma(\omega)\Big(\big(\sin(\pi\omega)\big)^{-1}E\Big(\omega,\frac{\overline{k}}{l},0\Big)\\
+\big(\cot(\pi\omega)+\delta i\big)E\Big(\omega,-\frac{\overline{k}}{l},0\Big)\Big)z^{1-\omega}\,d\omega-\operatorname*{Res}_{\omega=1}(\ldots).\end{aligned}\tag{9}$$

Obviously,

$$\begin{aligned}\sin(\pi\omega)&=\sin\pi(1-\omega)=-\sin\pi(\omega-1),\\
\cot(\pi\omega)&=-\cot\pi(1-\omega)=\cot\pi(\omega-1),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sin(\pi\omega)}&=\frac{-2i}{e^{\pi i(\omega-1)}-e^{-\pi i(\omega-1)}}=-\frac{1}{\pi(\omega-1)(1+o(1))},\\
\cot(\pi\omega)=i\frac{e^{\pi i(\omega-1)}+e^{-\pi i(\omega-1)}}{e^{\pi i(\omega-1)}-e^{-\pi i(\omega-1)}}&=\frac{1}{\pi(\omega-1)(1+o(1))}-\frac{\pi(\omega-1)}{2(1+o(1))}+o(1)\end{aligned}$$

as $\omega\to 1$. Therefore, using (2), (5) and

$$z^{1-\omega}=1-(\omega-1)\log z+o(\omega-1),$$

we find that

$$\begin{aligned}&\operatorname*{Res}_{\omega=1}\Big(\frac{2\pi}{l}\Big)^{1-2\omega}\Gamma(\omega)\Big(\big(\sin(\pi\omega)\big)^{-1}E\Big(\omega,\frac{\overline{k}}{l},0\Big)+\big(\cot(\pi\omega)+\delta i\big)E\Big(\omega,-\frac{\overline{k}}{l},0\Big)\Big)z^{1-\omega}\\
&\quad=-c_0'\frac{l}{2\pi^2}+c_0''\frac{l}{2\pi^2}+\frac{\delta i}{2\pi}(\gamma-2\log l-\log z)-\frac{1}{4}.\end{aligned}$$

Thus, by (9),

$$I = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{l}\right)^{1-2\omega}\Gamma(\omega)\biggl(\bigl(\sin(\pi\omega)\bigr)^{-1}E\Bigl(\omega,\frac{\overline{k}}{l},0\Bigr)$$
$$+\bigl(\cot(\pi\omega)+\delta i\bigr)E\Bigl(\omega,-\frac{\overline{k}}{l},0\Bigr)\biggr)z^{1-\omega}\,d\omega + c_0'\frac{l}{2\pi^2} - c_0''\frac{l}{2\pi^2}$$
$$+\frac{\delta i}{2\pi}(\gamma - 2\log l - \log z) + \frac{1}{4}.$$

This together with (8) proves the theorem.

References

[1] T. Estermann. On the representation of a number as the sum of two products. *Proc. London Math. Soc.*, **31**:123–133, 1930.

[2] I. Kiuchi. On an exponential sum involving the arithmetic function $\sigma_a(m)$. *Math. J. Okayma Kluw.*, **29**:193–205, 1987.

[3] A. Laurinčikas ir R. Garunkštis. *The Lerch Zeta-Function.* Kluwer, Dordrecht, 2002.

[4] A. Laurinčikas. One transformation formula related to the Riemann zeta-function. *Integral Transf. Spec. Funct.*, **19**:577–583, 2008.

[5] M. Lukkarinen. The Mellin transform of the square of Riemann's zeta-function and Atkinson's formula. *Ann. Acad. Scie. Fenn. Math. Diss.*, **140**, 2005, Suomalainen Tiedeakatemia, Helsinki.

[6] E.C. Titchmarsh. *The Theory of Riemann Zeta-Function.* Clarendor Press, Oxford, 2nd ed., revised by D.R. Heath-Brown edition, 1986.

REZIUMĖ

Transformacijos formulė, susijusi su Dirichlė L-funkcija, su pagrindiniu charakteriu

A. Balčiūnas

Straipsnyje gauta formulė eksponentinei eilutei su dalikliklių funkcija. Ši formulė gali būti pritaikyta L-funkcijos su pagrindiniu charakteriu kvadrato Melino transformacijos meromorfiniam tęsiniui gauti.

Raktiniai žodžiai: Dirichlė L-funkcija, Estermano dzeta-funkcija, Lercho dzeta-funkcija.