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Value distribution theorems for the periodic zeta-function

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Reikšmių pasiskirstymo teoremos periodinei dzeta funkcijai

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Notation

j, k, l, m, n	natural numbers
p	prime number
(m, n)	greatest common divisor of naturals m and n
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unity: $i = \sqrt{-1}$
$s = \sigma + it$	complex variable
$\bigoplus\limits_m A_m$	direct sum of sets A_m
$A \times B$	Cartesian product of the sets A and B
A^m	Cartesian product of m copies of the set A
$\text{meas}A$	Lebesgue measure of the set A
$\#A$	cardinality of the set A
$H(D)$	space of analytic functions on D
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\mathcal{B}(S)$	class of Borel sets of the space S
χ	Dirichlet character
$L(s, \chi)$	Dirichlet L-function
$F(z)$	cusp form
$f(x) = \mathrm{O}(g(x))$, $x \in I$	means that $\lvert f(x)\rvert \leqslant Cg(x)$, $x \in I$
$\zeta(s)$	Riemann zeta-function defined by $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$, for $\sigma > 1$, and by analytic continuation elsewhere

$\zeta(s;\mathfrak{a})$	periodic zeta-function defined by $\zeta(s;\mathfrak{a}) = \sum\limits_{m=1}^{\infty} \frac{a_m}{m^s}$, for $\sigma > 1$, and by analytic continuation elsewhere
$\zeta(s,\alpha)$	Hurwitz zeta-function defined by $\zeta(s,\alpha) = \sum\limits_{m=1}^{\infty} \frac{1}{(m+\alpha)^s}$, for $\sigma > 1$, and by analytic continuation elsewhere
$\zeta(s,\alpha;\mathfrak{a})$	periodic Hurwitz zeta-function defined by $\zeta(s,\alpha;\mathfrak{a}) = \sum\limits_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}$, for $\sigma > 1$, and by analytic continuation elsewhere
$\Gamma(s)$	Euler gamma-function defined by $\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-x} x^{s-1} \mathrm{d}x$ for $\sigma > 0$ and by analytic continuation elsewhere

Introduction

Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. In the thesis, the value-distribution of the periodic zeta-function $\zeta(s; \mathfrak{a})$ is considered. The function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}. \tag{1}$$

The periodicity of the sequence $\mathfrak{a}$ implies that there exists a constant $c_{\mathfrak{a}} > 0$ such that, for all $m \in \mathbb{N}$,

$$|a_m| \leqslant c_{\mathfrak{a}}.$$

Clearly, we can take, for example

$$c_{\mathfrak{a}} = \max(|a_1|, \ldots, |a_q|).$$

This shows that the series (1) is absolutely convergent in the half-plane $\sigma > 1$, and defines there an analytic function.

The function $\zeta(s; \mathfrak{a})$ also has a meromorphic continuation to the whole complex plane. For this, the classical Hurwitz zeta-function is applied. Let α, $0 < \alpha \leqslant 1$, be a fixed parameter. We recall that the Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and can be continued analytically to the whole complex plane, except for a simple pole at the point $s = 1$, and

$$\operatorname*{Res}_{s=1} \zeta(s, \alpha) = 1.$$

The function $\zeta(s, \alpha)$ was introduced and studied by A. Hurwitz in [9], and has various applications in analytic number theory. From the periodicity of the sequence $\mathfrak{a}$, we have that, for $\sigma > 1$,

$$\zeta(s; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right).$$

This equality and the properties of the Hurwitz zeta-function show that the function $\zeta(s; \mathfrak{a})$ has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$, and

$$\operatorname*{Res}_{s=1} \zeta(s; \mathfrak{a}) = \frac{1}{q} \sum_{l=1}^{q} a_l \stackrel{\text{def}}{=} r.$$

If $r = 0$, then $\zeta(s; \mathfrak{a})$ is an entire function.

We recall that the Riemann zeta-function $\zeta(s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

or, equivalently, by the Euler product over primes

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$, and

$$\operatorname*{Res}_{s=1} \zeta(s) = 1.$$

From definitions of $\zeta(s; \mathfrak{a})$ and $\zeta(s)$, it follows that $\zeta(s; \mathfrak{a}) = \zeta(s)$ if $\mathfrak{a} = \{a_m : a_m \equiv 1\}$. Therefore, the periodic zeta-function is a generalization of the famous Riemann zeta-function.

Let χ be a Dirichlet character modulo q. Roughly speaking, a character χ is an arithmetic function $\chi : \mathbb{N} \to \mathbb{C}$ which is periodic with period q ($\chi(m+q) = \chi(m)$ for all $m \in \mathbb{N}$), completely multiplicative ($\chi(mn) = \chi(m)\chi(n)$ for all

$m, n \in \mathbb{N}$), $\chi(m) = 0$ if $(m, q) > 1$, and $\chi(m) \neq 0$ if $(m, q) = 1$. We recall that the Dirichlet L-function $L(s, \chi)$ with a Dirichlet character χ is defined, for $\sigma > 1$, by the Dirichlet series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

or by the Euler product over primes

$$L(s, \chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If $\chi = \chi_0$ is the principal Dirichlet character modulo q ($\chi_0(m) = 1$ for all m, $(m, q) = 1$), then the function $L(s, \chi_0)$ is meromorphic, it has the unique simple pole at the point $s = 1$, and

$$\operatorname*{Res}_{s=1} L(s, \chi_0) = \prod_{p|q} \left(1 - \frac{1}{p}\right),$$

where p denotes a prime number. If $\chi \neq \chi_0$, then the function $L(s, \chi)$ is entire.

The definitions of the functions $\zeta(s; \mathfrak{a})$ and $L(s, \chi)$ show that the periodic zeta-function is a generalization of Dirichlet L-functions. Thus, the function $\zeta(s; \mathfrak{a})$ is a generalization of very important in analytic number theory functions, the Riemann zeta-function and Dirichlet L-functions. This remark shows the importance of the function $\zeta(s; \mathfrak{a})$.

Let $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be an another periodic sequence of complex numbers with minimal period $q_1 \in \mathbb{N}$. The results of the thesis also are related to the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{b})$, where α is the same fixed parameter as in the classical Hurwitz zeta-function, which is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s},$$

and, in view of the equality

$$\zeta(s, \alpha; \mathfrak{b}) = \frac{1}{q_1^s} \sum_{l=0}^{q_1 - 1} b_l \zeta\left(s, \frac{l + \alpha}{q_1}\right), \sigma > 1,$$

can be continued analytically to the whole complex plane, except for a simple pole at the point $s = 1$ with

$$\operatorname*{Res}_{s=1} \zeta(s, \alpha; \mathfrak{b}) = \frac{1}{q_1^s} \sum_{l=0}^{q_1-1} b_l \stackrel{\text{def}}{=} r_1.$$

If $r_1 = 0$, then the function $\zeta(s, \alpha; \mathfrak{b})$ is entire.

Since $\zeta(s, \alpha; \mathfrak{b}) = \zeta(s, \alpha)$ with $\mathfrak{b} = \{b_m : b_m \equiv 1\}$, the function $\zeta(s, \alpha; \mathfrak{b})$ is a generalization of the classical Hurwitz zeta-function $\zeta(s, \alpha)$.

Aims and problems

The aim of the thesis is the approximations of a wide class of analytic functions by shifts $\zeta(s + i\tau; \mathfrak{a})$ of the periodic zeta-function with a multiplicative sequence $\mathfrak{a}$ ($a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$, $(m, n) = 1$), i.e., the aim of this thesis are universality theorems for the function $\zeta(s; \mathfrak{a})$. The problems investigated in the thesis are the following:

1. Universality of the function $\zeta(s; \mathfrak{a})$ with multiplicative coefficients.
2. Universality of the function $\zeta(s; \mathfrak{a})$ with a special sequence $\mathfrak{a}$.
3. Weighted universality of the function $\zeta(s; \mathfrak{a})$.
4. Weighted discrete universality of the function $\zeta(s; \mathfrak{a})$.
5. Value distribution of certain compositions involving periodic zeta-functions.

Actuality

Value distribution of zeta and L-functions is one of the most important problems of analytic number theory and occupies a honorable place in mathematics in general. One of the most important seven Millennium problems is devoted to zero-distribution of the Riemann zeta-function, more precisely, to the Riemann hypothesis which asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. In 1975, the universality of zeta-functions was discovered [52] and this strengthened still the positions of zeta-functions because new theoretical and practical applications appeared. The approximation of complicated analytic functions by shifts of comparatively simple zeta-functions found deep applications in quantum mechanics [3], [7]. It became known that the Riemann hypothesis is equivalent to the self-approximation problem for the function $\zeta(s)$. Finally, the last Fermat theorem was proved by using connection between zeta-functions of certain cusp forms and L-functions of elliptic curves [3]. All these examples show the importance of investigations of value distribution of zeta-functions, and stimulate the researches in the field.

Investigations of value distribution of zeta-functions is one of the priority successfully cultivated directions of Lithuanian mathematicians. The latter direction was began to study by Professor Jonas Kubilius and is continued by his students. Applications of probabilistic methods in number theory plays a significant role in the Kubilius school, and it is our obligation to develop this interesting direction of the theory of zeta-functions.

Methods

For the proof of universality theorems for the periodic zeta-function, the method of limit theorems on weakly convergent probability measures in the space of analytic functions is applied, while the proofs of limit theorems are based on the Fourier transform method and other classical approaches of the weak convergence theory.

Novelty

All results of the thesis are new. The first part of Theorem 1.1 under a certain additional condition on the sequence $\mathfrak{a}$ was proved in [31]. Weighted universality theorems for the periodic zeta-function earlier were not known.

History of the problem and the results

The periodic zeta-function $\zeta(s;\mathfrak{a})$ is an attractive analytic object, and it was studied in various aspects by many mathematicians. To our knowledge, the first important result was obtained by W. Schnee in [45]. It is well known that zeta-functions usually satisfy certain functional equations. In [45], such an equation was proved for the function $\zeta(s;\mathfrak{a})$. Let $\mathfrak{b}=\{b_m : m \in \mathbb{Z}\}$ be a sequence related to $\mathfrak{a}$, and be defined by

$$b_m = \frac{1}{q}\sum_{k=0}^{q-1} a_k \mathrm{e}^{2\pi i k \frac{m}{q}}.$$

Moreover, let $\hat{\mathfrak{b}} = \{\hat{b}_m : m \in \mathbb{Z}\}$, where $\hat{b}_m = b_{-m}$. Then the main result of [45] is the following functional equation. As usual, $\Gamma(s)$ denotes the Euler gamma-function.

Theorem A. *For all $s \in \mathbb{C}$,*

$$\zeta(1-s;\mathfrak{a}) = \left(\frac{q}{2\pi}\right)^s \Gamma(s)\Big(\mathrm{e}^{\frac{\pi is}{2}}\zeta(s;\mathfrak{b}) + \mathrm{e}^{-\frac{\pi is}{2}}\zeta(s;\hat{\mathfrak{b}})\Big).$$

The paper [10] is also devoted to the value distribution of the function $\zeta(s;\mathfrak{a})$. In that paper, the Laurent expansion at the point $s=1$ is presented, the Dirichlet series for powers $\zeta^r(s;\mathfrak{a})$, $r \in \mathbb{N}$, are obtained, and a certain approximation for $\zeta(s;\mathfrak{a})$ by the Riemann zeta-function is given.

Further investigations of the function $\zeta(s;\mathfrak{a})$ are related to the name of J. Steuding. In [46], he created the zero-distribution theory for $\zeta(s;\mathfrak{a})$. He proved that there exists a positive constant depending on the sequence $\mathfrak{a}$, say, $A(\mathfrak{a})$, such that $\zeta(s;\mathfrak{a})$ has no zeros for $\sigma > 1 + A(\mathfrak{a})$. J. Steuding also introduced the notions of trivial and non-trivial zeros of $\zeta(s;\mathfrak{a})$ and obtained [46] the Mangoldt type formula for the number of non-trivial zeros of $\zeta(s;\mathfrak{a})$. Let

$$\hat{a}_m^{\pm} = \frac{1}{\sqrt{q}}\sum_{k=1}^{q} a_k \mathrm{e}^{\pm 2\pi ik\frac{m}{q}},$$

and let $\mathfrak{a}^{\pm} = \{\hat{a}_m^{\pm} : m \in \mathbb{N}\}$, $B(\mathfrak{a}) = \max(A(\mathfrak{a}^+), A(\mathfrak{a}^-))$. Then the zeros $\rho = \beta + i\gamma$ of the function $\zeta(s;\mathfrak{a})$ are called trivial if $\beta < -B(\mathfrak{a})$, and remained zeros are called non-trivial. They lie in the region

$$\{s \in \mathbb{C} : -B(\mathfrak{a}) \leqslant \sigma \leqslant 1 + B(\mathfrak{a})\}.$$

Let $N(T;\mathfrak{a})$ be the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s;\mathfrak{a})$ with $|\gamma| \leqslant T$ counted with multiplicity. Then the most interesting result of [46] is the following formula.

Theorem B. *Suppose that $T \to \infty$. Then*

$$N(T;\mathfrak{a}) = \frac{T}{\pi}\log\frac{qT}{2\pi m_{\mathfrak{a}}\sqrt{m_{\mathfrak{a}^-} + m_{\mathfrak{a}^+}}} + \mathrm{O}(\log T),$$

where $m_{\mathfrak{a}} = \min\{1 \leqslant m \leqslant q : a_m \neq 0\}$ and $m_{\mathfrak{a}^{\pm}} = \min\{1 \leqslant m \leqslant q : a_m^{\pm} \neq 0\}$.

The first results on the moments of the function $\zeta(s;\mathfrak{a})$,

$$I_k(T,\sigma;\mathfrak{a}) = \int\limits_0^T |\zeta(\sigma+it;\mathfrak{a})|^{2k}\mathrm{d}t,$$

where k is a non-negative integer and $\sigma \geqslant \frac{1}{2}$, were obtained in [13]. However, the best moment results for $\zeta(s;\mathfrak{a})$ were given by D. Šiaučiūnas in his thesis [50] and the corresponding papers. For this aim, the approximate functional equation for the function $\zeta(s;\mathfrak{a})$ was applied [50].

Theorem C. *Suppose that* $t \geqslant 1$, $y = \sqrt{\frac{t}{2\pi}}$, $n = [y]$, $r = \left[y - \frac{k}{q}\right]$, $l = n - r$, *and* $\frac{1}{2} \leqslant \sigma \leqslant 1$. *Then*

$$\begin{aligned}\zeta(s;\mathfrak{a}) &= \frac{1}{q^s}\sum_{k=1}^{q} a_k \sum_{0\leqslant m\leqslant r} \frac{1}{\left(1+\frac{k}{q}\right)^s} \\ &+ \frac{1}{q^s}\left(\frac{2\pi}{t}\right)^{s-\frac{1}{2}} \mathrm{e}^{i\left(t+\frac{\pi}{4}\right)} \sum_{k=1}^{q} a_k \sum_{1\leqslant m\leqslant n} \frac{\mathrm{e}^{-2\pi i m\frac{k}{q}}}{m^{1-s}} \\ &+ \frac{1}{q^s}\left(\frac{2\pi}{t}\right)^{\frac{\sigma}{2}} \sum_{k=1}^{q} a_k \mathrm{e}^{if\left(\frac{k}{q},t\right)} \psi\left(2y - 2n + l - \frac{k}{q}\right) + \frac{1}{q^s}R(s,q),\end{aligned}$$

where

$$\psi(a) = \frac{\cos\pi\left(\frac{a^2}{2} - a - \frac{1}{8}\right)}{\cos\pi a},$$

$$f(\alpha,t) = t\log\left(\frac{2\pi}{t}\right) + \frac{t}{2} - \frac{7\pi}{8} + \frac{\pi\alpha^2}{2} + \frac{\pi l}{2} + \pi n - \pi\alpha l + 2\pi y(l-\alpha)$$

and

$$R(s,q) = \mathrm{O}\left(t^{\frac{\sigma}{2}-1}\sum_{k=1}^{q}|a_k|\right).$$

Applications of Theorem C led to a series of results for the moments $I_k(T,\sigma;\mathfrak{a})$. Their statements are sufficiently complicated, therefore, we recall only one mean square estimate [50]. Let γ_0 be the Euler constant.

Theorem D. *Suppose that* $T \to \infty$. *Then*

$$\begin{aligned}I_1\left(T,\frac{1}{2};\mathfrak{a}\right) &= q^{-1}K(q)T\log T + q^{-1}K(q)T(2\gamma_0 - \log\pi - 1) \\ &- q^{-1}T(K_1(q) - K_2(q)) + \mathrm{O}\left(q^{\frac{1}{2}}K(q)T^{\frac{1}{2}}\log T\right) + \mathrm{O}(qK(q)),\end{aligned}$$

where

$$K(q) = \sum_{k=1}^{q} |a_k|^2,$$

$$K_1(q) = \sum_{k=1}^{q} k|a_k|^2 \sum_{m=1}^{\infty} \frac{1}{m(mq+k)}$$

and

$$K_2(q) = q \sum_{k=1}^{q} \frac{|a_k|^2}{k}.$$

The first result of probabilistic type for the function $\zeta(s;\mathfrak{a})$ was given in [13]. Denote by $H(G)$ the space of analytic functions on the region $G \subset \mathbb{C}$ endowed with the topology of uniform convergence on compacta. Let $\mathcal{B}(\mathbb{X})$ stand for the Borel σ-field of the space $\mathbb{X}$, and let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. For $A \in \mathcal{B}(H(D))$, define

$$P_T(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau;\mathfrak{a}) \in A\},$$

where $\text{meas}A$ denotes the Lebegue measure of a measurable set $A \subset \mathbb{R}$. In [14], the weak convergence for P_T, as $T \to \infty$, was considered. Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We recall that P_n converges weakly to P, as $n \to \infty$, if, for every real bounded continuous function f on $\mathbb{X}$,

$$\lim_{n\to\infty} \int\limits_{\mathbb{X}} f \mathrm{d}P_n = \int\limits_{\mathbb{X}} f \mathrm{d}P.$$

Let $\gamma = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication, the torus Ω is a compact topological group, therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the pth component of an element $\omega \in \Omega$, and, for $m \in \mathbb{N}$, define

$$\omega(m) = \prod_{\substack{p^\alpha | m \\ p^{\alpha+1} \nmid m}} \omega^\alpha(p).$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega; \mathfrak{a})$ by the formula

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}.$$

Then in [14], Theorem 4, the following statement has been proved.

Theorem E. *P_T converges weakly to the distribution of the random element $\zeta(s, \omega; \mathfrak{a})$ as $T \to \infty$.*

However, since $\omega(m)$ are dependent random variable, Theorem E can't be applied for the investigation of universality for the function $\zeta(s; \mathfrak{a})$.

Limit theorems of the type of Theorem E also were considered in [15], including multidimensional limit theorems and limit theorems in the space of meromorphic functions.

Now, we pass to the main problem of the thesis, i.e., to the universality of $\zeta(s; \mathfrak{a})$. We recall that the universality of the Riemann zeta-function $\zeta(s)$ was discovered by S. M. Voronin in [52]. He proved that if $0 < r < \frac{1}{4}$, the function $f(s)$ is continuous and non-vanishing on the disc $|s| \leqslant r$, and is analytic in the interior of this disc, then, for every $\varepsilon > 0$, there exists $\tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Later, the famous Voronin theorem was improved. We will recall its modern statement. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. The following version of the Voronin theorem is known, see, for example [22].

Theorem F. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Theorem F shows that there are infinitely many of shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K)$.

J. Steuding was the first who began to study the universality of the function $\zeta(s; \mathfrak{a})$. In [46], he proved the following theorem. Let $H(K)$, $K \in \mathcal{K}$, be the class of continuous functions on K that are analytic in the interior of K.

Theorem G. *Suppose that q is an odd prime number, a_m is not multiple of a Dirichlet character modulo q, and $a_q = 0$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Note that, under conditions of Theorem G, the periodic sequence $\mathfrak{a}$ is not multiplicative. This follows from the characterization of periodic multiplicative functions given in [32].

In [47], J. Steuding extended Theorem G in the following manner. He proved that the assertion of Theorem G is valid if $\mathfrak{a}$ is a periodic sequence with minimal period $q > 2$, not a multiple of a Dirichlet character modulo q, satisfying $a_m = 0$ for $(m,q) > 1$.

Universality of the periodic zeta-function is not a simple problem. J. Kaczorowski observed in [12] that not all functions $\zeta(a;\mathfrak{a})$ are universal in the above sense. He presented the following example of the periodic sequence $\mathfrak{a}_0 = \{a_{0m} : m \in \mathbb{N}\}$. Let $q = 2$, and $a_{01} = 1$, and $a_{02} = 2^{\frac{3}{4}} + 1$. Then we have that

$$\zeta(s;\mathfrak{a}_0) = \bigl(1 + 2^{\frac{3}{4}-s}\bigr)\zeta(s).$$

Moreover, let

$$K = \left[\frac{5}{8}, \frac{7}{8}\right] \times [-c_0, c_0]$$

be a rectangle in the right-half of the critical strip. If $c > \frac{\pi}{\log 2}$, then every shift $\zeta(s+i\tau;\mathfrak{a}_0)$ has a zero inside K, therefore, it can't approximate uniformly functions which do not vanish inside K, for example, it can't approximate the constant function $f(s) \equiv 1$ in K.

J. Kaczorowski also introduced [12] a certain restricted universality property. Let $K \in \mathcal{K}$. Then the number

$$h(K) = \max_{s\in K} \Im s - \min_{s\in K} \Im s$$

is called the height of K. The Kaczorowski theorem is the following statement [12].

Theorem H. *There exists a positive constant $c_0 = c_0(\mathfrak{a})$ such that, for $K \in \mathcal{K}$*

with $h(K) \leqslant c_0$, $f(s) \in H_0(K)$ *and every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

The thesis is devoted to the universality of the periodic zeta-function $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$. We recall that the sequence $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for $(m,n) = 1$.

An universality theorem for the function $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ was proved in [31] with additional condition that

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\frac{\alpha}{2}}} \leqslant c < 1 \tag{2}$$

for all primes p. In the thesis, the latter condition is removed, and in Chapter 1, the following theorem is obtained.

Theorem 1.1. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative. Let* $K \in \mathcal{K}$ *and* $f(s) \in H_0(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the same inequality with "$\lim$*" holds for all but at most countably many* $\varepsilon > 0$.

We note that the second fact of Theorem 1.1 for the Riemann zeta-function was independently obtained in [29] and [37], see also the thesis of L. Meška [39].

A proof of Theorem 1.1 is probabilistic and is based on a limit theorem of type of Theorem E. First, in view of multiplicativity of the sequence $\mathfrak{a}$, it is proved that, for almost all $\omega \in \Omega$, the equality

$$\zeta(s,\omega;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s} = \prod_{p\in\mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l}\omega^l(p)}{p^{ls}}\right), \sigma > \frac{1}{2},$$

is valid. Let P_ζ be the distribution of the random element $\zeta(s,\omega;\mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s,\omega;\mathfrak{a}) \in A\}, A \in \mathcal{B}(H(D)).$$

Then a limit theorem for P_T is of the following form.

Theorem 1.6. *The measure P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the set*

$$S \stackrel{def}{=} \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Chapter 2 of the thesis is devoted to universality of $\zeta(s; \mathfrak{a})$ with a special periodic sequence. For this, universality of Dirichlet L-functions is applied. We recall that an analogue of Theorem G for Dirichlet L-functions $L(s, \chi)$ is known [52].

Theorem I. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |L(s + i\tau, \chi) - f(s)| < \varepsilon\right\} > 0.$$

Suppose that $a_m \not\equiv 0$, the period q of the sequence $\mathfrak{a}$ is a prime number, and

$$a_q = \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l, \tag{2.1}$$

where $\varphi(q)$ is the Euler totient function. The main result of the chapter is the following statement.

Define

$$b(q, \chi) = \sum_{l=1}^{q-1} a_l \chi(l),$$

where χ is a Dirichlet character modulo q. We say, that the function is universal if the inequality of universality

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon\right\} > 0$$

with every ε is satisfied for $f(s) \in H_0(K)$, $K \in \mathcal{K}$. If the above inequality holds for $f(s) \in H(K)$, $K \in \mathcal{K}$, then we say that the function $\zeta(s; \mathfrak{a})$ is strongly universal.

Theorem 2.2. *Suppose that the periodic sequence $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ with minimal period q satisfies equality (2.1), and that q is a prime number.*

1° If the sequence $\mathfrak{a}$ satisfies at least one of the hypothesis

i) $a_m \equiv c$, $m \in \mathbb{N}$;

ii) a_m *is a multiple of a Dirichlet character modulo* q*;*

iii) $q = 2$*;*

iv) only one of the numbers $b(q, \chi) \neq 0$*,* $q > 2$*,*

then the function $\zeta(s, \mathfrak{a})$ *is universal.*

2° *If* $q > 2$ *and at least two numbers* $b(q, \chi) \neq 0$*, then the function* $\zeta(s; \mathfrak{a})$ *is strongly universal.*

The results of the chapter are published in [49].

In Chapter 3, a weighted universality theorem for the periodic zeta-function is proved. A theorem of such a type for the Riemann zeta-function was obtained in [21] by using an additional condition for the weight function related to the Birkhoff-Khintchine ergodic theorem (see Lemma 1.12 bellow). Suppose that $\zeta(\tau, \omega)$ is an ergodic process on a certain probability space $(\Omega, \mathfrak{A}, \mu)$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\mathbb{E}|\zeta(\tau, \omega)| < \infty$, and sample paths are integrable almost surely over any finite interval. In [21], it was assumed that

$$\frac{1}{U(T, \omega)} \int\limits_{T_0}^{T} w(\tau)\zeta(t + \tau, \omega)\mathrm{d}\tau = \mathbb{E}(\zeta(0, \omega)) + \mathrm{o}((1 + |t|)^{\alpha})$$

almost surely for any $t \in \mathbb{R}$, with $\alpha > 0$, as $T \to \infty$. The same condition was used in [21] for the weighted universality of the Matsumoto zeta-function. In the thesis, the latter condition is removed.

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, such that the variation $V_a^b w$ on the interval $[a, b]$ satisfies the inequality

$$V_a^b w \leqslant c w(a)$$

with a certain positive constant c for any subinterval $[a, b] \subset [T_0, \infty)$. Denote

$$U(T, w) = \int\limits_{T_0}^{T} w(t)\mathrm{d}t,$$

and suppose that

$$\lim_{T \to \infty} U(T, w) = +\infty.$$

Denote the class of the functions $w(t)$ satisfying the above conditions by W. Moreover, let $I(A)$ be the indicator function of the set A. Then the main result for Chapter 3 is the following universality theorem of continuous type (τ in shifts $\zeta(s+\tau;\mathfrak{a})$ takes arbitrary real values).

Theorem 3.1. *Suppose that $w \in W$, and the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U(T,w)} \int_{T_0}^{T} w(t) I\Big(\Big\{\tau \in [T_0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}\Big) \mathrm{d}\tau > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

The proof of Theorem 3.1 uses a weighted limit theorem for the function $\zeta(s;\mathfrak{a})$ in the space of analytic functions $H(D)$ (Theorem 3.2). The results of Chapter 3 are published in [34].

Chapter 4 of the thesis is devoted to the weighted discrete universality of periodic zeta-function. In this case, in approximating shifts $\zeta(s+i\tau;\mathfrak{a})$, τ takes values from certain discrete sets, for example, from the arithmetic progression $\{kh : k \in \mathbb{N}_0\}$ with fixed $h > 0$. The discrete universality for zeta-functions was proposed by A. Reich. In [44], he obtained the following statement. Let N run over non-negative integers, and $\#A$ denotes the cardinality of the set A.

Theorem J. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $h > 0$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \Big\} > 0.$$

Theorem J by another method was independently proved in [1].

The first weighted discrete universality theorem of the thesis deals with the arithmetic progression $\{kh\}$. Suppose that $w(t)$ is a non-increasing positive function for $t \geqslant 1$ having a continuous derivative such that, for $h > 0$, $w(t) \ll_h w(ht)$ and $(w'(t))^2 \ll w(t)$. Moreover, let

$$V(N,w) = \sum_{k=1}^{N} w(k)$$

be such that

$$\lim_{N\to\infty} V(N,w) = +\infty.$$

The class of the above function $w(t)$ is denoted by V_1. Then the first weighted discrete universality theorem for the function $\zeta(s;\mathfrak{a})$ has the following form.

Theorem 4.1. *Suppose that $w \in V_1$, the sequence $\mathfrak{a}$ is multiplicative, and the set*

$$L(\mathbb{P}, h, \pi) = \left\{(\log p : p \in \mathbb{P}), \frac{2\pi}{h}\right\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a})-f(s)| < \varepsilon\Big\}\Big) > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

It is well known that, in view of the Lindemann theorem, the number e^k with $k \in \mathbb{Z}\backslash\{0\}$ is transcendental. Therefore, in Theorem 4.1, we can take, for example, $h = \pi$ and $w(t) = \frac{1}{t}$.

The second weighted discrete universality theorem for the function $\zeta(s;\mathfrak{a})$ of the thesis uses a more complicated discrete set $\{k^\alpha h\}$ with fixed α, $0 < \alpha < 1$, and $h > 0$. Suppose that the weight $w(t)$ be such that $\lim\limits_{N\to\infty} V(N,w) = +\infty$, and has a continuous derivative such that

$$\int_1^N t|w'(t)|\mathrm{d}t \ll V(N,w).$$

We denote the class of above functions $w(t)$ by V_2. Then the following theorem is true.

Theorem 4.7. *Suppose that $w \in V_2$, the sequence $\mathfrak{a}$ is multiplicative, and $0 < \alpha < 1$ is fixed. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I(\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ik^\alpha h;\mathfrak{a})-f(s)| < \varepsilon\}) > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

For the proof of Theorem 4.7, the uniform distribution modulo 1 of the sequence $\{ak^\alpha\}$ with fixed $0 < \alpha < 1$ and every real $a \neq 0$ is applied. We recall that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $[a, b) \subset (0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I_{[a,b)}\big(\{x_k\}\big) = b - a,$$

where $\{x_k\}$ denotes the fractional part of x_k, and $I_{[a,b)}$ is the indicator function of the interval $[a, b)$.

The results of Chapter 4 are published in [35], [36] and [48].

In the last chapter, Chapter 5, of the thesis, joint universality of the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$ is considered. The joint theorem of such a type was known under certain additional restrictions. For example, in [16], the following theorem was proved.

Theorem K. *Suppose that α is a transcendental number, the sequence $\mathfrak{a}$ is multiplicative and the condition (2) is satisfied. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s\in K_1} |\zeta(s + i\tau; \mathfrak{a}) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon\Big\} > 0.$$

In the thesis, the transcendence of α is replaced by a weaker requirement that the set $L(\mathbb{P}; \alpha) = \{(\log p : p \in \mathbb{P}), (\log(m + \alpha) : m \in \mathbb{N}_0)\}$ is linear independent over the field of rational numbers. Moreover, the condition (2) is removed. Thus, the following statement is true.

Theorem 5.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P}; \alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s\in K_1} |\zeta(s + i\tau; \mathfrak{a}) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon\Big\} > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

Also, in Chapter 5, the value distribution of some compositions of the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ is discussed. The first composition is

$$\underline{\zeta}(s;\alpha;\mathfrak{a},\mathfrak{b}) = c_1\zeta(s;\mathfrak{a}) + c_2\zeta(s,\alpha;\mathfrak{b}), c_1, c_2 \in \mathbb{C}\backslash\{0\}.$$

The following theorem on the number of zeros for the function $\underline{\zeta}(s;\alpha;\mathfrak{a},\mathfrak{b})$ is obtained.

Theorem 5.8. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$, and the sequence $\mathfrak{a}$ is multiplicative. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1,\sigma_2,\alpha,\mathfrak{a},\mathfrak{b}) > 0$ such that, for sufficiently large T, the function $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ has more than cT zeros in the rectangle*

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

Note that the first theorem of type of Theorem 5.8 was obtained by S. M. Voronin [54] for the Hurwitz zeta-function $\zeta(s,\alpha)$ with rational parameter α. For other zeta-functions and their compositions, the lower estimates for the number of zeros were considered in [6], [41] and [42].

In the thesis, also a more complicated composition than $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ is considered. We say that the operator $F : H^2(D) \to H(D)$ belongs to class $\mathrm{Lip}(\beta_1,\beta_2)$, $\beta_1 > 0$, $\beta_2 > 0$, if the following conditions are satisfied:

1° For each polynomial $p = p(s)$ and any set $K \in \mathcal{K}$, there exists an element $(g_1, g_2) \in F^{-1}\{p\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on K;

2° For any set $K \in \mathcal{K}$, there exist a positive constant c and sets $K_1, K_2 \in \mathcal{K}$ such that

$$\sup_{s\in K}\left|F(g_{11}(s),g_{12}(s))-F(g_{21}(s),g_{22}(s))\right| \leqslant c \sup_{1\leqslant j\leqslant 2}\sup_{s\in K_j}\left|g_{1j}(s)-g_{2j}(s)\right|^{\beta_j}$$

for all $(g_{j1}, g_{j2}) \in H^2(D)$, $j = 1, 2$.

For the composition $F(\zeta(s,\mathfrak{a}),\zeta(s,\alpha;\mathfrak{b}))$ with $F \in \mathrm{Lip}(\beta_1,\beta_2)$, the following universality theorem in the thesis is obtained.

Theorem 5.9. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$, the sequence $\mathfrak{a}$ is multiplicative and $F \in \mathrm{Lip}(\beta_1,\beta_2)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\{\tau\in[0,T] : \sup_{s\in K}|F(\zeta(s+i\tau;\mathfrak{a}),\zeta(s+i\tau,\alpha;\mathfrak{b}))-f(s)| < \varepsilon\} > 0.$$

In [18], the condition (2) and the transcendence of α were assumed.

We observe that the universality for certain composition of zeta-functions has proposed by A. Laurinčikas in [24] and [27]. For example, there the following theorem was proved. We recall that $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Theorem L. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 5.9 implies the last result of the thesis for the composition $F(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$.

Theorem 5.10. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$, the sequence $\mathfrak{a}$ is multiplicative, and $F \in \text{Lip}(\beta_1,\beta_2)$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1,\sigma_2,\alpha,\mathfrak{a},\mathfrak{b},F) > 0$ such that, for sufficiently large T, the function $F(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$ has more than cT zeros in the rectangle*

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

The result of Chapter 5 are published in [30].

Approbation

The results of the thesis were presented at the International MMA (Mathematical Modeling and Analysis) conferences (MMA 2015, May 26-29, 2015, Sigulda, Latvia), (MMA 2016, June 1-4, 2016 Tartu, Estonia), (MMA 2017, May 30 - June 2, 2017, Druskininkai), (MMA 2018, May 29 - June 1, 2018, Sigulda, Latvia), the 14th International Conference "Algebra and Number Theory: Modern Problems and Applications" (September 12-15, 2016, Saratov, Russia), Vilnius Conference in Combinatorics and Number Theory (July 16-22, 2017, Vilnius), the 15th International Conference "Algebra, Number Theory and Discrete Geometry: Modern Problems and Applications" (May 28-31, 2018, Tula, Russia), the International Conference on Number Theory dedicated to the 70th birthdays of Professors Antanas Laurinčikas and Eugenijus

Manstavičius (September 9-15, 2018, Palanga), The Conferences of Lithuanian Mathematical Society (LMS 2017, June 21-22, 2017, Vilnius), (LMS 2018, June 18-19, 2018, Vilnius), (LMS 2019, June 19-20, 2019, Vilnius), as wel as at the Number Theory Seminar of Vilnius University.

Principal publications

The main results of the thesis are published in the following papers:

1. A. Laurinčikas, M. Stoncelis, D. Šiaučiūnas. On the Zeros of Some Functions Related to Periodic Zeta-functions. Chebyshevskii Sbornik **15** (1) (2014), 121–130.

2. R. Macaitienė, M. Stoncelis, D. Šiaučiūnas. A Weighted Universality Theorem for Periodic Zeta-functions. Math. Mod. Analysis **22** (1) (2017), 95-105.

3. R. Macaitienė, M. Stoncelis, D. Šiaučiūnas. A Weighted Discrete Universality Theorem for Periodic Zeta-function. Analytic and Prob. Meth. in Numb. Th., A.Dubickas et.al. (Eds), Vilnius University, (2017), pp. 97–107.

4. R. Macaitienė, M. Stoncelis, D. Šiaučiūnas. A Weighted Discrete Universality Theorem for Periodic Zeta-functions. II. Math. Mod. Analysis **22** (6) (2017), 750-762.

5. M. Stoncelis, D. Šiaučiūnas. On the Periodic Zeta-function. Chebyshevskii Sbornik **15** (4) (2014), 139–147.

6. M. Stoncelis. Weighted Universality of Periodical Zeta-function. Algebra, Numb. Th. Discr. Geom.: Modern Probl. App. XV International Conference, Tula, TSPU of L. N. Tolstoy, 2018, 241-244.

Abstracts for conferences

1. M. Stoncelis, D. Šiaučiūnas. On Universal Class of Periodic Zeta-functions. Abstracts of MMA2015, May 26-29, 2015, Sigulda, Latvia, p. 80.

2. M. Stoncelis, D. Šiaučiūnas. A Weighted Universality Theorem for the Periodic Zeta-function. Abstracts of MMA2016, June 1-4, 2016, Tartu, Estonia, p. 72.

3. M. Stoncelis. Weighted Universality of Periodic Zeta-function. Abstracts of the 14th International Conference "Algebra and Number Theory: Modern Problems and Applications", September 12-15, 2016, Saratov, Russia, p 118.

4. M. Stoncelis, D. Šiaučiūnas. On Weighted Discrete Universality of Periodic Zeta-functions. Abstracts of Vilnius Conference in Combinatorics and Number Theory, July 16-July 22, 2017, Vilnius, Lithuania, p. 25-26.

5. M. Stoncelis, D. Šiaučiūnas. A Weighted Discrete Universality Theorem for the Periodic Zeta-function. Abstracts of MMA2017, May 30–June 2, 2017, Druskininkai, Lithuania, p. 62.

6. V. Garbaliauskienė, M. Stoncelis. On Weighted Universality for Composite Functions of Periodic Zeta-functions. Abstracts of MMA2018, May 29–June 1, 2018, Sigulda, Latvia, p. 24.

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Chapter 1

Universality of the periodic zeta-function with multiplicative coefficients

Let $s = \sigma + it$ be a complex variable, and let $\mathfrak{a} = \{a_m : \ m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

In virtue of the periodicity of the sequence $\mathfrak{a}$, we have that, for $\sigma > 1$,

$$\begin{aligned} \zeta(s; \mathfrak{a}) &= \sum_{l=1}^{q} \sum_{\substack{m=1 \\ m \equiv l (\operatorname{mod} q)}}^{\infty} \frac{a_m}{m^s} = \sum_{l=1}^{q} a_l \sum_{k=0}^{\infty} \frac{1}{(kq+l)^s} \\ &= \frac{1}{q^s} \sum_{l=1}^{q} a_l \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{l}{q}\right)^s} = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right), \end{aligned} \tag{1.1}$$

where $\zeta(s, \alpha)$, $0 < \alpha \leqslant 1$, is the Hurwitz zeta-function, i.e. for $\sigma > 1$,

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{a}{(m+\alpha)^s}.$$

Since the function $\zeta(s, \alpha)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1, the equality (1.1) gives analytic continuation for the function $\zeta(s; \mathfrak{a})$ to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\frac{1}{q} \sum_{l=1}^{q} a_l.$$

If $\sum\limits_{l=1}^{q} a_l = 0$, then the function $\zeta(s; \mathfrak{a})$ is entire.

We suppose additionally that the sequence $\mathfrak{a}$ is multiplicative, i.e., $a_{mn} = a_m a_n$ for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$ and $a_m \not\equiv 0$.

In this chapter, we prove an universality theorem for the function $\zeta(s; \mathfrak{a})$ with a multiplicative sequence $\mathfrak{a}$ on the approximation of analytic functions by shifts $\zeta(s + i\tau; \mathfrak{a})$, $\tau \in \mathbb{R}$.

1.1 Statement of the main theorem

Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Moreover, denote by $\text{meas}A$ Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

The main result of this chapter is the following universality theorem.

Theorem 1.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

We notice that Theorem 1.1, under an additional condition that, for every prime number p,

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\frac{\alpha}{2}}} \leqslant c < 1, \tag{1.2}$$

was proved in [31]. We observe that if

$$\mathfrak{c}_{\mathfrak{a}} = \max\left(|a_1|, \ldots, |a_q|\right) < \sqrt{2} - 1,$$

then (1.2) holds.

1.2 Definition of one random element

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. In this topology, $\{g_n\} \in H(D)$ converges to $g \in H(D)$ if and only if, for every compact set $K \in D$,

$$\lim_{n\to\infty} \sup_{s\in K} |g_n(s) - g(s)| = 0.$$

In this section, we will define an $H(D)$-valued random element connected to the function $\zeta(s; \mathfrak{a})$.

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, i.e., the minimal σ-field generated by open sets of the space $\mathbb{X}$. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ ($\mathbb{P}$ is the set of all prime numbers). With the product topology and positive multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined. We note that the measure m_H has the invariance property, i.e., for every $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega$,

$$m_H(A) = m_H(\omega A) = m_H(A\omega).$$

Thus we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the pth component of the element $\omega \in \Omega$, $p \in \mathbb{P}$, and extend the function $\omega(p)$ to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1}_{m_H} \nmid m}} \omega^l(p), m \in \mathbb{N}.$$

Since the Haar measure is the product of Haar measures on $(\gamma_p, \mathcal{B}(\gamma_p))\,, p \in \mathbb{P}$, we have that $\{\omega(p) : p \in \mathbb{P}\}$ is a sequence of independent random complex-valued random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Proposition 1.2. *Define*

$$X(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}. \tag{1.3}$$

Then $X(s, \omega; \mathfrak{a})$ is an $H(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

For the proof of Proposition 1.2, we will use the Rademacher theorem, see [33], which is stated as the following lemma. Denote by $\mathbb{E}X$ the expectation of the random variable X, and recall that the random variables X and Y we said to be orthogonal if $\mathbb{E}XY = 0$.

Lemma 1.3. *Suppose that $\{X_m : m \in \mathbb{N}\}$ is the sequence of positive orthogonal random variables (real or complex), and*

$$\sum_{m=1}^{\infty} \mathbb{E}\left|X_m\right|^2 \log^2 m < \infty.$$

Then the series

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely.

Proof of Proposition 1.2. It suffices to show that, for almost all $\omega \in \Omega$, the series (1.3) converges uniformly on compact subsets of D. Let $\sigma_0 > \frac{1}{2}$ be fixed, and

$$X_m = X_m(\omega) = \frac{a_m \omega(m)}{m^{\sigma_0}}.$$

Then $\{X_m(\omega) : m \in \mathbb{N}\}$ is a sequence of complex-valued random elements on the probability space $(\Omega, B\,(\Omega)\,, m_H)$. We have that

$$\int\limits_{\Omega} \omega(k)\overline{\omega(l)} \mathrm{d}m_H = \int\limits_{\Omega} \mathrm{d}m_H = 1$$

for $k = l$. If $k \neq l$, then

$$\int\limits_{\Omega} \omega(k)\overline{\omega(l)}\mathrm{d}m_H = 0$$

because always will be an integral with $k \in \mathbb{Z}$

$$\int\limits_0^1 \mathrm{e}^{2\pi ikx}\mathrm{d}x = 0.$$

Here, $\{\omega(m) : m \in \mathbb{Z}\}$ is a sequence of pairwise orthogonal random variables. Therefore,

$$\mathbb{E}\left(X_k\overline{X}_l\right) = \frac{a_k\overline{a}_l}{k^{\sigma_0}l^{\sigma_0}}\int\limits_{\Omega} \omega(k)\overline{\omega(l)}\mathrm{d}m_H = \begin{cases} 0 & \text{if} \quad k \neq l, \\ \frac{|a_k|^2}{k^{2\sigma_0}} & \text{if} \quad k = l, \end{cases}$$

and $\{X_m(\omega) : m \in \mathbb{N}\}$ is sequence of pairwise orthogonal random variables. Moreover,

$$\mathbb{E}|X_m|^2 = \frac{|a_m|^2}{m^{2\sigma_0}}.$$

Therefore,

$$\sum_{m=1}^{\infty} \mathbb{E}|X_m|^2 \log^2 m = \sum_{m=1}^{\infty} \frac{|a_m|^2 \log^2 m}{m^{2\sigma_0}} \leqslant c_{\mathfrak{a}}^2 \sum_{m=1}^{\infty} \frac{\log^2 m}{m^{2\sigma}} < \infty,$$

and, by Lemma 1.3, the series

$$\sum_{m=1}^{\infty} X_m = \sum_{m=1}^{\infty} \frac{a_m\omega(m)}{m^{\sigma_0}}$$

converges almost surely. The latter series is a Dirichlet series, and its convergence for $s = \sigma_0$ implies the uniform convergence on compacta in the half-plane $\sigma > \sigma_0$. This shows that the series (1.3), for almost all $\omega \in \Omega$, converges on compacta of the half-plane $\sigma > \sigma_0$.

The number $\sigma_0 > \frac{1}{2}$ is arbitrary. We take $\sigma_0 = \frac{1}{2} + \frac{1}{n}$, and denote by A_n the set of all $\omega \in \Omega$ such that the series (1.3) converges uniformly on compacta of the half-plane $\sigma > \frac{1}{2} + \frac{1}{n}$. Then we have that $m_H(A_n) = 1$. We set $A = \bigcap\limits_{n=1}^{\infty} A_n$. Then again we have that $m_H(A) = 1$, and, for $\omega \in A$, the series (1.3) converges uniformly on compacta of the half-plane $\sigma > \frac{1}{2}$.

Of course, this implies that, for almost all $\omega \in \Omega$, the series (1.3) converges uniformly on compact subsets of the strip D.

The terms of the series (1.3) are entire functions. Therefore, the above uniform convergence shows that the random element $X(s, \omega; \mathfrak{a})$ is $H(D)$-valued for almost all $\omega \in \Omega$.

$\square$

For the proof of the next proposition, we will apply the three series theorem, and state it as the following lemma. For $c > 0$ and a random variable X, denote

$$X^c = \begin{cases} X & \text{if} \quad |X| \leqslant c, \\ 0 & \text{if} \quad |X| > c. \end{cases}$$

Moreover, let DX be the variance of X.

Lemma 1.4. *Suppose that $\{X_m : m \in \mathbb{N}\}$ is the sequence of independent random variables on the probability space with the measure P. Moreover suppose that, for some $c > 0$, the series*

$$\sum_{m=1}^{\infty} P\big(|X_m| > c\big),$$

$$\sum_{m=1}^{\infty} \mathbb{E}X_m^c,$$

and

$$\sum_{m=1}^{\infty} DX_m^c$$

are convergent. Then the series

$$\sum_{m=1}^{\infty} X_m$$

is convergent almost surely.

Proof of the lemma 1.4 can be found, for example, in [19].

Now, we use the multiplicity of the sequence $\mathfrak{a}$.

Proposition 1.5. *For almost all $\omega \in \Omega$, the product*

$$\prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} \omega^l(p)}{p^{ls}}\right)$$

converges uniformly on compact subsets of D. Moreover, for almost all $\omega \in \Omega$, the equality

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s} = \prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} \omega^l(p)}{p^{ls}} \right)$$

holds.

Proof. Since $|a_m| < c_{\mathfrak{a}}$, the series and product, for all $\omega \in \Omega$, are absolutely convergent for $\sigma > 1$. Therefore, the equality of the proposition follows by the multiplicativity of the function $\frac{a_m \omega(m)}{m^s}$.

For brevity, denote, for $p \in \mathbb{P}$,

$$x_p(s, \omega) = \sum_{l=1}^{\infty} \frac{a_{p^l} \omega^l(p)}{p^{ls}}$$

and

$$y_p(s, \omega) = \frac{a_p \omega(p)}{p^s}.$$

The series for $x_p(s, \omega)$ converges uniformly for $\sigma > \frac{1}{2}$, thus, $x_p(s, \omega)$ is a $H(D)$-valued random element. Moreover,

$$|x_p(s, \omega)| \leqslant \sum_{l=1}^{\infty} \frac{c_{\mathfrak{a}}}{p^{l\sigma}} = \frac{c_{\mathfrak{a}}}{p^{\sigma} - 1}.$$

Therefore, the series

$$\sum_{p \in \mathbb{P}} |x_p(s, \omega)|^2$$

converges uniformly on compact subsets of D. From this, it follows that to prove almost sure convergence of the product

$$\prod_{p \in \mathbb{P}} \big(1 + x_p\left(s, \omega\right) \big)$$

it is sufficient to prove that the series

$$\sum_{p \in \mathbb{P}} x_p(s, \omega)$$

converges almost surely. By the definitions of $x_p(s, \omega)$ and $y_p(s, \omega)$, we have that

$$|x_p(s,\omega) - y_p(s,\omega)| \leqslant \sum_{l=2}^{\infty} \frac{c_{\mathfrak{a}}}{p^{l\sigma}} = \frac{c_{\mathfrak{a}}}{p^{2\sigma} - p^{\sigma}}.$$

Hence, the series

$$\sum_{p \in \mathbb{P}} |x_p(s,\omega) - y_p(s,\omega)|$$

converges uniformly on compact subsets of D for all $\omega \in \Omega$. Thus, it suffices to prove that the series

$$\sum_{p \in \mathbb{P}} y_p(s,\omega) \tag{1.4}$$

converges almost surely on compact subsets of D. We have seen above that $\{\omega(p) : p \in \mathbb{P}\}$ is a sequence of independent random variables, thus, the terms of series (1.4) are independent random elements. Moreover,

$$\mathbb{E} y_p(s,\omega) = \int_{\Omega} \frac{a_p \omega(p)}{p^s} \mathrm{d}m_H = \frac{a_p}{p^s} \int_0^1 \mathrm{e}^{2\pi i x} \mathrm{d}x = 0$$

and

$$\mathbb{E} |y_p(s,\omega)|^2 = \int_{\Omega} \frac{|a_p|^2 |\omega(p)|^2}{p^{2\sigma}} \mathrm{d}m_H \leqslant \frac{c_{\mathfrak{a}}}{p^{2\sigma}}.$$

Hence, for $\sigma > \frac{1}{2}$,

$$\sum_{p \in \mathbb{P}} \mathbb{E} |y_p(s,\omega)|^2 < \infty.$$

Now, an application of Lemma 1.4 shows that the series (1.4) almost surely converges for every fixed $s \in D$. Let $s = \sigma_0 > \frac{1}{2}$. Then the series (1.4) converges uniformly on compact subsets of the half-plane $\sigma > \sigma_0$. Thus, the series (1.4) converges almost sure uniformly on compact subsets of the half-plane $\sigma > \sigma_0$. We take $\sigma_0 = \frac{1}{2} + \frac{1}{n}$, and let A_n be the set of all $\omega \in \Omega$ such that the series (1.4) converges uniformly on compact sets of the half-plane $\sigma_0 > \frac{1}{2} + \frac{1}{n}$. Then, for every n, $m_H(A_n) = 1$. Define

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Then $m_H(A) = 1$, and the series (1.4), for $\omega \in \Omega$, converges uniformly on compact subsets of D.

Thus, we proved that the product of the proposition is almost sure convergent on compact subsets of D, and the equality of the proposition follows by analytic continuation using Proposition 1.2.

□

1.3 Statement of a limit theorem

For the proof of Theorem 1.1, we will apply a probabilistic approach based on a limit theorem for weakly convergent probability measures in the space $H(D)$. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We recall that P_n converges weakly to P as $n \to \infty$ if, for every real continuous bounded function g on $\mathbb{X}$

$$\lim_{n\to\infty} \int\limits_{\mathbb{X}} g \mathrm{d}P_n = \int\limits_{\mathbb{X}} g \mathrm{d}P.$$

In the proof of universality theorems for zeta-functions, the notion of the support of a probability measure plays an important role. We recall that the support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is a minimal closed set S_P such that $P(S_P) = 1$. The set S_P consists of all elements $x \in \mathbb{X}$ such that, for every open neighborhood G of x, the inequality $P(G) > 0$ is satisfied.

For $A \in \mathcal{B}(H(D))$, define

$$P_T(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(s + i\tau; \mathfrak{a}) \in A\}.$$

We will consider the weak convergence for P_T as $T \to \infty$. Denote by P_ζ the distribution of the $H(D)$-valued random element $X(s, \omega, \mathfrak{a})$ defined in Proposition 1.2, i.e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : X(s, \omega; \mathfrak{a}) \in A\}, A \in \mathcal{B}(H(D)).$$

We will prove the following theorem.

Theorem 1.6. *The measure P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the set*

$$S \overset{def}{=} \{g \in H(D) : g(s) \neq 0 \textit{ or } g(s) \equiv 0\}.$$

We divide the proof of Theorem 1.6 into several steps. First we will obtain limit theorems for absolutely convergent Dirichlet series connected to the function $\zeta(s;\mathfrak{a})$. After this, we will prove certain approximation results and limit theorems for $\zeta(s;\mathfrak{a})$. The next step of the proof be devoted to the identification of the limit measure. In the last step, we will consider the support of the limit measure.

1.4 Limit theorems for absolutely convergent Dirichlet series

Let $\theta > \frac{1}{2}$ be a fixed number, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}.$$

Define the functions

$$\zeta_n(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s}$$

and

$$\zeta_n(s,\omega;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s}, \omega \in \Omega.$$

Then the latter series are absolutely convergent for $\sigma > \frac{1}{2}$ [11]. For $A \in \mathcal{B}(H(D))$, define

$$P_{T,n}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s+i\tau;\mathfrak{a}) \in A\}$$

and

$$\hat{P}_{T,n}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s+i\tau,\omega;\mathfrak{a}) \in A\}.$$

In this section, we will prove the following theorem.

Theorem 1.7. *On $(H(D), \mathcal{B}(H(D)))$, there exist a probability measure V_n such that the measures $P_{T,n}(A)$ and $\hat{P}_{T,n}(A)$ both converges weakly to V_n as $T \to \infty$.*

We start the proof with a limit theorem on the torus Ω. For $A \in \mathcal{B}(\Omega)$, define

$$Q_T(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : (p^{-i\tau} : p \in \mathbb{P}) \in A\}.$$

Lemma 1.8. *Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We apply the Fourier transform method. The dual group (character group) of the torus Ω is isomorphic to

$$\mathcal{D} = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p,$$

where $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathbb{P}$. An element $\underline{k} = (k_p : p \in \mathbb{P}) \in \mathcal{D}$, where only a finite number of integers k_p are distinct from zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = \prod_{p\in\mathbb{P}} \omega^{k_p}(p).$$

Therefore, the characters of Ω are of the form

$$\prod_{p\in\mathbb{P}} \omega^{k_p}(p).$$

Hence, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is given by the formula

$$g_T(\underline{k}) = \int_\Omega \prod_{p\in\mathbb{P}}{}' \omega^{k_p}(p) \mathrm{d}Q_T,$$

where the sign "$'$" means that only a finite number of integers k_p are distinct from zero. Thus, by the definition of Q_T, we have that

$$g_T(\underline{k}) = \frac{1}{T}\int_0^T \prod_{p\in\mathbb{P}}{}' p^{-ik_p\tau}\mathrm{d}\tau = \frac{1}{T}\int_0^T \exp\Big\{-i\tau\sum_{p\in\mathbb{P}}{}' k_p\log p\Big\}\mathrm{d}\tau.$$

It is well known that the logarithms $\log p$ of prime numbers are linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, if $\underline{k} \neq \underline{0}$, then

$$\sum_{p\in\mathbb{P}}{}' k_p \log p \neq 0.$$

Thus, after integration, we find that

$$g_T(\underline{k}) = \frac{1 - \exp\Big\{-iT\sum\limits_{p\in\mathbb{P}} k_p\log p\Big\}}{iT\sum\limits_{p\in\mathbb{P}} k_p\log p} \tag{1.5}$$

for $\underline{k} \neq \underline{0}$, Obviously, if $\underline{k} = \underline{0}$, then $g_T(\underline{k}) = 1$. This and (1.5) imply

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq 0. \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the assertion of the lemma follows from the continuity theorem for probability measures on compact topological groups, see for example, Theorem 1.4.2 in [8]. □

In the sequel, one property of weak convergence of probability measures will be useful for us. We recall it. Let P be a probability measure on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$, and $u : \mathbb{X}_1 \to \mathbb{X}_2$. The mapping u is called $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable if $u^{-1}\mathcal{B}(\mathbb{X}_2) \subset \mathcal{B}(\mathbb{X}_1)$. Suppose that u has the latter property. Then the measure P defines the unique probability measure Pu^{-1} on $(\mathbb{X}_2, \mathcal{B}(\mathbb{X}_2))$ by the formula

$$Pu^{-1}(A) = P(u^{-1}A), A \in \mathcal{B}(\mathbb{X}_2).$$

Here $u^{-1}A$ is the preimage of the set A. It is well known that every continuous mapping $u : \mathbb{X}_1 \to \mathbb{X}_2$ is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable. The following lemma often is useful.

Lemma 1.9. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$, $u : \mathbb{X}_1 \to \mathbb{X}_2$ is a continuous mapping, and P_n converges weakly to P as $n \to \infty$. Then also $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$.*

Proof of the lemma can be find in [2], Section 1.5.

Proof of Theorem 1.6. Define the mapping $u_n : \Omega \to H(D)$ by the formula

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s} = \zeta_n(s, \omega; \mathfrak{a}).$$

Since the latter series is absolutely convergent for $\sigma > \frac{1}{2}$, and uniformly on compact subsets of the strip D, the mapping u_n is continuous. Moreover,

$$u_n(p^{-i\tau} : p \in \mathbb{P}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^{s+i\tau}} = \zeta(s + i\tau; \mathfrak{a}).$$

Therefore, from the definitions of Q_T and $P_{T,n}$, we have that, for $A \in \mathcal{B}(H(D))$,

$$P_{T,n}(A) = \frac{1}{T}\text{meas}\big\{\tau \in [0,T] : (p^{-i\tau} : p \in \mathbb{P}) \in u_n^{-1}A\big\}$$
$$= Q_T(u^{-1}A) = Q_T u_n^{-1}(A),$$

i.e., $P_{T,n} = Q_T u_n^{-1}$. This equality, the continuity of u_n, and Lemmas 1.8 and 1.9 show that $P_{T,n}$ converges weakly to the measure $V_n \stackrel{\text{def}}{=} m_H u_n^{-1}$ as $T \to \infty$.

It remains to prove that the measure $\hat{P}_{T,n}$ converges weakly to V_n as well as $T \to \infty$. Define the mapping $\hat{u}_n : \Omega \to H(D)$ by the formula

$$\hat{u}_n(\hat{\omega}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)\hat{\omega}(m)v_n(m)}{m^s} = \zeta(s, \omega\hat{\omega}; \mathfrak{a}), \hat{\omega} \in \Omega.$$

Then, as above, we have that the mapping $\hat{u}_n$ is continuous, and

$$\hat{u}_n(p^{-i\tau} : p \in \mathbb{P}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^{s+i\tau}} = \zeta(s + i\tau, \omega; \mathfrak{a}).$$

Therefore, similarly as in the case of $P_{T,n}$, we find that $\hat{P}_{T,n} = m_H \hat{u}_n^{-1}$. Thus, we have to show that $m_H \hat{u}_n^{-1} = m_H u_n^{-1}$. For this, we will apply the invariance property of the Haar measure m_H. Define the mapping $u : \Omega \to \Omega$ by the formula

$$u(\hat{\omega}) = \omega\hat{\omega}, \omega, \hat{\omega} \in \Omega,$$

Then

$$m_H \hat{u}_n^{-1} = m_H(u_n(u)) = (m_H u^{-1})u_n^{-1} = m_H u_n^{-1} = V_n,$$

since, by invariance of m_H, the equality $m_H = m_H u^{-1}$ holds. □

1.5 Approximation in the mean

To pass from $\zeta_n(s; \mathfrak{a})$ and $\zeta_n(s, \omega; \mathfrak{a})$ to $\zeta(s; \mathfrak{a})$ and $\zeta(s, \omega; \mathfrak{a})$, respectively, we have to show that $\zeta_n(s; \mathfrak{a})$ and $\zeta_n(s, \omega; \mathfrak{a})$ are, in a certain sense, near the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \omega; \mathfrak{a})$, respectively. This is the aim of the present section.

Denote by $\Gamma(s)$ the Euler gamma-function, an define

$$l_n(s) = \frac{s}{\theta}\Gamma\Big(\frac{s}{\theta}\Big)n^s, n \in \mathbb{N},$$

where the fixed number θ is the same as in the definition of $v_n(m)$. We also need the metric in the space $H(D)$ inducing its topology of uniform convergence on compacta. It is well known, see, for example, [3], that there exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\}$ of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. For $g_1, g_2 \in H(D)$, we set

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-l} \frac{\sup\limits_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s\in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is the desired metric in $H(D)$ inducing its topology.

Now, we are ready to state a lemma on the approximation of $\zeta(s; \mathfrak{a})$ by $\zeta_n(s; \mathfrak{a})$ in the mean.

Lemma 1.10. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s + i\tau; \mathfrak{a}), \zeta_n(s + i\tau; \mathfrak{a})\big) \mathrm{d}\tau = 0$$

holds.

Proof. Using the Mellin formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s)a^{-s} \mathrm{d}s = \mathrm{e}^{-a}, a, b > 0,$$

we find that, for $\sigma > \frac{1}{2}$,

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z;\mathfrak{a})l_n(z)\frac{\mathrm{d}z}{z} &= \sum_{m=1}^{\infty}\frac{a_m}{m^s}\left(\frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty}\frac{1}{m^z}l_n(z)\frac{\mathrm{d}z}{z}\right)\\
&= \sum_{m=1}^{\infty}\frac{a_m}{m^s}\left(\frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty}\Gamma\Big(\frac{z}{\theta}\Big)\Big(\frac{m}{n}\Big)^{\left(-\frac{z}{\theta}\right)\theta}\mathrm{d}\Big(\frac{z}{\theta}\Big)\right)\\
&= \sum_{m=1}^{\infty}\frac{a_m}{m^s}\left(\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\Gamma(z)\left(\Big(\frac{m}{n}\Big)^{\theta}\right)^{-z}\mathrm{d}z\right)\\
&= \sum_{m=1}^{\infty}\frac{a_m}{m^s}\exp\left\{-\Big(\frac{m}{n}\Big)^{\theta}\right\} = \zeta_n(s;\mathfrak{a}).
\end{aligned} \tag{1.6}$$

Let $K \subset D$ be an arbitrary compact set. We fix a positive ε such that $\frac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for points $s \in K$. We take $\hat{\theta} > 0$. Then the equality (1.6) and the residue theorem yield

$$\zeta_n(s;\mathfrak{a}) - \zeta(s;\mathfrak{a}) = \frac{1}{2\pi i}\int\limits_{-\hat{\theta}-i\infty}^{-\hat{\theta}+i\infty}\zeta(s+z;\mathfrak{a})l_n(z)\frac{\mathrm{d}z}{z} + R_n(s), \tag{1.7}$$

where

$$R_n(s) = \begin{cases} 0 & \text{if} \quad \sum\limits_{l=1}^{q} a_l = 0, \\ \frac{1}{q}\sum\limits_{l=1}^{q} a_l\frac{l_n(1-s)}{1-s}, & \text{otherwise.}\end{cases}$$

Denote the point of the set K by $s = \sigma + iv$, and suppose that $\hat{\theta} = \sigma - \varepsilon - \frac{1}{2}$. Then, for $s \in K$, we derive from (1.7)

$$\begin{aligned}
&|\zeta(s+i\tau;\mathfrak{a}) - \zeta_n(s+i\tau;\mathfrak{a})|\\
&\leqslant \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left|\zeta(s+i\tau-\hat{\theta}+it;\mathfrak{a})\right|\frac{|l_n(-\hat{\theta}+it)|}{|-\hat{\theta}+it|}\mathrm{d}t + |R_n(s+i\tau)|
\end{aligned}$$

Writing t in place of $v+t$, gives

$$
\begin{aligned}
&|\zeta(s+i\tau;\mathfrak{a})-\zeta_n(s+i\tau;\mathfrak{a})|\\
&\leqslant\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau);\mathfrak{a}\right)\right|\frac{\left|l_n\left(\frac{1}{2}+\varepsilon-s+it\right)\right|}{\left|\frac{1}{2}+\varepsilon-s+it\right|}\mathrm{d}t+|R_T(s+i\tau)|
\end{aligned}
$$

Hence,

$$
\frac{1}{T}\int\limits_{0}^{T}\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a})-\zeta_n(s+i\tau;\mathfrak{a})|\mathrm{d}\tau\ll I_1+I_2, \tag{1.8}
$$

where

$$
I_1=\frac{1}{T}\int\limits_{-\infty}^{\infty}\left(\int\limits_{0}^{T}\left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau);\mathfrak{a}\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\frac{\left|l_n\left(\frac{1}{2}+\varepsilon-s+it\right)\right|}{\left|\frac{1}{2}+\varepsilon-s+it\right|}\mathrm{d}t
$$

and

$$
I_2=\frac{1}{T}\int\limits_{0}^{T}\sup_{s\in K}|R_n(s+i\tau)|\mathrm{d}\tau.
$$

It is well known that uniformly in $\sigma_1\leqslant\sigma\leqslant\sigma_2$

$$
\Gamma(\sigma+it)\ll\exp\{-c|t|\}, c>0.
$$

Therefore, taking $\theta=\frac{1}{2}+\varepsilon$, we find, by the definition of $l_n(s)$,

$$
\begin{aligned}
\frac{\left|l_n\left(\frac{1}{2}+\varepsilon-s+it\right)\right|}{\left|\frac{1}{2}+\varepsilon-s+it\right|}&=\frac{n^{\frac{1}{2}+\varepsilon-\sigma}}{\theta}\left|\Gamma\left(\frac{\frac{1}{2}+\varepsilon-\sigma}{\theta}+\frac{i(t-v)}{\theta}\right)\right|\\
&\ll_K\frac{n^{-\varepsilon}}{\theta}\exp\left\{-\frac{c|t|}{\theta}\right\}\\
&\ll_K n^{-\varepsilon}\exp\{-c_1|t|\}, c_1>0.
\end{aligned} \tag{1.9}
$$

Similarly, we find the estimate

$$
\begin{aligned}
|R_n(s+i\tau)|&\ll_K n^{1-\sigma}\exp\left\{-c\frac{|\tau-v|}{\theta}\right\}\\
&\ll_K n^{1-\sigma}\exp\left\{-c\frac{|\tau|}{\theta}\right\}\ll_K n^{1-\sigma}\exp\{-c_2|\tau|\}, c_2>0.
\end{aligned} \tag{1.10}
$$

It is known [3] that, for $\sigma > \frac{1}{2}$,

$$\int_0^T |\zeta(\sigma + it, \alpha)|^2 \mathrm{d}t \ll T.$$

Hence, using (1.1), we obtain, for $\sigma > \frac{1}{2}$,

$$\int_0^T |\zeta(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t \ll \frac{1}{q^{2\sigma}} \sum_{l=1}^{q} |a_l|^2 \int_0^T \left|\zeta\Big(\sigma + it, \frac{l}{q}\Big)\right|^2 \mathrm{d}t \ll_{\mathfrak{a}} T.$$

Therefore,

$$\begin{aligned}\int_0^T \left|\zeta\Big(\frac{1}{2} + \varepsilon + i(t+\tau); \mathfrak{a}\Big)\right| \mathrm{d}\tau &\leqslant \left(T \int_0^T \left|\zeta\Big(\frac{1}{2} + \varepsilon + i(t+\tau); \mathfrak{a}\Big)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\ll T(1+|t|)^{\frac{1}{2}} \ll T(1+|t|).\end{aligned}$$

This together with (1.9) shows that

$$I_1 \ll_K n^{-\varepsilon} \int_{-\infty}^{\infty} (1+|t|) \exp\{-c_1|t|\} \mathrm{d}t \ll_K n^{-\varepsilon}. \tag{1.11}$$

The estimate (1.10) gives

$$I_2 \ll_K \frac{n^{1-\sigma}}{T} \int_0^T \exp\{-c_2|t|\} \mathrm{d}\tau \ll_K \frac{n^{1-\sigma}}{T} \ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{T}.$$

Thus, in view of (1.8) and (1.11),

$$\frac{1}{T} \int_0^T \sup_{s \in K} |\zeta(s + i\tau; \mathfrak{a}) - \zeta_n(s + i\tau; \mathfrak{a})| \mathrm{d}\tau \ll_K n^{-\varepsilon} + \frac{n^{\frac{1}{2}-2\varepsilon}}{T}.$$

Now, taking $T \to \infty$ and then $n \to \infty$, we obtain that, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \big|\zeta(s+i\tau; \mathfrak{a}), \zeta_n(s+i\tau;\mathfrak{a})\big| \mathrm{d}\tau = 0.$$

This together with the metric ρ proves the lemma. $\square$

The case of the functions $\zeta(s, \omega; \mathfrak{a})$ and $\zeta_n(s, \omega; \mathfrak{a})$ is more complicated because we have not any information about the mean square

$$\int_0^T |\zeta(\sigma + it, \omega; \mathfrak{a})|^2 \mathrm{d}t, \sigma > \frac{1}{2}.$$

To obtain an estimate for the above mean square, we will apply some elements of the ergodic theory.

Let, for brevity,

$$a_\tau = (p^{-i\tau} : p \in \mathbb{P}), \tau \in \mathbb{R}.$$

Then $\{a_\tau : \tau \in \mathbb{R}\}$ is an one-parametric group. Define the transformation φ_τ on the torus Ω by

$$\varphi_\tau(\omega) = a_\tau \omega, \omega \in \Omega.$$

Since the Haar measure m_H is invariant with respect to translations by point of Ω, we have that $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is an one-parameter group of measurable, measure preserving transformations on Ω. Recall that a set $A \in \mathcal{B}(\Omega)$ is called invariant with respect to the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ if, for every $\tau \in \mathbb{R}$, the sets A and $A_\tau = \varphi_\tau(A)$ may differ one from another at most by a set of m_H-measure zero. All invariant sets form a σ-field that is a sub-σ-field of $\mathcal{B}(\Omega)$. The group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is called ergodic if its σ-field of invariant sets consists only of the sets of m_H-measure zero or one.

Lemma 1.11. *The group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof. The lemma already was used in the theory of the Riemann zeta-function, see, for example [22]. However, for fullness, we will present its modified proof.

Let $\chi : \Omega \to \gamma$ be a character of the group Ω. We have seen in the proof of Lemma 1.8 that

$$\chi(\omega) = {\prod_{p \in \mathbb{P}}}' \omega^{k_p}(p),$$

where "$'$" means that only a finite number of integers k_p are distinct from zero. Suppose that χ is a non-trivial character ($\chi(\omega) \not\equiv 1$). Then we have

$$\chi(a_\tau) = {\prod_{p \in \mathbb{P}}}' p^{i\tau k_p} = \exp\Big\{ - i\tau {\sum_{p \in \mathbb{P}}}' k_p \log p \Big\}.$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, there exists a real number $\tau_0 \neq 0$ such that

$$\chi(a_{\tau_0}) \neq 1. \tag{1.12}$$

Let A be an invariant set of the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$, and let I_A be the indicator function of A. Then, by the definition of invariant set, we have that, for almost all $\omega \in \Omega$,

$$I_A(a_\tau \omega) = I_A(\omega). \tag{1.13}$$

Denote by $\hat{I}_A$ Fourier transform of I_A. Then, in view of invariance of the measure m_H and (1.13), we find that

$$\begin{aligned}\hat{I}_A(\chi) &= \int\limits_\Omega \chi(\omega) I_A(\omega) m_H(\mathrm{d}\omega) \\ &= \int\limits_\Omega \chi(a_{\tau_0}\omega) I_A(a_{\tau_0}\omega) m_H(\mathrm{d}\omega) \\ &= \chi(a_{\tau_0}) \int\limits_\Omega \chi(\omega) I_A(\omega) m_H(\mathrm{d}\omega) = \chi(a_{\tau_0}) \hat{I}_A(\chi).\end{aligned}$$

Hence, by (1.12), we find that

$$\hat{I}_A(\chi)\big(1 - \chi(a_{\tau_0})\big) = 0$$

implies the equality

$$\hat{I}_A(\chi) = 0 \tag{1.14}$$

for all non-trivial characters χ of the group Ω. Now let χ_0 be the trivial character ($\chi_0(\omega) \equiv 0$) of Ω. Suppose that $\hat{I}_A(\chi_1) = a$. Then, using the orthogonality of characters,

$$\int\limits_\Omega \chi(\omega) m_H(\mathrm{d}\omega) = \begin{cases} 1 & \text{if} \quad \chi = \chi_0, \\ 0 & \text{if} \quad \chi \neq \chi_0, \end{cases}$$

and (1.14), we obtain that, for every character χ of the group Ω,

$$\hat{I}_A(\chi) = a \int\limits_\Omega \chi(\omega) m_H(\mathrm{d}\omega) = a\hat{1}\chi = \hat{a}(\chi).$$

Since $I_A(\omega)$ is uniquely determined by the Fourier transform $\hat{I}_A(\chi)$, from this it follows that $I_A(\omega) = a$ for almost all $\omega \in \Omega$. However, $I_A(\omega)$ is the indicator function, thus, $a = 0$, or $a = 1$. Therefore, either $I_A(\omega) = 0$, or $I_A(\omega) = 1$ for almost all $\omega \in \Omega$. This shows that either $m_H(A) = 0$, or $m_H(A) = 1$, and the lemma is proved. $\square$

Also, we will use the notion of the ergodic process. Let $X(t, \omega)$, $t \in \mathcal{T}$, be a random process defined on a certain probability space with measure P. Let $t_1, t_1, \dots, t_n$ be arbitrary values of t. Then the family of distributions

$$P(X(t_1, \omega) < x_1, \dots, X(t_n, \omega) < x_n), n \in \mathbb{N},$$

is called a family of finite-dimensional distributions of $X(t, \omega)$. Moreover, let $\mathbb{Y}$ be a space of all functions, $t \in \mathcal{T}$. Then the family of finite-dimensional distributions of $X(t, \omega)$ defines a probability measure Q on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$, and, on the probability space $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}), \mathbb{Q})$, the translation $g_u : y(t) \to y(t+u)$, $y \in \mathbb{Y}$, can be defined random process is said to be strongly stationary if its finite-dimensional distributions are invariant with respect to translations g_u. If a process is strongly stationary, then the translation g_u is measure preserving, i.e., for each $A \in \mathcal{B}(\mathbb{Y})$ and $u \in \mathbb{R}$, the equality $Q(A) = Q(A_u)$ holds, where $A_u = g_u(A)$.

A set $A \in \mathcal{B}(\mathbb{Y})$ is called invariant of the process if, for each u, the sets A and A_u can differ one from another at most by a set of Q-measure zero. All invariant sets form a σ-field wich is a sub-σ-field of the σ-field $\mathcal{B}(\mathbb{Y})$. A strongly stationary random process is ergodic if its σ-field of invariant sets consists only if the sets having Q-measure 0 or 1.

Now, we state the classical Birkhoff-Khintchine theorem for ergodic processes

Lemma 1.12. *Suppose that $X(t, \omega)$ is ergodic process, $\mathbb{E}|X(t, \omega)| < \infty$, with sample paths integrable in the Riemann sense over every finite interval. Then, for almost all ω,*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, \omega) \mathrm{d}t = \mathbb{E}X(0, \omega).$$

Proof of the lemma can be found, for example, in [5].

Lemma 1.12 allows to estimate the mean square for $\zeta(s, \omega; \mathfrak{a})$.

Lemma 1.13. *Suppose that $\sigma > \frac{1}{2}$ is fixed. Then, for almost all $\omega \in \Omega$,*

$$\int_0^T |\zeta(\sigma + it, \omega; \mathfrak{a})|^2 \mathrm{d}t \ll T, T \to \infty.$$

Proof. By the definition of $\zeta(s, \omega; \mathfrak{a})$,

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} X_m,$$

where

$$X_m = X(s, \omega) = \frac{a_m \omega(m)}{m^s}.$$

We have seen in the proof of Lemma 1.3 that X_m, $m \in \mathbb{N}$, are pairwise orthogonal random variables such that

$$\mathbb{E}|X|^2 = \frac{|a_m|^2}{m^{2\sigma}}.$$

Since $|a_m| \leqslant c_{\mathfrak{a}}$, we have that, for $\sigma > \frac{1}{2}$,

$$\sum_{m=1}^{\infty} \mathbb{E}|X_m|^2 < \infty.$$

Therefore, using the orthogonality and applying the Perseval identity, we find that, for $\sigma > \frac{1}{2}$,

$$\mathbb{E}|\zeta(\sigma, \omega; \mathfrak{a})|^2 = \sum_{m=1}^{\infty} \mathbb{E}|X_m|^2 < \infty. \tag{1.15}$$

In the view of Lemma 1.11, the group $\{\varphi_t : t \in \mathbb{R}\}$ is ergodic. Therefore, the random process $\zeta(\sigma, \varphi_t(\omega); \mathfrak{a})$ is ergodic as well. Hence, by Lemma 1.12 and (1.15), for $\sigma > \frac{1}{2}$ and almost all $\omega \in \Omega$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\zeta(\sigma + it, \omega; \mathfrak{a})|^2 \mathrm{d}t = \mathbb{E}|\zeta(\sigma, \omega; \mathfrak{a})|^2 < \infty$$

because $\zeta(\sigma, \varphi_t(\omega); \mathfrak{a}) = \zeta(\sigma + it, \omega; \mathfrak{a})$. Hence, for $\sigma > \frac{1}{2}$ and almost all

$\omega \in \Omega$,
$$\int_0^T |\zeta(\sigma + it, \omega; \mathfrak{a})|^2 \mathrm{d}t \ll T, T \to \infty.$$

$\square$

Now, we are in position to prove an analogue of Lemma 1.10 for the functions $\zeta(\sigma, \omega; \mathfrak{a})$ and $\zeta_n(\sigma, \omega; \mathfrak{a})$.

Lemma 1.14. *For almost all $\omega \in \Omega$, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s + i\tau, \omega; \mathfrak{a}), \zeta_n(s + i\tau, \omega; \mathfrak{a})\big) \mathrm{d}\tau = 0$$

holds.

Proof. We repeat the proof of Lemma 1.10 and use the estimate

$$\int_0^T \left|\zeta\left(\frac{1}{2} + \varepsilon + i(t + \tau), \omega; \mathfrak{a}\right)\right| \mathrm{d}t \ll T(1 + |t|)$$

which is implied, for all $t \in \mathbb{R}$ and almost all $\omega \in \Omega$, by Lemma 1.13. $\square$

1.6 Limit theorems for $\zeta(s; \mathfrak{a})$ and $\zeta(s, \omega; \mathfrak{a})$

In this section, together with P_T we consider the measure

$$P_{T,\Omega}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \zeta(s + i\tau, \omega; \mathfrak{a}) \in A\}, a \in \mathcal{B}(H(D)).$$

Some of assertions will be true for almost all $\omega \in \Omega$, however, this has no any influence for final results, therefore, we often will omit phrase "for almost all $\omega \in \Omega$".

Theorem 1.15. *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure P such that the measures P_T and $P_{T,\Omega}$ both converges to P as $T \to \infty$.*

In the proof of Theorem 1.15, we will use two notions of the weak convergence of probability measures. Let $\{P\}$ be a family of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. The family $\{P\}$ is called relatively compact if every sequence

$\{P_n\} \subset \{P\}$ contains a subsequence $\{P_{n_k}\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $k \to \infty$. The family $\{P\}$ is tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$. The notions of the relative compactness and tightness are connected by the Prokhorov theorem which we state as the following lemma

Lemma 1.16. *If the family $\{P\}$ is tight, then it is relatively compact.*

Proof of the lemma is given in [2], Theorem 6.1.

Sometimes, in place of the weak convergence of probability measures it is convenient to use the notion of the convergence in distribution. We recall that the random element X_n converges to X in distribution as $n \to \infty$ $\left(X_n \xrightarrow[n\to\infty]{\mathcal{D}} X\right)$ if the distribution of X_n converges weakly to the distribution of X as $n \to \infty$.

The next lemma is very important for the proof of Theorem 1.15.

Lemma 1.17. *Suppose that the space $(\mathbb{X}, d)$ is separable, the $\mathbb{X}$-valued elements Y_n, X_{1n}, X_{2n}, …, $n \in \mathbb{N}$, are defined on the same probability space with measure μ, for any $k \in \mathbb{N}$,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

Moreover, if, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(X_{kn}, Y_n) \geqslant \varepsilon\} = 0,$$

then

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

The lemma is Theorem 4.2 in [2], where its proof is given.

We recall that V_n is the limit measure in Lemma 1.9

Lemma 1.18. *The sequence $\{V_n : n \in \mathbb{N}\}$ is tight.*

Proof. Let ξ be a random variable defined on a certain probability space with measure μ, and uniformly distributed on $[0,1]$. Define the $H(D)$-valued random element $X_{T,n}$ by the formula

$$X_{T,n} = X_{T,n}(s) = \zeta_n(s + i\xi T; \mathfrak{a}).$$

Moreover, let X_n be the $H(D)$-valued random element having the distribution V_n. Then the assertion of Lemma 1.9 can be written as

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n. \tag{1.16}$$

The series for $\zeta_n(s;\mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$. Therefore, by the well-known property of Dirichlet series, we know, for $\sigma > \frac{1}{2}$, that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma+it;\mathfrak{a})|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)}{m^{2\sigma}} \leqslant \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} < \infty$$

for all $n \in \mathbb{N}$. Consequently, for $\sigma > \frac{1}{2}$,

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma+it;\mathfrak{a})| \mathrm{d}t \leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \left(\frac{1}{T} \int_0^T |\zeta_n(\sigma+it;\mathfrak{a})|^2 \mathrm{d}t \right)^{\frac{1}{2}} < \infty.$$

Thus, for all $n \in \mathbb{N}$ and $\sigma > \frac{1}{2}$,

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma+it;\mathfrak{a})| \mathrm{d}t \leqslant C < \infty. \tag{1.17}$$

Let K_l be compact set from the definition of the metric ρ. Then an application of the Cauchy integral formula and (1.17) shows that, for all $n \in \mathbb{N}$,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau;\mathfrak{a})| \mathrm{d}\tau \leqslant C_l < \infty. \tag{1.18}$$

For an arbitrary fixed $\varepsilon > 0$, let $M_l = M_l(\varepsilon) = 2^l C_l \varepsilon^{-1}$. Then, in view of

(1.18), we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{T\to\infty} \mu\Big\{ \sup_{s\in K_l} |X_{T,n}(s)| > M_l \Big\} \\
&= \limsup_{T\to\infty} \frac{1}{T}\text{meas}\Big\{ \tau \in [0,T] : \sup_{s\in K_l} |\zeta_n(s+i\tau;\mathfrak{a})| > M_l \Big\} \\
&\leqslant \limsup_{T\to\infty} \frac{1}{TM_l} \int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau;\mathfrak{a})| \mathrm{d}\tau \leqslant \frac{C_l}{TM_l} = \frac{\varepsilon}{2^l}.
\end{aligned}$$

Hence, in virtue of (1.16),

$$\mu\Big\{ \sup_{s\in K_l} |X_n(s)| > M_l \Big\} \leqslant \frac{\varepsilon}{2^l}. \tag{1.19}$$

Define the set $K = K(\varepsilon) = \Big\{ g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_l, l \in \mathbb{N} \Big\}$. Then K is a compact set in the space $H(D)$. Moreover, by (1.19), for all $n \in \mathbb{N}$,

$$\mu(X_n \in K) \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon.$$

Since V_n is the distribution of X_n, this shows that

$$V_n(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, i.e., the sequence $\{V_n : n \in \mathbb{N}\}$ is tight. □

Proof of Theorem 1.15. By Lemma 1.18, the sequence $\{V_n : n \in \mathbb{N}\}$ is tight, hence, in view of Lemma 1.16, it is relatively compact. Therefore, there exists a subsequence $\{V_{n_k}\} \subset \{V_n\}$ such that V_{n_k} converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. Hence, using the notation of Lemma 1.18, we have that

$$X_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{1.20}$$

Define one more $H(D)$-valued random element $Y_T = Y_T(s)$ by the formula

$$Y_T(s) = \zeta(s + i\xi T; \mathfrak{a}).$$

Then Lemma 1.10 implies, for every $\varepsilon > 0$, the equality

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\{\rho(Y_T, X_{T,n}) \geqslant \varepsilon\} \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \text{meas}\big\{\tau \in [0,T] : \rho(\zeta(s+i\tau;\mathfrak{a}), \zeta_n(s+i\tau;\mathfrak{a})) \geqslant \varepsilon\big\} \\
&\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \rho(\zeta(s+i\tau;\mathfrak{a}), \zeta_n(s+i\tau;\mathfrak{a}))\mathrm{d}\tau = 0.
\end{aligned} \tag{1.21}$$

Now, the relations (1.16) and (1.20) together with equality (1.21) show that all conditions of Lemma 1.16 are satisfied. Therefore

$$Y_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{1.22}$$

in other words, the measure P_T converges weakly to P as $T \to \infty$. Moreover, the relation (1.22) shows that the limit measure P is independent of the sequence $\{X_{n_k}\}$. Since the sequence $\{X_n\}$ is relatively compact, we deduce from this that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{1.23}$$

It remains to prove that the measure $P_{T,\Omega}$ also converges weakly to the measure P as $T \to \infty$. For this, we define two $H(D)$-valued random elements $X_{T,n,\Omega} = X_{T,n,\Omega}(s)$ and $Y_{T,\Omega} = Y_{T,\Omega}(s)$ by the formulas

$$X_{T,n,\Omega}(s) = \zeta_n(s + i\xi T, \omega; \mathfrak{a})$$

and

$$Y_{T,\Omega}(s) = \zeta(s + i\xi T, \omega; \mathfrak{a}).$$

Then, by Lemma 1.9, we have that

$$X_{T,n,\Omega} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \tag{1.24}$$

and Lemma 1.14 implies, for every $\varepsilon > 0$, the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mu\{\rho(Y_{T,\Omega}, X_{T,n,\Omega}) \geqslant \varepsilon\} = 0. \tag{1.25}$$

From (1.23)-(1.25) and Lemma 1.16, it follows that

$$Y_{T,\Omega} \xrightarrow[T\to\infty]{\mathcal{D}} P,$$

and this is equivalent to weak convergence of $P_{T,\Omega}$ to P as $T \to \infty$. The theorem is proved. $\square$

1.7 Proof of the first part of Theorem 1.6

In this section, we will prove that P_T converges weakly to the measure P_ζ, in other words, we will identify the limit measure P in Theorem 1.15.

For this, we recall an equivalent of weak convergence in terms of continuity sets. We remind that $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A.

Lemma 1.19. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 of [2].

Let A be an arbitrary fixed continuity set of the limit measure P. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define a random variable η by the formula

$$\eta(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(s, \omega; \mathfrak{a}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously,

$$\mathbb{E}_\eta = \int\limits_\Omega \eta \mathrm{d} m_H = m_H\{\omega \in \Omega : \zeta(s, \omega; \mathfrak{a}) \in A\} = P_\zeta(A). \tag{1.26}$$

Moreover, by Theorem 1.15 and Lemma 1.19, we have the equality

$$\begin{aligned} \lim_{T\to\infty} P_{T,\Omega}(A) &= \lim_{T\to\infty} \frac{1}{T} \text{meas}\{\tau \in [0,T] : \zeta(s+i\tau, \omega; \mathfrak{a}) \in A\} \\ &= P(A). \end{aligned} \tag{1.27}$$

Since, in view of Lemma 1.11, the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic, we have that the random process $(\varphi_\tau(\omega))$ is ergodic as well. Therefore, Lemma 1.12 shows that

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \eta(\varphi_\tau(\omega))\mathrm{d}\tau = \mathbb{E}\eta. \tag{1.28}$$

However, by the definition of φ_τ and η,

$$\begin{aligned} \frac{1}{T} \int\limits_0^T \eta(\varphi_\tau(\omega))\mathrm{d}\tau &= \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s, \varphi_\tau(\omega); \mathfrak{a}) \in A\} \\ &= \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s + i\tau, \omega; \mathfrak{a}) \in A\}. \end{aligned}$$

This, (1.28) and (1.26) show that

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta(s + i\tau, \omega; \mathfrak{a}) \in A\} = P(A).$$

Hence, in view of (1.27), we obtain that $P(A) = P_\zeta(A)$. Since A was an arbitrary continuity set of the measure P, we have that $P(A) = P_\zeta(A)$ for all continuity sets A of P. However, it is known [2] that all continuity sets constitute a determining class. Therefore, the equality $P(A) = P_\zeta(A)$ holds for every $A \in \mathcal{B}(H(D))$. Thus, we have that $P = P_\zeta$, and the first part of Theorem 1.6 is proved.

1.8 The support of the measure P_ζ

In this section, we will prove that the support of the limit measure in Theorem 1.4 is the set

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

We start with several statements of known results.

Recall that the support of the distribution of a random element X is called a support of X, and will be denoted by S_X.

Lemma 1.20. *Suppose that $\{X_m : m \in \mathbb{N}\}$ is a sequence of independent*

$H(D)$-valued random elements such that the series

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely. Then the support of the sum of this series is equal to the closure of the set of all $g \in H(D)$ that can be written as the sum of a convergent series

$$g = \sum_{m=1}^{\infty} g_m, g_m \in S_{X_m}.$$

The lemma is Theorem 1.7.10 of [22].

Lemma 1.21. *Suppose that the sequence $\{g_m : m \in \mathbb{N}\} \subset H(D)$ satisfies the following conditions:*

1° If μ is a complex-valued Borel measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D such that

$$\left| \sum_{m=1}^{\infty} g_m d\mu \right| < \infty,$$

then

$$\int\limits_{\mathbb{C}} s^l \mathrm{d}\mu(s) = 0$$

for all $l \in \mathbb{N}_0$;

2° For every compact subset $K \subset D$,

$$\sum_{m=1}^{\infty} \sup_{s \in K} |g_m(s)|^2 < \infty;$$

3° The series

$$\sum_{m=1}^{\infty} g_m$$

is convergent in $H(D)$.

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m g_m$$

with $|a_m| = 1$ is dense in $H(D)$.

The lemma is Theorem 6.3.10 of [22].

Lemma 1.22. *Let μ be a complex-valued Borel measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_0\}$, and*

$$g(s) = \int\limits_{\mathbb{C}} \mathrm{e}^{sz} \mathrm{d}\mu(z).$$

If $g(s) \not\equiv 0$, then

$$\limsup_{x \to \infty} \frac{\log |g(x)|}{x} > \sigma_0.$$

The lemma is Lemma 6.4.10 from [22].

Recall that an analytic function $g(s)$ in an angular region $|\arg s| \leqslant \theta_0$, $0 < \theta_0 \leqslant \pi$, is called a function of exponential type if

$$\limsup_{r \to \infty} \frac{\log |g(r\mathrm{e}^{i\theta})|}{r} < \infty$$

uniformly in θ, $|\theta| \leqslant \theta_0$.

Lemma 1.23. *Suppose that $g(s)$ is a function of exponential type, and*

$$\limsup_{x \to \infty} \frac{\log |g(x)|}{x} > -1.$$

Then, for all coprime l an q,

$$\sum_{p \equiv l \bmod q} |g(\log p)| < \infty$$

The lemma is Lemma 4.1 of [28].

We also recall the Hurwitz theorem which we state as the next lemma.

Lemma 1.24. *Suppose that $\{g_n(s) : n \in \mathbb{N}\}$ is a sequence of analytic functions in a region $\mathcal{G}$ bounded by a simple closed contour, and that*

$$\lim_{n \to \infty} g_n(s) = g(s)$$

uniformly in $\mathcal{G}$, where $g(s) \not\equiv 0$. Then an interior point s_0 of $\mathcal{G}$ is a zero of $g(s)$ if and only if here exist a sequence $\{s_n\} \subset \mathcal{G}$ such that $s_n \to s_0$ as $n \to \infty$, and $g(s_n) = 0$ for $n > n_0 = n_0(s_0)$.

Proof of lemma is given in [51].

Now, we are ready to prove the second statement of Theorem 1.6 on the support of the measure P_ζ.

Lemma 1.25. *The support of the measure P_ζ is the set S.*

Proof. By Proposition 1.5, we have that

$$\zeta(s,\omega;\mathfrak{a}) = \prod_{p\in\mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_p l \omega^l(p)}{p^{ls}}\right),$$

where, for almost all $\omega \in \Omega$, the product converges uniformly on compact subsets of D. Let p_0 be such that, for $p > p_0$,

$$\left|\sum_{l=1}^{\infty} \frac{a_p l \omega^l(p)}{p^{ls}}\right| \leqslant \frac{1}{2}$$

for all $s \in D$. Such a number p_0 exists because

$$\left|\sum_{l=1}^{\infty} \frac{a_p l \omega^l(p)}{p^{ls}}\right| \leqslant \sum_{l=1}^{\infty} \left|\frac{a_p l \omega^l(p)}{p^{ls}}\right| \leqslant c_{\mathfrak{a}} \sum_{l=1}^{\infty} \frac{1}{p^{\frac{l}{2}}} = c_{\mathfrak{a}} \frac{\frac{1}{\sqrt{p}}}{1 - \frac{1}{\sqrt{p}}} = \frac{c_{\mathfrak{a}}}{\sqrt{p} - 1}.$$

Next, we consider, for $\hat{p}_0 > p_0$, the product

$$\prod_{p>\hat{p}_0} \left(1 + \sum_{l=1}^{\infty} \frac{a_p l \omega^l(p)}{p^{ls}}\right).$$

For brevity, let, as in the proof of Proposition 1.5,

$$x_p(s,\omega) = \sum_{l=1}^{\infty} \frac{a_{p^l} \omega^l(p)}{p^{ls}}, p > \hat{p}_0$$

For $|z| < 1$, define

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots .$$

Then the functions $\log(1 + x_p(s,\omega))$ are well defined for $s \in D$. In the proof of Proposition 1.5, it was obtained, that

$$\sum_{p\in\mathbb{P}} x_p(s,\omega)$$

converges uniformly on compact sets of D for almost all $\omega \in \Omega$. Therefore, there exists a sequence $\mathfrak{b} = \{b_p : |b_p| = 1\}$ such that the series

$$\sum_{p > \hat{p}_0} x_p(s, \mathfrak{b}) \tag{1.29}$$

converges in the space $H(D)$. Moreover, it was observed in the proof of Proposition 1.5 that

$$\sum_{p \in \mathbb{P}} |x_p(s, \omega)|^2$$

converges uniformly on compact sets of the strip D. Thus, for every compact set $K \in D$,

$$\sum_{p > \hat{p}_0} \sup_{s \in K} |x_p(s, \mathfrak{b})|^2 < \infty.$$

This and the convergence of the series (1.29) show that the conditions 2° and 3° of Lemma 1.21 are satisfied by the sequence $\{x_p(s, \mathfrak{b})\}$. It remains to check the condition 1°.

Suppose that μ is a complex-valued Borel measure $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D such that

$$\sum_{p > \hat{p}_0} \left| \int_{\mathbb{C}} x_p(s, \mathfrak{b}) \mathrm{d}\mu(s) \right| < \infty. \tag{1.30}$$

Write

$$x_p(s, \mathfrak{b}) = \frac{a_p b_p}{p^s} + y_p(s, \mathfrak{b}),$$

where, by the proof of Proposition 1.5,

$$\sum_{p > \hat{p}_0} |x_p(s, \mathfrak{b}) - y_p(s, \mathfrak{b})|$$

converges uniformly on compact subsets of D. Thus, in view of (1.30),

$$\sum_{p > \hat{p}_0} \left| a_p \int_{\mathbb{C}} \frac{1}{p^s} \mathrm{d}\mu(s) \right| < \infty.$$

Hence, by the periodicity of the sequence $\mathfrak{a}$, we find that

$$\sum_{\substack{p > \hat{p}_0 \\ p \equiv l (\mathrm{mod} q)}} \left| a_l \int_{\mathbb{C}} \frac{1}{p^s} \mathrm{d}\mu(s) \right| < \infty \tag{1.31}$$

for all $l = 1, \dots, q$, $(l, q) = 1$. Since the sequence $\mathfrak{a}$ is multiplicative, we have that $a_1 = 1$. Therefore, (1.31) implies the inequality

$$\sum_{\substack{p>\hat{p}_0 \\ p\equiv 1(\mathrm{mod} q)}} |\rho(\log p)| < \infty, \tag{1.32}$$

where $\rho(z) = \int\limits_{\mathbb{C}} e^{-sz} \mathrm{d}\mu(s)$, $l = 1, ..., q$. The function $\rho(z)$ is of exponential type, therefore, in virtue of Lemma 1.22, we have that either $\rho(z) \equiv 0$, or

$$\limsup_{x\to\infty} \frac{\log|\rho(x)|}{x} > -1.$$

If the latter in equality holds, by Lemma 1.22

$$\sum_{\substack{p>\hat{p}_0 \\ p\equiv 1(\mathrm{mod} q)}} |\rho(\log p)| = \infty, \tag{1.33}$$

and this is contradicts (1.32). Thus, we have that $\rho(z) \equiv 0$, i.e.,

$$\int\limits_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\mu(s) \equiv 0.$$

Differentiating the latter equality m times and then taking $z = 0$, we find that

$$\int\limits_{\mathbb{C}} s^m \mathrm{d}\mu(s) = 0$$

for all $m \in \mathbb{N}_0$. This means that the condition 1° of Lemma 1.21 also holds for the sequence $\{x_p(s, \mathfrak{b}) : p > \hat{p}_0\}$. Therefore, the set of all convergent series

$$\sum_{p>\hat{p}_0} \hat{b}(p) x_p(s, \mathfrak{b}) \tag{1.34}$$

with $|\hat{b}(p)| = 1$, $p > \hat{p}_0$, is dense in the space $H(D)$.

Let $x_0(s)$ be an arbitrary point of $H(D)$, $\varepsilon > 0$ is an arbitrary number, and

$K \subset D$ be an arbitrary compact set. We have that, for $p > \hat{p}_0$,

$$\log(1 + x_p(s,\omega)) = \sum_{k=1}^{\infty}\left(\sum_{l=1}^{\infty}\frac{a_{p^l}a^l(p)}{p^{ls}}\right)^k\frac{(-1)^{k-1}}{k}$$
$$= \frac{a_p\omega(p)}{p^s} + \sum_{l=2}^{\infty}\frac{a_{p^l}l\omega^l(p)}{p^{ls}} + \sum_{k=2}^{\infty}\left(\sum_{l=1}^{\infty}\frac{a_{p^l}\omega^l(p)}{p^{ls}}\right)^k\frac{(-1)^{k-1}}{k}.$$

Then there exist $\hat{p}_0$ such that

$$\sup_{s\in K}\left|\sum_{p>\hat{p}_0}\left(\sum_{l=2}^{\infty}\frac{a_{p^l}a^l(p)}{p^{ls}} + \sum_{k=2}^{\infty}\left(\sum_{l=1}^{\infty}\frac{a_{p^l}a^l(p)}{p^{ls}}\right)^k\frac{(-1)^{k-1}}{k}\right)\right| < \frac{\varepsilon}{2} \tag{1.35}$$

with every $\mathfrak{a} = \{a(p) : |a(p)| = 1\}$. The denseness of the set of the series (1.34) implies that there exists $\hat{\mathfrak{a}} = \{\hat{a}(p) : |\hat{a}(p)| = 1\}$ such that

$$\sup_{s\in K}\left|x_o(s) - \sum_{p_0\leqslant p<\hat{p}_0}\log\big(1 + x_p(s,\mathfrak{b})\big) - \sum_{p>\hat{p}_0}\hat{a}(p)x_p(s,\mathfrak{b})\right| < \frac{\varepsilon}{2}. \tag{1.36}$$

Now, let

$$a(p) = \begin{cases} \hat{a}(p)b(p) & \text{if} \quad p > \hat{p}_0, \\ b(p) & \text{if} \quad p_0\leqslant p < \hat{p}_0. \end{cases}$$

Then we deduce from (1.35) and (1.36) that

$$\sup_{s\in K}\left|x_o(s) - \sum_{p>\hat{p}_0}\log\big(1 + x_p(s,a)\big)\right|$$
$$\leqslant \sup_{s\in K}\left|x_o(s) - \sum_{p_0\leqslant p<\hat{p}_0}\log\big(1 + x_p(s,\mathfrak{b})\big) - \sum_{p>\hat{p}_0}\hat{a}(p)x_p(s,\mathfrak{b})\right|$$
$$+ \sup_{s\in K}\left|\sum_{p>\hat{p}_0}\left(\sum_{l=2}^{\infty}\frac{a_p l a^l(p)}{p^{ls}} + \sum_{k=2}^{\infty}\left(\sum_{l=1}^{\infty}\frac{a_{p^l}a^l(p)}{p^{ls}}\right)^k\frac{(-1)^{k-1}}{k}\right)\right| < \varepsilon.$$

This shows that the set of all convergent series

$$\sum_{p>p_0}\log\big(1 + x_p(s,\hat{\mathfrak{a}})\big) \tag{1.37}$$

with $\hat{\mathfrak{a}} = \{a(p) : |a(p)| = 1\}$ is dense in the space $H(D)$.

We already have mention that $\{\omega(p) : p \in \mathbb{P}\}$ is a sequence of independent random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Hence

$$\left\{ \log\left(1 + x_p(s,\omega)\right) : p \in \mathbb{P} \right\}$$

is a sequence of independent $H(D)$-valued elements on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. The support of each random variable $\omega(p)$ is the unit circle on the complex plane. Therefore, the set

$$\left\{ g \in H(D) : g(s) = \log\left(1 + x_p(s, \hat{\mathfrak{a}})\right) \right\}$$

with $\hat{\mathfrak{a}} = \{a(p) : |a(p)| = 1\}$ is the support of the $H(D)$-valued random element

$$\log\left(1 + x_p(s,\omega)\right).$$

Consequently, by Lemma 1.20, the support of the random element

$$\sum_{p>p_0} \log\left(1 + x_p(s,\omega)\right) \tag{1.38}$$

is the closure of the set if all convergent series (1.37). Since this set is dense in $H(D)$, we have that the support of the random element (1.38) in the whole of $H(D)$.

Now, let the function $u : H(D) \to H(D)$ be given by the formula

$$u(g) = \mathrm{e}^g, g \in H(D).$$

Then u is a continuous function sending

$$\sum_{p>p_0} \log\left(1 + x_p(s,\omega)\right)$$

to

$$\prod_{p>p_0} \log\left(1 + x_p(s,\omega)\right), \tag{1.39}$$

and mapping $H(D)$ onto $S \backslash \{0\}$. This shows that the support of the random element (1.39) contains the set $S \backslash \{0\}$. However, the support of the random elements (1.39) is a closed set. By Lemma 1.23, the closure of the set $S \backslash \{0\}$ is S. This, the support of the random element (1.39) contains the set S.

The product (1.39) consists of non-zero factors and converges uniformly on compact subsets of the strip D for almost all $\omega \in \Omega$. Therefore, in view of Lemma 1.23 again, the set S contains the support of the random element (1.39). This and the opposite inclusion shows that the support of the random element (1.39) is the set S.

Write

$$\prod_{p\in\mathbb{P}} \big(1 + x_p(s,\omega)\big) = X_1 X_2,$$

where

$$X_1 = \prod_{p\leqslant p_0} \big(1 + x_p(s,\omega)\big), X_2 = \prod_{p>p_0} \big(1 + x_p(s,\omega)\big).$$

Then we have that X_1 and X_2 are independent random elements. Since the product is uniformly convergent on compact subsets of D for almost all $\omega \in \Omega$, the random element X_1 is not degenerated at zero. Hence, the support of X_1X_2 is the same as X_2, i.e., it is the set S. The lemma is proved. □

Proof of Theorem 1.6. The theorem follows from Section 1.7 and Lemma 1.25. □

1.9 Proof of Theorem 1.1

We recall one more equivalent of weak convergent of probability measures, in this case, in terms of open sets.

Lemma 1.26. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every open set $\mathcal{G}$ of $\mathbb{X}$,*

$$\liminf_{n\to\infty} P_n(\mathcal{G}) \geqslant P(\mathcal{G}).$$

The lemma is a part of Theorem 2.1 of [2].

We also will use the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 1.27. *Let $K \subset \mathbb{C}$ be a compact set with connected complements, and the function $f(s)$ be continuous on K and analytic in the interior if K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof of the lemma can be found in [38].

Proof of Theorem 1.1. The case of "$\liminf$". Since the function $f(s)$ is non-vanishing on K, by Lemma 1.27, there exist a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s)-\mathrm{e}^{p(s)}|<\frac{\varepsilon}{2}. \tag{1.40}$$

Define the set

$$\mathcal{G}_\varepsilon=\Big\{g\in H(D):\sup_{s\in K}|g(s)-\mathrm{e}^{p(s)}|<\frac{\varepsilon}{2}\Big\}.$$

Then $\mathcal{G}_\varepsilon$ is an open neighborhood of $\mathrm{e}^{p(s)}$ which, in view of Lemma 1.25, is an element of the support of the measure P_ζ. Thus

$$P_\zeta(\mathcal{G}_\varepsilon)>0.$$

Hence, by Theorem 1.6 and Lemma 1.26,

$$\liminf_{T\to\infty}P_T(\mathcal{G}_\varepsilon)\geqslant P_\zeta(\mathcal{G}_\varepsilon)>0.$$

This and the definition of P_T and $\mathcal{G}_\varepsilon$ give the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}\big|\zeta(s+i\tau;\mathfrak{a})-e^{p(s)}\big|<\frac{\varepsilon}{2}\Big\}>0. \tag{1.41}$$

Suppose that $\tau\in\mathbb{R}$ satisfy the inequality

$$\sup_{s\in K}\big|\zeta(s+i\tau;\mathfrak{a})-e^{p(s)}\big|<\frac{\varepsilon}{2}.$$

Then, for these τ, taking into account (1.40), we find

$$\sup_{s\in K}\big|\zeta(s+i\tau;\mathfrak{a})-f(s)\big|\leqslant\sup_{s\in K}\big|\zeta(s+i\tau;\mathfrak{a})-e^{p(s)}\big|+\sup_{s\in K}\big|f(s)-e^{p(s)}\big|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This shows that

$$\Big\{\tau\in[0,T]:\sup_{s\in K}\big|\zeta(s+i\tau;\mathfrak{a})-e^{p(s)}\big|<\frac{\varepsilon}{2}\Big\}$$
$$\subset\Big\{\tau\in[0,T]:\sup_{s\in K}\big|\zeta(s+i\tau;\mathfrak{a})-f(s)\big|<\varepsilon\Big\}.$$

Hence, by the monotonicity of the Lebegue measure and (1.41), we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} \big|\zeta(s+i\tau;\mathfrak{a}) - f(s)\big| < \varepsilon\Big\} > 0.$$

The case of "lim". Define the set

$$\hat{\mathcal{G}}_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\Big\}.$$

The boundary $\partial\hat{\mathcal{G}}_\varepsilon$ of $\hat{\mathcal{G}}_\varepsilon$ lies in the set

$$\big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\big\},$$

therefore, the boundaries $\partial\hat{\mathcal{G}}_{\varepsilon_1}$ and $\partial\hat{\mathcal{G}}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. This shows, that $P_\zeta(\partial\hat{\mathcal{G}}_\varepsilon)$ is positive for at most countably many $\varepsilon > 0$, in other words, the set $\hat{\mathcal{G}}_\varepsilon$ is the continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 1.6 and Lemma 1.19, the limit

$$\lim_{T\to\infty} P_T(\hat{\mathcal{G}}_\varepsilon) = P_\zeta(\hat{\mathcal{G}}_\varepsilon)$$

exists for all but at most countably many $\varepsilon > 0$. By the distributions of P_T and $\hat{\mathcal{G}}_\varepsilon$, the limit

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} \big|\zeta(s+i\tau;\mathfrak{a}) - f(s)\big| < \varepsilon\Big\} = P_\zeta(\hat{\mathcal{G}}_\varepsilon). \quad (1.42)$$

exists for all but at most countably many $\varepsilon > 0$. Thus, it remains to prove that $P_\zeta(\hat{\mathcal{G}}_\varepsilon) > 0$. Let $\mathcal{G}_\varepsilon$ be the same as in the case of "$\liminf$". Suppose that $g \in \mathcal{G}_\varepsilon$. then

$$\sup_{s\in K} |f(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}.$$

Hence, and (1.40), for such $g(s)$,

$$\sup_{s\in K} \big|g(s) - f(s)\big| \leqslant \sup_{s\in K} \big|g(s) - e^{p(s)}\big| + \sup_{s\in K} \big|f(s) - e^{p(s)}\big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $g \in \hat{\mathcal{G}}_\varepsilon$, i.e., $\mathcal{G}_\varepsilon \subset \hat{\mathcal{G}}_\varepsilon$. Since $P_\zeta(\mathcal{G}_\varepsilon) > 0$, hence, we have the inequality $P_\zeta(\hat{\mathcal{G}}_\varepsilon > 0$, and the theorem follows by (1.42).

The theorem is proved. $\square$

Chapter 2

Special case of the sequence $\mathfrak{a}$

In this chapter, we consider the universality of the function $\zeta(s;\mathfrak{a})$ with a special periodic sequence $\mathfrak{a}$, such that $a_m \not\equiv 0$. We suppose that the period q of the sequence $\mathfrak{a}$ is a prime number, and

$$a_q = \frac{a}{\varphi(q)} \sum_{l=1}^{q-1} a_l, \tag{2.1}$$

where $\varphi(q) = \#\{1\leqslant l\leqslant q : (l,q) = 1\}$ is the Euler totient function. Clearly, the equality (2.1) defines a non-trivial sequence for $q\geqslant 3$. If $q = 2$, then $\varphi(2) = 1$, and (2.1) implies $a_2 = a_1$. Thus, by periodicity of $\mathfrak{a}$, we have that $a_m = a_1$ for all $m \in \mathbb{N}$.

In this chapter, we do not require the multiplicativity of the sequence $\mathfrak{a}$. For the proofs, we will apply the approach based on properties of Dirichlet L-functions.

2.1 Statement of the results

Dirichlet characters and Dirichlet L-functions were shortly described in Introduction, therefore, we recall only that if χ is a Dirichlet characters modulo q, then the corresponding Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma > 1$, by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

and has a meromorphic continuation to the whole complex plane. The function $L(s,\chi)$ is entire if χ is the non-principal character modulo q, and has the unique simple pole at the point $s=1$ if χ is the principal character modulo q.

If $a_m \equiv c$, $m \in \mathbb{N}$, with $c \in \mathbb{C}\backslash\{0\}$, then the sequence $\mathfrak{a}$ is periodic with period $q=1$. In this case, we have

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{c}{m^s}, \sigma > 1.$$

thus, $\zeta(s;\mathfrak{a}) = c\zeta(s)$. Similarly, if a_m is a multiple of a Dirichlet character χ modulo q, i.e., $a_m = c\chi(m)$, $m \in \mathbb{N}$, with a certain constant $c \in \mathbb{C}\backslash\{0\}$, then

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{c\chi(m)}{m^s}, \sigma > 1.$$

Thus, $\zeta(a;\mathfrak{a}) = cL(s,\chi)$. Since the functions $\zeta(s)$ and $L(s,\chi)$ are universal in the Voronin sense, Theorems G and I, in the above cases, the function $\zeta(s;\mathfrak{a})$ is also universal. Thus, we have the following statement.

Theorem 2.1. *Suppose that $a_m \equiv c \neq 0$, or a_m is a multiple of a Dirichlet character modulo q. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\{\tau \in [0,T] : \sup_{s\in k} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\} > 0.$$

For the statement of the main theorem, we need some notation. It is well known, see, for example, [43], that, for $1 \leqslant b < q$, $(b,q)=1$, $b \in \mathbb{N}$,

$$\zeta\left(s, \frac{b}{q}\right) = \frac{q^s}{\varphi(q)} \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(b) L(s,\chi),$$

where the summing runs over all $\varphi(q)$ Dirichlet characters modulo q. Therefore, denoting by (l,q) the greatest common divisor of the numbers l and q, and using (1.1), we find

$$
\begin{aligned}
\zeta(s;\mathfrak{a}) &= \frac{1}{q^s}\sum_{l=1}^{q} a_l\zeta\left(s,\frac{l}{q}\right) = \frac{1}{q^s}\sum_{l=1}^{q} a_l\zeta\left(s,\frac{\frac{l}{(l,q)}}{\frac{q}{(l,q)}}\right)\\
&= \frac{1}{q^s}\sum_{l=1}^{q}\frac{a_l\left(\frac{q}{(l,q)}\right)^s}{\varphi\left(\frac{q}{(l,q)}\right)}\sum_{\chi=\chi\left(\mathrm{mod}\,\frac{q}{(l,q)}\right)}\overline{\chi}\left(\frac{l}{(l,q)}\right)L(s,\chi) \qquad (2.2)\\
&= \sum_{l=1}^{q}\frac{a_l}{\varphi\left(\frac{q}{(l,q)}\right)(l,q)^s}\sum_{\chi=\chi\left(\mathrm{mod}\,\frac{q}{(l,q)}\right)}\overline{\chi}\left(\frac{l}{(l,q)}\right)L(s,\chi).
\end{aligned}
$$

In this chapter, we do not require the multiplicativity of the sequence $\mathfrak{a}$. Therefore, it is convenient to separate two types of universality. We say that $\zeta(s;\mathfrak{a})$ is universal if inequality of universality

$$
\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\{\tau\in[0,T]:\sup_{s\in k}|\zeta(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\}>0.
$$

with every $\varepsilon>0$ is satisfied for all $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. If the latter inequality is satisfied for all $K\in\mathcal{K}$ and $f(s)\in H(K)$ ($H(K)$ is the class of continuous functions on K that are analytic in the interior of K), then we say, that the function $\zeta(s;\mathfrak{a})$ is strongly universal. For brevity, let

$$
b(q,\chi)=\sum_{l=1}^{q-1}a_l\chi(l),
$$

where χ is a Dirichlet character modulo q. We suppose that $a_m\not\equiv 0$, $m\in\mathbb{N}$. Then the following statement is true.

Theorem 2.2. *Suppose that the periodic sequence $\mathfrak{a}=\{a_m : m\in\mathbb{N}\}$ with minimal period q satisfies equality (2.1), and that q is a prime number.*

1° If the sequence $\mathfrak{a}$ satisfies at least one of the hypothesis

i) $a_m\equiv c$, $m\in\mathbb{N}$;

ii) a_m is a multiple of a Dirichlet character modulo q;

iii) $q=2$;

iv) only one of the numbers $b(q,\chi)\neq 0$, $q>2$,

then the function $\zeta(s,\mathfrak{a})$ is universal.

2° *If $q > 2$ and at least two numbers $b(q, \chi) \neq 0$, then the function $\zeta(s; \mathfrak{a})$ is strongly universal.*

For the proof of Theorem 2.2, we will use the Voronin joint universality theorem for Dirichlet L-functions.

2.2 The Voronin theorem

First we remind the notion of equivalent Dirichlet characters. Let χ_1 and χ_2 two Dirichlet characters modulo q_1 and q_2, respectively. Denote by $[q_1, q_2]$ the least common multiple of q_1 and q_2. For $m \in \mathbb{N}$ such that $(m, q_1) = 1$ and $(m, q_2) = 1$, we have that $(m, [q_1, q_2]) = 1$. Then, for such m, by the definition of a character,

$$\chi_1(m) \neq 0 \text{ and } \chi_2(m) \neq 0.$$

The characters χ_1 and χ_2 are called equivalent if

$$\chi_1(m) = \chi_2(m)$$

for $(m, [q_1, q_2]) = 1$, or, in other words, if χ_1 is equal to χ_2 for m such that the values $\chi_1(m) \neq 0$ and $\chi_2(m) \neq 0$.

S. M. Voronin in [53], see also [17] and [54], obtained the joint universality of Dirichlet L-functions. Roughly speaking, he proved that a collection on analytic functions can be simultaneously approximated by the collection of shifts of Dirichlet L-functions. We state a modern version of the Voronin theorem as the following lemma.

Lemma 2.3. *Suppose that $\chi_1, \ldots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and $L(s, \chi_1), \ldots, L(s, \chi_r)$ are the corresponding Dirichlet L-functions. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\{\tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in k} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon\} > 0.$$

Proof of the lemma can be found in [25].

The initial Voronin theorem for a collection of Dirichlet L-functions with pairwise non-equivalent characters was proved for closed circle in D, $j =$

$1, \dots, r$. B. Bagchi obtained [1] a joint universality for Dirichlet L-functions with different character modulo q.

2.3 Proofs of universality

Theorem 2.1 is trivial, however, we present some remarks.

Proof of Theorem 2.1. 1. The case $a_m \equiv c$, $c \neq 0$, $m \in \mathbb{N}$. By Theorem G stated in Introduction, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \frac{1}{c} f(s)| < \frac{\varepsilon}{|c|}\right\} > 0.$$

Thus,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |c\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0,$$

and the theorem follows by the equality $\zeta(s+i\tau;\mathfrak{a}) = c\zeta(s+i\tau)$.

2. Similarly, if $a_m = c\chi(m)$, $c \neq 0$, $m \in \mathbb{N}$, where χ is a Dirichlet character modulo q, then, by Theorem I of Introduction, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(s+i\tau,\chi) - \frac{1}{c} f(s)| < \frac{\varepsilon}{|c|}\right\} > 0.$$

Hence,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |cL(s+i\tau,\chi) - f(s)| < \varepsilon\right\} > 0,$$

and the theorem follows by the equality $\zeta(s+i\tau;\mathfrak{a}) = c\zeta(s+i\tau,\chi)$. $\square$

The proof of Theorem 2.2 is based on a partial case of equality (2.2) and Lemma 2.3.

Proof of Theorem 2.2. The cases i) - iii) of the assertion 1° are contained in Theorem 2.1. Thus, it remains to consider the case iv) of 1°.

Since the modulo q is prime, we have that $(l,q) = 1$ for $l = 1, \dots, q-1$, and $(l, q) = q$ for $l = q$. Therefore, we deduce from the identity (2.2) that

$$\zeta(s;\mathfrak{a}) = \frac{a_q}{q^s} \sum_{\chi=\chi(\mathrm{mod} 1)} \overline{\chi}(1)L(s,\chi) + \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l)L(s,\chi). \quad (2.3)$$

However, $\chi(m) \equiv 1$ for $\chi = \chi(\mathrm{mod}1)$, thus,

$$L(s,\chi_0) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s), \sigma > 1.$$

Therefore, by (2.3),

$$\zeta(s;\mathfrak{a}) = \frac{a_q\zeta(s)}{q^s} + \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l)L(s,\chi). \quad (2.4)$$

It is well known that, for the principal characters χ_0 modulo q

$$L(s,\chi) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) = \zeta(s)\left(1 - \frac{1}{q^s}\right)$$

because, in this case, q is a prime number. This and (2.4) show that

$$\begin{aligned}\zeta(s;\mathfrak{a}) &= \frac{1}{q^s}\left(a_l - \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l\right)\zeta(s) \\ &+ \frac{\zeta(s)}{\varphi(q)} \sum_{l=1}^{q-1} a_l + \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l)L(s,\chi) \\ &= \frac{\zeta(s)}{\varphi(q)} \sum_{l=1}^{q-1} a_l + \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l)L(s,\chi)\end{aligned} \quad (2.5)$$

in view of the equality (2.1). Now, in the set $\{\chi : \chi = \chi(\mathrm{mod}q)\}$, we replace the principal character χ_0 modulo q by the character $\tilde{\chi}(\mathrm{mod}1)$. Then the equality (2.5) can be rewritten in the form

$$\begin{aligned}\zeta(s;\mathfrak{a}) &= \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l)L(s,\chi) \\ &= \frac{1}{\varphi(q)} \sum_{\chi=\chi(\mathrm{mod} q)} L(s,\chi)b(q,\chi),\end{aligned} \quad (2.6)$$

where

$$b(q, \chi) = \sum_{l=1}^{q-1} a_l \chi(l).$$

We note that at least one number $b(q, \chi)$ in (2.6) is non-zero. Actually, if all the numbers $b(q, \chi) = 0$, then $\zeta(s; \mathfrak{a}) \equiv 0$, and this contradicts the assumption that $a_m \not\equiv 0$.

Now, let one number $b(q, \chi) \neq 0$. Then the definition of $b(q, \chi)$ and (2.6) give

$$\zeta(s; \mathfrak{a}) = \frac{1}{\varphi(q)} L(s, \chi) \sum_{l=1}^{q-1} a_l \overline{\chi}(l).$$

Hence, by the uniqueness theorem for Dirichlet series, we find that

$$a_m = \frac{1}{\varphi(q)} \chi(m) \sum_{l=1}^{q-1} a_l \overline{\chi}(l)$$

for all $m \in \mathbb{N}$. Thus,

$$a_1 \overline{\chi}(1) = \ldots = a_{q-1} \overline{\chi}(q-1) = \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \overline{\chi}(l),$$

and, by periodicity,

$$a_m = a_1 \chi(m), m \in \mathbb{N}.$$

Therefore, this case reduces to case ii).

It remains to prove the assertion $2°$. Let, as in Chapter 1, $H(D)$ denote the space of analytic functions on D. We also preserve the above notation, i.e., in place of $\chi_0(\mathrm{mod} q)$ taking $\widetilde{\chi}(\mathrm{mod} 1)$. Define the operator $F : H^{\varphi(q)}(D) \to H(D)$ by the formula

$$F(g_\chi(s) : \chi = \chi(\mathrm{mod} q)) = \frac{1}{\varphi(q)} \sum_{\chi = \chi(\mathrm{mod} q)} g_\chi(s) b(q, \chi),$$

where $(g_\chi(s) : \chi = \chi(\mathrm{mod} q)) \in H^{\varphi(q)}(D)$. First we will prove that, for every $K \in \mathcal{K}$ and a polynomial $p = p(s)$, there exists $(g_\chi(s) : \chi(\mathrm{mod} q)) \in F^{-1}\{\rho\}$ such that $g_\chi(s) \neq 0$ on K for all $\chi(\mathrm{mod} q)$. Suppose that

$$b(q, \chi_j) \neq 0, j = 1, 2.$$

Since the set K is bounded as a compact set, there exists a constant $C \in \mathbb{C}$ such that

$$p(s) + C \neq 0$$

on K, and

$$-C - \frac{1}{\varphi(q)} \sum_{\substack{\chi=\chi(\mathrm{mod} q) \\ \chi \neq \chi_1, \chi_2}} b(q, \chi) \neq 0.$$

We take

$$g_{\chi_1}(s) = \varphi(q) b^{-1}(q, \chi_1)\big(p(s) + C\big)$$

and

$$g_{\chi_2}(s) = \varphi(q) b^{-1}(q, \chi_2)\Bigg(-C - \frac{1}{\varphi(q)} \sum_{\substack{\chi=\chi(\mathrm{mod} q) \\ \chi \neq \chi_1, \chi_2}} b(q, \chi)\Bigg),$$

and $g_\chi(s) = 1$ for $\chi \neq \chi_1, \chi_2$. Then we have that $g_\chi(s) \neq 0$ on K for all $\chi = \chi(\mathrm{mod} q)$, and

$$F(g_\chi(s) : \chi = \chi(\mathrm{mod} q)) = p(s),$$

i.e., $(g_\chi(s) : \chi = \chi(\mathrm{mod} q)) \in F^{-1}\{\rho\}$.

For brevity, let

$$M = \sum_{l=1}^{q-1} |a_l|,$$

and let $\tau \in \mathbb{R}$ satisfy the inequality

$$\sup_{\chi=\chi(\mathrm{mod} q)} \sup_{s \in K} |L(s + i\tau, \chi) - g_\chi(s)| < \frac{\varepsilon}{2M}, \tag{2.7}$$

where the functions $g_\chi(s)$ have the above properties. Then, for such τ, in view of (2.6),

$$\begin{aligned} &\sup_{s \in K} |\zeta(s + i\tau, \mathfrak{a}) - p(s)| \\ &= \sup_{s \in K} \big|F\big(L(s + i\tau, \chi) : \chi = \chi(\mathrm{mod} q)\big) - F\big(g_\chi(s) : \chi = \chi(\mathrm{mod} q)\big)\big| \end{aligned} \tag{2.8}$$

$$\leqslant \sup_{s\in K} \frac{M}{\varphi(q)} \sum_{\chi=\chi(\mathrm{mod}q)} |L(s+i\tau,\chi)-g_{\chi}(s)|$$
$$\leqslant \sup_{\chi=\chi(\mathrm{mod}q)} \sup_{s\in K} |L(s+i\tau,\chi)-g_{\chi}(s)| < \frac{\varepsilon}{2}.$$

The characters $\chi = \chi(\mathrm{mod}q)$, where χ_0 is replaced by $\widetilde{\chi}$, are different (all $\varphi(q)$ characters modulo q are different), i.e., they are pairwise non-equivalent. Therefore, by Lemma 2.3, the set of $\tau \in \mathbb{R}$ satisfying the inequality (2.7) has a positive lower density, i.e.,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T] : \sup_{\chi=\chi(\mathrm{mod}q)} \sup_{s\in K} |L(s+i\tau,\chi)-g_{\chi}(s)| < \frac{\varepsilon}{2M}\Big\} > 0. \tag{2.9}$$

Moreover, we have seen that (2.7) implies (2.8). Therefore,

$$\Big\{\tau \in [0,T] : \sup_{\chi=\chi(\mathrm{mod}q)} \sup_{s\in K} |L(s+i\tau,\chi)-g_{\chi}(s)| < \frac{\varepsilon}{2M}\Big\}$$
$$\subset \Big\{\tau \in [0,T] : \sup_{\chi=\chi(\mathrm{mod}q)} \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-p(s)| < \frac{\varepsilon}{2}\Big\}.$$

Hence, by(2.9) and the monotonicity of the Lebesgue measure

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-p(s)| < \frac{\varepsilon}{2}\Big\} > 0 \tag{2.10}$$

for every polynomial $p(s)$.

It remains to replace the polynomial $p(s)$ by $f(s)$ in (2.10). By Lemma 1.27, we may find a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s)-p(s)| < \frac{\varepsilon}{2}. \tag{2.11}$$

If $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-p(s)| < \frac{\varepsilon}{2},$$

then, in view of (2.11),

$$\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-f(s)| \leqslant \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-p(s)| + \sup_{s\in K} |f(s)-p(s)| < \varepsilon.$$

This shows that

$$\Big\{\tau \in [0,T] : \sup_{s\in K}\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}\Big\}$$
$$\subset \Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}.$$

Therefore, by the inequality (2.10),

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

The theorem is proved. □

Chapter 3

Weighted universality of periodic zeta-function

The aim of this chapter is a generalization of Theorem 1.1. More precisely, this chapter is devoted to a weighted universality theorem for the periodic zeta-function with multiplicative coefficients.

3.1 Statement of the theorem

First of all, we define the weight function. Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, such that the variation $V_a^b w$ on the interval $[a, b]$ satisfies the inequality

$$V_a^b w \leqslant c w(a)$$

with a certain constant $c > 0$ for any subinterval $[a, b] \subset [T_0, \infty)$. Define

$$U(T, w) = \int_{T_0}^{T} w(t) \mathrm{d}t,$$

and suppose that

$$\lim_{T \to \infty} U(T, w) = +\infty.$$

Denote the class of the above functions $w(t)$ by W, and by $I(A)$ the indicator function of the set A.

Theorem 3.1. *Suppose that $w \in W$, and the sequence $\mathfrak{a}$ is multiplicative. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U(T,w)} \int_{T_0}^{T} w(t) I\Big(\Big\{\tau \in [T_0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}\Big) \mathrm{d}\tau > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

We note that Theorem 3.1 does not use any additional condition related to the Birkhoff-Khintchine ergodic theorem. The mentioned condition was involved in the papers [21], [23]. The proof of Theorem 3.1 is based on a weighted limit theorem in the space of analytic function for the function $\zeta(s;\mathfrak{a})$. This theorem will be obtained in the next section.

3.2 Weighted limit theorem

We preserve the notation of previous chapters. For $A \in \mathcal{B}(H(D))$, define

$$P_{T,w}(A) = \frac{1}{U(T,w)} \int_{T_0}^{T} w(t) I\big(\{\tau \in [T_0,T] : \zeta(s+i\tau;\mathfrak{a}) \in A\}\big) \mathrm{d}\tau.$$

Moreover, as in Chapter 1, P_ζ is the distribution of the $H(D)$-valued random element

$$\zeta(s,\omega;\mathfrak{a}) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{a_{p^\alpha}\omega^\alpha(p)}{p^{\alpha s}}\right).$$

Theorem 3.2. *The measure $P_{\zeta,w}$ converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the set*

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

We start the proof of Theorem 3.2 with a weighted limit theorem on the torus Ω. For $A \in \mathcal{B}(\Omega)$, define

$$Q_{T,w}(A) = \frac{1}{U(T,w)} \int_{T_0}^{T} w(t) I\big(\{\tau \in [T_0, T] : (p^{-i\tau} : p \in \mathbb{P}) \in A\}\big) \mathrm{d}\tau.$$

Lemma 3.3. *$Q_{T,w}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. Denote by $g_{T,w}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, the Fourier transform of the measure $Q_{T,w}$. As in the proof of Lemma 1.8, we have that

$$g_{T,w}(\underline{k}) = \int_{\Omega} \prod_{p \in \mathbb{P}}{}' \omega^{k_p}(p) \mathrm{d}Q_{T,w},$$

where the sign "$'$" means that only a finite number of integers k_p are distinct from zero. Hence, by the definition of $Q_{T,w}$, we find that

$$\begin{aligned} g_{T,w}(\underline{k}) &= \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) \prod_{p \in \mathbb{P}}{}' p^{ik_p\tau} \mathrm{d}\tau \\ &= \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) \exp\Big\{ - i\tau \sum_{p \in \mathbb{P}}{}' k_p \log p \Big\} \mathrm{d}\tau. \end{aligned} \tag{3.1}$$

Clearly,

$$g_{T,w}(\underline{0}) = \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) \mathrm{d}\tau = 1. \tag{3.2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$,

$$\sum_{p \in \mathbb{P}}{}' k_p \log p \neq 0$$

for $\underline{k} \neq \underline{0}$. Using properties of the weight function $w(t)$, we find by (3.1) that in the case $\underline{k} \neq \underline{0}$

$$
\begin{aligned}
g_{T,w}(\underline{k}) &= -\frac{1}{iU(T,w)\sum\limits_{p\in\mathbb{P}}{}' k_p \log p} \int\limits_{T_0}^{T} w(\tau) \mathrm{d} \exp\Big\{ - i\tau \sum_{p\in\mathbb{P}}{}' k_p \log p \Big\} \\
&= -\frac{w(\tau)\exp\Big\{ - i\tau \sum\limits_{p\in\mathbb{P}}{}' k_p \log p \Big\}}{iU(T,w)\sum\limits_{p\in\mathbb{P}}{}' k_p \log p} \Bigg|_{T_0}^{T} \\
&+ \frac{1}{iU(T,w)\sum\limits_{p\in\mathbb{P}}{}' k_p \log p} \int\limits_{T_0}^{T} \exp\Big\{ - i\tau \sum_{p\in\mathbb{P}}{}' k_p \log p \Big\} \mathrm{d} w(\tau) \\
&= O\Bigg(\Big| U(T,w) \sum_{p\in\mathbb{P}}{}' k_p \log p \Big|^{-1} \Bigg) + O\Bigg(\Big| U(T,w) \sum_{p\in\mathbb{P}}{}' k_p \log p \Big|^{-1} \int\limits_{T_0}^{T} \mathrm{d} w(\tau) \Bigg) \\
&= O\Bigg(\Big| U(T,w) \sum_{p\in\mathbb{P}}{}' k_p \log p \Big|^{-1} \Bigg) + O\Bigg(\Big| U(T,w) \sum_{p\in\mathbb{P}}{}' k_p \log p \Big|^{-1} V_{T_0}^{T} w \Bigg) \\
&= O\Bigg(\Big| U(T,w) \sum_{p\in\mathbb{P}}{}' k_p \log p \Big|^{-1} \Bigg).
\end{aligned}
$$

Since $U(T,w) \to \infty$ as $T \to \infty$, this shows that

$$
\lim_{T\to\infty} g_{T,w}(\underline{k}) = 0
$$

for $\underline{k} \neq \underline{0}$. Thus, in view of (3.2),

$$
\lim_{T\to\infty} g_{T,w}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}
$$

Consequently, the Fourier transform of $Q_{T,w}$ converges to the Fourier transform of the Haar measure m_H, and the lemma follows from a continuity theorem for probability measures on compact groups. □

The next lemma is a weighted limit theorem for the function

$$
\zeta_n(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s},
$$

whose Dirichlet series is absolutely convergent for $\sigma > \frac{1}{2}$, where $\zeta_n(s;\mathfrak{a})$ is the same as in Section 1.4. For $A \in \mathcal{B}(H(D))$, define

$$P_{T,n,w}(A) = \frac{1}{U(T,w)} \int\limits_{T_0}^{T} w(\tau) I\big(\{\tau \in [T_0, T] : \zeta_n(s + i\tau; \mathfrak{a}) \in A\}\big) \mathrm{d}\tau.$$

Lemma 3.4. *The measure $P_{T,n,w}$ converges weakly to the measure $V_n = m_H u_n^{-1}$ as $T \to \infty$, where the function $u_n : \Omega \to H(D)$ is defined by the formula*

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s}, \omega \in \Omega.$$

Proof. We apply similar arguments to those used in the proof of Theorem 1.6.

First, we observe that

$$u_n(p^{-i\tau} : p \in \mathbb{P}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^{s+i\tau}} = \zeta_n(s + i\tau; \mathfrak{a}).$$

Therefore, for $A \in \mathcal{B}(H(D))$,

$$\begin{aligned} P_{T,n,w}(A) &= \frac{1}{U(T,w)} \int\limits_{T_0}^{T} w(\tau) I\big(\{\tau \in [T_0, T] : (p^{-i\tau} : p \in \mathbb{P}) \in u_n^{-1} A\}\big) \mathrm{d}\tau \\ &= Q_{T,w}(u_n^{-1} A) = Q_{T,w} u_n^{-1}(A). \end{aligned}$$

Thus, we have the relation

$$P_{T,n,w} = Q_{T,w} u_n^{-1}. \tag{3.3}$$

The absolute convergence of the series

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s}$$

for $\sigma > \frac{1}{2}$, implies the continuity of the function u_n. Therefore, the equality (3.3), Lemmas 3.3 and 1.9 show that $P_{T,n,w}$ converges weakly to the measure $V_n \stackrel{\text{def}}{=} m_H u_n^{-1}$. $\square$

The next lemma is devoted to the weighted approximation of the function $\zeta(s; \mathfrak{a})$ by $\zeta_n(s; \mathfrak{a})$ in the mean. For this, we need the estimate for the weighted mean square of the periodic zeta-function. We start with some results for the

Hurwitz zeta-function $\zeta(s, \alpha)$.

Lemma 3.5. *Suppose that* $\sigma \geqslant \sigma_0 > 0$ *and* $2\pi \leqslant |t| \leqslant \pi x$. *Then*

$$\zeta(s, \alpha) = \sum_{0 \leqslant m \leqslant x} \frac{1}{(m+\alpha)^s} + \frac{(x+\alpha)^{1-s}}{s-1} + O_{\sigma_0}(x^{-\sigma}).$$

A proof of the lemma can be found in [27], Theorem 3.1.3.

Lemma 3.6. *Suppose that* $w \in W\ \sigma$, $\frac{1}{2} < \sigma < 1$, *is fixed, and* $\tau \in \mathbb{R}$. *Then*

$$\int_{T_0}^{T} w(t) |\zeta(\sigma + it + i\tau, \alpha)|^2 \mathrm{d}t \ll U(T, w)(1 + |\tau|).$$

Proof. We use Lemma 3.5 with $x = t + |\tau|$. Then, by Lemma 3.5, we have

$$\int_{T_0}^{T} w(t) |\zeta(\sigma + it + i\tau, \alpha)|^2 \mathrm{d}t \ll \int_{T_0}^{T} w(t) \Bigg| \sum_{0 \leqslant m \leqslant t + |\tau|} \frac{1}{m^{\sigma + it + i\tau}} \Bigg|^2 \mathrm{d}t + \int_{T_0}^{T} w(t) \frac{(t + |\tau| + \alpha)^{2-2\sigma}}{(t+\tau)^2 + (\sigma - 1)^2} \mathrm{d}t + \int_{T_0}^{T} w(t) \big(t + |\tau|\big)^{-2\sigma} \mathrm{d}t. \quad (3.4)$$

It is not difficult to see that, for $T_0 < 2|\tau| < T$,

$$\begin{aligned} \int_{T_0}^{T} w(t) \frac{(t + |\tau| + \alpha)^{2-2\sigma}}{(t+\tau)^2 + (\sigma-1)^2} \mathrm{d}t &\ll \left(\int_{T_0}^{2|\tau|} + \int_{2|\tau|}^{T} \right) w(t) \frac{(t + |\tau|)^{2-2\sigma}}{(t+\tau)^2 + (\sigma-1)^2} \mathrm{d}t \\ &\ll_\sigma \int_{T_0}^{2|\tau|} w(t) \big(t + |\tau|\big)^{2-2\sigma} \mathrm{d}t + \int_{2|\tau|}^{T} w(t) t^{-2} \big(t + |\tau|\big)^{2-2\sigma} \mathrm{d}t \\ &\ll_\sigma |\tau| \int_{T_0}^{T} w(t) \mathrm{d}t + \int_{2|\tau|}^{T} w(t) t^{-2\sigma} \mathrm{d}t + \int_{2|\tau|}^{T} w(t) t^{-2} |\tau|^{2-2\sigma} \mathrm{d}t \\ &\ll_\sigma U(T, w)(1 + |\tau|). \end{aligned} \quad (3.5)$$

If $2|\tau| \geqslant T$, then $t + |\tau| \leqslant 3|\tau|$, thus, again

$$\int_{T_0}^{T} w(t) \frac{(t + |\tau| + \alpha)^{2-2\sigma}}{(t+\tau)^2 + (\sigma-1)^2} \mathrm{d}t \ll_\sigma |\tau| \int_{T_0}^{T} w(t) \mathrm{d}t = U(T, w)|\tau|. \quad (3.6)$$

Thus, estimates (3.5) and (3.6) imply

$$\int_{T_0}^{T} w(t)\frac{(t+|\tau|+\alpha)^{2-2\sigma}}{(t+\tau)^2+(\sigma-1)^2}\mathrm{d}t \ll_\sigma U(T,w)\big(1+|\tau|\big). \tag{3.7}$$

Obviously,

$$\int_{T_0}^{T} w(t)\big(t+|\tau|\big)^{-2\sigma}\mathrm{d}t \ll u(T,w). \tag{3.8}$$

Denote $\max(m,k)=T_1+|\tau|$, with $T_1=T_1(m,k)$. Then

$$\begin{aligned}
&\int_{T_0}^{T} w(t)\bigg|\sum_{0\leqslant m\leqslant t+|\tau|}\frac{1}{m^{\sigma+it+i\tau}}\bigg|^2\mathrm{d}t\\
&=\int_{T_0}^{T} w(t)\sum_{0\leqslant m\leqslant t+|\tau|}\frac{1}{(m+\alpha)^{\sigma+it+i\tau}}\sum_{0\leqslant k\leqslant t+|\tau|}\frac{1}{(k+\alpha)^{\sigma+it+i\tau}}\mathrm{d}t \qquad (3.9)\\
&=\sum\sum_{T_0+|\tau|\leqslant m,k\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{\sigma+i\tau}(k+\alpha)^{\sigma-i\tau}}\int_{T_1}^{T} w(t)\bigg(\frac{k+\alpha}{m+\alpha}\bigg)^{it}\mathrm{d}t\\
&\quad+O(U(T,w)(1+|\tau|))\\
&=\sum_{T_0+|\tau|\leqslant m,k\leqslant T+|\tau|}\frac{1}{(m+\alpha)^{2\sigma}}\int_{T_1}^{T} w(t)\mathrm{d}t\\
&\quad+\sum\sum_{\substack{T_0+|\tau|\leqslant m,k\leqslant T+|\tau|\\ m\neq k}}\frac{1}{(m+\alpha)^{\sigma+i\tau}(k+\alpha)^{\sigma-i\tau}}\int_{T_1}^{T} w(t)\bigg(\frac{k+\alpha}{m+\alpha}\bigg)^{it}\mathrm{d}t.
\end{aligned}$$

Integrating by parts, we find

$$\int\limits_{T_1}^{T} w(t)\left(\frac{k+\alpha}{m+\alpha}\right)^{it}\mathrm{d}t = \int\limits_{T_1}^{T} w(t)\exp\left\{it\log\frac{k+\alpha}{m+\alpha}\right\}\mathrm{d}t$$

$$= \frac{1}{i\log\frac{k+\alpha}{m+\alpha}}\int\limits_{T_1}^{T} w(t)\mathrm{d}\exp\left\{it\log\frac{k+\alpha}{m+\alpha}\right\}$$

$$= \frac{1}{i\log\frac{k+\alpha}{m+\alpha}}\left(w(t)\exp\left\{it\log\frac{k+\alpha}{m+\alpha}\right\}\Bigg|_{T_1}^{T} - \int\limits_{T_1}^{T}\exp\left\{it\log\frac{k+\alpha}{m+\alpha}\right\}\mathrm{d}w(t)\right)$$

$$\ll \frac{w(T)+w(T_1)+V_{T_1}^{T}w}{|\log\frac{k+\alpha}{m+\alpha}|}.$$

Therefore, the properties of the class W imply

$$\begin{aligned}&\sum\sum_{\substack{T_0+|\tau|\leqslant m,k\leqslant T+|\tau|\\ m\neq k}} \frac{1}{(m+\alpha)^{\sigma+i\tau}(k+\alpha)^{\sigma-i\tau}}\int\limits_{T_1}^{T}\left(\frac{k+\alpha}{m+\alpha}\right)^{it}\mathrm{d}tw(t)\\ &\ll \sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{(m+\alpha)^{\sigma}(k+\alpha)^{\sigma}\log\frac{k+\alpha}{m+\alpha}}.\end{aligned} \tag{3.10}$$

If $m+\alpha<\frac{k+\alpha}{2}$, then

$$\log\frac{k+\alpha}{m+\alpha}>\log 2,$$

and

$$\begin{aligned}&\sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{(m+\alpha)^{\sigma}(k+\alpha)^{\sigma}\log\frac{k+\alpha}{m+\alpha}}\\ &\ll \sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{m^{\sigma}k^{\sigma}} \ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|}\frac{w(k-|\tau|)}{k^{2\sigma-1}}\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} w(k-|\tau|) = \int\limits_{T_0+|\tau|}^{T+|\tau|} w(u-|\tau|)\mathrm{d}[u] \\
&= [u]w(u-|\tau|)\Big|_{T_0+|\tau|}^{T+|\tau|} - \int\limits_{T_0+|\tau|}^{T+|\tau|} (u-\{u\})\mathrm{d}w(u-|\tau|) \\
&= [T+|\tau|]w(T) - [T_0+|\tau|]w(T_0) - (uw(u-|\tau|))\Big|_{T_0+|\tau|}^{T+|\tau|} \\
&+ \int\limits_{T_0+|\tau|}^{T+|\tau|} w(u-|\tau|)\mathrm{d}u + \int\limits_{T_0+|\tau|}^{T+|\tau|} \{u\}\mathrm{d}w(u-|\tau|) \\
&= -\{T+|\tau|\}w(T) - [T_0+|\tau|]w(T_0) + (T_0+\tau)w(T_0) \\
&+ \int\limits_{T_0}^{T} w(t)\mathrm{d}t + \int\limits_{T_0+|\tau|}^{T+|\tau|} \{u\}\mathrm{d}w(u-|\tau|) \ll |\tau| + U(T,w).
\end{aligned}
$$

If $m+\alpha \geqslant \frac{k+\alpha}{2}$, then we denote $m=k-r$, where $1\leqslant r\leqslant \frac{k}{2}+\frac{\alpha}{2}$. Hence,

$$
\log\frac{k+\alpha}{m+\alpha} = -\log\frac{k-r+\alpha}{k+\alpha} = -\log\left(1-\frac{r}{k+\alpha}\right) > \frac{r}{k+\alpha} \geqslant \frac{r}{k+1},
$$

and

$$
\begin{aligned}
&\sum\sum_{T_0+|\tau|\leqslant m<k\leqslant T+|\tau|} \frac{w(k-|\tau|)}{(m+\alpha)^\sigma (k+\alpha)^\sigma \log\frac{k+\alpha}{m+\alpha}} \\
&\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \sum_{r\leqslant \frac{k}{2}+\frac{\alpha}{2}} \frac{kw(k-|\tau|)}{rk^\sigma (k-r)^\sigma} \\
&\ll \sum_{T_0+|\tau|\leqslant k\leqslant T+|\tau|} \frac{w(k-|\tau|)\log k}{k^{2\sigma-1}} \ll |\tau| + U(T,w).
\end{aligned}
\tag{3.12}
$$

Clearly

$$
\sum_{T_0+|\tau|\leqslant m\leqslant T+|\tau|} \frac{1}{(m+\alpha)^{2\sigma}} \int\limits_{T_1}^{T} w(t)\mathrm{d}t \ll U(T,w).
$$

This, (3.4), and (3.7)-(3.12) prove the lemma. □

Lemma 3.7. *Suppose that $w \in W$, σ, $\frac{1}{2} < \sigma < 1$, is fixed, and $\tau \in \mathbb{R}$. Then*

$$\int_{T_0}^{T} w(t)\big|\zeta(\sigma + it + i\tau; \mathfrak{a})\big|^2 \mathrm{d}t \ll U(T, w)\big(1 + |\tau|\big).$$

Proof. From the equality (1.1), it follows that

$$\zeta(s; \mathfrak{a}) \ll \sum_{l=1}^{q} \left| a_l \zeta\left(s, \frac{l}{q}\right)\right| \ll \sum_{l=1}^{q} \left|\zeta\left(s, \frac{l}{q}\right)\right|$$

because $a_l \ll 1$. Therefore, using Lemma 3.6, we find

$$\begin{aligned}
\int_{T_0}^{T} w(t)\big|\zeta(\sigma + it + i\tau; \mathfrak{a})\big|^2 \mathrm{d}t &\ll \int_{T_0}^{T} w(t)\left(\sum_{l=1}^{q}\left|\zeta\left(\sigma + it + i\tau, \frac{l}{q}\right)\right|\right)^2 \mathrm{d}t \\
&\ll \sum_{l=1}^{q} \int_{T_0}^{T} w(t)\left|\zeta\left(\sigma + it + i\tau, \frac{l}{q}\right)\right|^2 \mathrm{d}t \\
&\ll qU(T, w)\big(1 + |\tau|\big) \ll U(T, w)\big(1 + |\tau|\big).
\end{aligned}$$

□

Lemma 3.8. *The equality*

$$\lim_{n\to\infty} \liminf_{T\to\infty} \frac{1}{U(T, w)} \int_{T_0}^{T} w(\tau)\rho\big(\zeta(s + i\tau; \mathfrak{a}), \zeta_n(s + i\tau; \mathfrak{a})\big)\mathrm{d}\tau = 0$$

holds.

Here ρ is the metric in the space $H(D)$ defined in Section 1.5.

Proof of Lemma 3.8. We use the integral representation (1.6)

$$\zeta_n(s; \mathfrak{a}) = \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z; \mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z}$$

for the function $\zeta_n(s; \mathfrak{a})$ which is valid for $\sigma > \frac{1}{2}$.

Further, we follow the proof of Lemma 1.10. Thus let $K \subset D$ be an arbitrary compact set. We fix a positive ε such that $\frac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for points

$s \in K$. Then, using (1.6), and repeating the proof of Lemma 1.10, we obtain that

$$\frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \zeta_n(s+i\tau;\mathfrak{a})| \mathrm{d}\tau \ll J_1 + J_2, \quad (3.13)$$

where

$$J_1 = \frac{1}{U(T,w)} \int_{-\infty}^{\infty} \left(\int_{T_0}^{T} w(\tau) \left| \zeta\Big(\frac{1}{2}+\varepsilon+i(t+\tau);\mathfrak{a}\Big) \right| \mathrm{d}\tau \right) \sup_{s\in K} \frac{|l_n(\frac{1}{2}+\varepsilon-s+it)|}{|\frac{1}{2}+\varepsilon-s+it|} \mathrm{d}t$$

and

$$J_2 = \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |R_n(s+i\tau)| \mathrm{d}\tau.$$

Using Lemma 3.7 and the Cauchy inequality, we find that

$$\begin{aligned}
&\int_{T_0}^{T} w(\tau) \left| \zeta\Big(\frac{1}{2}+\varepsilon+i(t+\tau);\mathfrak{a}\Big) \right| \mathrm{d}\tau \\
&\ll \left(\int_{T_0}^{T} w(\tau)\mathrm{d}\tau \int_{T_0}^{T} w(\tau) \left| \zeta\Big(\frac{1}{2}+\varepsilon+i(t+\tau);\mathfrak{a}\Big) \right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\ll \big(U(T,w)U(T,w)(1+|t|)\big)^{\frac{1}{2}} \ll U(T,w)(1+|t|).
\end{aligned}$$

Therefore, in view of the estimate (1.9),

$$J_1 \ll_K n^{-\varepsilon} \int_{-\infty}^{\infty} (1+|t|) \exp\{-c_1|t|\} \mathrm{d}t \ll_K n^{-\varepsilon}. \quad (3.14)$$

The estimate (1.10) implies

$$J_2 \ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{U(T,w)} \int\limits_{T_0}^{T} w(\tau) \exp\{-c_2|\tau|\} \mathrm{d}\tau$$
$$\ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{U(T,w)} \int\limits_{T_0}^{T} \exp\{-c_2|\tau|\} \mathrm{d}\tau \ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{U(T,w)}.$$

Thus, in view of (3.13) and (3.14)

$$\begin{aligned} &\frac{1}{U(T,w)} \int\limits_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \zeta_n(s+i\tau;\mathfrak{a})| \mathrm{d}\tau \\ &\ll_K n^{-\varepsilon} + \frac{n^{\frac{1}{2}-2\varepsilon}}{U(T,w)}. \end{aligned} \tag{3.15}$$

Since $U(T,w) \to \infty$ as $T \to \infty$, taking $T \to \infty$ and then $n \to \infty$ we obtain that, for any compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U(T,w)} \int\limits_{T_0}^{T} w(\tau) \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \zeta_n(s+i\tau;\mathfrak{a})| \mathrm{d}\tau = 0.$$

This and the definition of the metric ρ prove the lemma. $\square$

Recall that $V_n = m_H u_n^{-1}$ is the limit measure in Lemma 3.4.

Lemma 3.9. *The family of probability measures* $\{V_n : n \in \mathbb{N}\}$ *is tight.*

Proof. By Lemma 3.7 with $\tau = 0$, we have that, for $\frac{1}{2} < \sigma < 1$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U(T,w)} \int\limits_{T_0}^{T} w(t) |\zeta(s+i\tau;\mathfrak{a})| \mathrm{d}\tau \ll 1. \tag{3.16}$$

Let K_l be a compact set from the definition of the metric ρ. Then, using the Cauchy integral formula and (3.16), we obtain that

$$\limsup_{T\to\infty} \frac{1}{U(T,w)} \int\limits_{T_0}^{T} w(t) |\zeta(s+i\tau;\mathfrak{a})| \mathrm{d}\tau \leqslant A_l < \infty. \tag{3.17}$$

Clearly, the estimates (3.15) and (3.17) imply

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{U(T,w)}\int\limits_{T_0}^{T} w(t)|\zeta_n(s+i\tau;\mathfrak{a})|\mathrm{d}\tau\\
&\leqslant\limsup_{T\to\infty}\frac{1}{U(T,w)}\int\limits_{T_0}^{T} w(t)|\zeta(s+i\tau;\mathfrak{a})|\mathrm{d}\tau \qquad (3.18)\\
&+\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{U(T,w)}\int\limits_{T_0}^{T} w(t)|\zeta(s+i\tau;\mathfrak{a})-\zeta_n(s+i\tau;\mathfrak{a})|\mathrm{d}\tau\leqslant B_l<\infty.
\end{aligned}$$

On a certain probability space with measure μ, define a random variable η_T by

$$\mu(\eta_T\in A)=\frac{1}{U(T,w)}\int\limits_{T_0}^{T} w(t)I(A)(t)\mathrm{d}t, A\in\mathcal{B}(\mathbb{R}).$$

By Lemma 3.4, we have that $P_{T,n,w}$ converges weakly to P_n as $T\to\infty$. Define

$$X_{T,n}=X_{T,n}(s)=\zeta_n(s+i\eta_T;\mathfrak{a}).$$

Then the assertion of Lemma 3.4 can be written as

$$X_{T,n}\xrightarrow[T\to\infty]{\mathcal{D}} X_n, \qquad (3.19)$$

where X_n is the $H(D)$-valued random element having the distribution V_n.

Now, let $\varepsilon>0$ be arbitrary fixed number, and $M_l=2^lB_l\varepsilon^{-1}$. Then

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\mu(\sup_{n\in\mathcal{K}_l}|X_{T,n}(s)|>M_l)\\
&=\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{U(T,w)}\int\limits_{T_0}^{T} w(\tau)I(\{\tau\in[0,T]:\sup_{n\in\mathcal{K}_l}|\zeta_n(s+i\tau;\mathfrak{a})|>M_l\})\mathrm{d}\tau\\
&\leqslant\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{M_lU(T,w)}\int\limits_{T_0}^{T} w(\tau)\sup_{n\in\mathcal{K}_l}|\zeta_n(s+i\tau;\mathfrak{a})|>M_l\})\mathrm{d}\tau\leqslant\frac{\varepsilon}{2^l}.
\end{aligned}$$

Therefore, in view of (3.19),

$$\mu\left(\sup_{n\in\mathcal{K}_l}|X_n(s)|>M_l\right)\leqslant\frac{\varepsilon}{2^l} \qquad (3.20)$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$. Let

$$H_\varepsilon = \left\{ g \in H(D) : \sup_{n \in \mathcal{K}_l} |g(s)| \leqslant M_l, l \in \mathbb{N} \right\}.$$

Then the set H_ε in uniformly bounded on every compact set of the strip D, thus, it is a compact subset of the space $H(D)$. Moreover, by (3.20)

$$\mu(X_n(s) \in H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence,

$$V_n(H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, i.e., the family $\{V_n : n \in \mathbb{N}\}$ is tight. □

Proof of Theorem 3.2. By Lemmas 3.9 and 1.16, the family $\{V_n : n \in \mathbb{N}\}$ is relatively compact. Thus, every sequence of $\{V_n\}$ contains a subsequence $\{V_{n_r}\}$ such that V_{n_r}converges to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$, i.e.,

$$X_{n_r} \xrightarrow[r \to \infty]{\mathcal{D}} P. \tag{3.21}$$

Moreover, using Lemma 3.8, we find that, for every $\varepsilon > 0$,

$$\sup_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U(T,w)}$$

$$\int\limits_{T_0}^{T} w(\tau) I(\{\tau \in [T_0, T] : \rho(\zeta(s+i\tau;\mathfrak{a}), \zeta_n(s+i\tau;\mathfrak{a})) > \varepsilon\}) \mathrm{d}\tau$$

$$\leqslant \sup_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon U(T,w)} \int\limits_{T_0}^{T} w(\tau) \rho(\zeta(s+i\tau;\mathfrak{a}), \zeta_n(s+i\tau;\mathfrak{a})) \mathrm{d}\tau = 0.$$

Now this, (3.21), (3.19) and Lemma 1.18 show that

$$X_{T,w}(s) = \zeta(s + i\eta_T; \mathfrak{a}) \xrightarrow[T \to \infty]{\mathcal{D}} P, \tag{3.22}$$

in other words, $P_{T,w}$ converges weakly to P as $T \to \infty$. Moreover, the relation (3.22) shows that the measure P in (3.22) is independent of the choice of the subsequence V_{n_r}. Thus

$$X_n \xrightarrow[r \to \infty]{\mathcal{D}} P,$$

or V_n converges weakly to P as $T \to \infty$. This means that $P_{T,w}$ as $T \to \infty$ converges weakly to the measure of P_n as $n \to \infty$. It remains to identify the measure P. For this, we apply Theorem 1.6. In its proof, it was obtained that the limit measure of V_n coincides with the measure P_ζ, and its support is the set S. $\square$

3.3 Proof of Theorem 3.1

Theorem 3.1, as Theorem 1.1, follows from a limit theorem and the Mergelyan theorem.

Proof of Theorem 3.1. Let $p(s)$ be a polynomial, and

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2} \right\}.$$

Then, by Theorem 3.2, the set $\mathcal{G}_\varepsilon$ is an open neighborhood of the element $\mathrm{e}^{p(s)}$ of the support of the measure P_ζ. Therefore, by properties of the support,

$$P_\zeta(\mathcal{G}_\varepsilon) > 0. \tag{3.23}$$

This, Theorem 3.2 and Lemma 1.26 show that

$$\liminf_{T \to \infty} P_{T,w}(\mathcal{G}_\varepsilon) \geqslant P_\zeta(\mathcal{G}_\varepsilon) > 0.$$

Hence, by the definitions of $P_{T,w}$ and $\mathcal{G}_\zeta$, we obtain the inequality

$$\begin{aligned} &\liminf_{T \to \infty} \frac{1}{U(T,w)} \\ &\int_{T_0}^{T} w(\tau) I\left(\left\{\tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\tau; \mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\right\}\right) \mathrm{d}\tau > 0. \end{aligned} \tag{3.24}$$

In view of Lemma 1.27, we can choose the polynomial $p(s)$ to satisfy

$$\sup_{s \in K} |f(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}. \tag{3.25}$$

Let $\tau \in \mathbb{R}$ satisfy the inequality

$$\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}.$$

Then, by (3.25), we have

$$\sup_{n\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| \leqslant \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \mathrm{e}^{p(s)}| + \sup_{s\in K} |f(s) - \mathrm{e}^{p(s)}| < \varepsilon.$$

This implies the inclusion

$$\begin{aligned} &\Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\Big\} \\ &\subset \Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\} \end{aligned}$$

Thus,

$$\begin{aligned} &I\Big(\Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\Big\}\Big) \\ &\leqslant I\Big(\Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) I\Big(\Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}\Big) \mathrm{d}\tau \\ &\geqslant \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) I\Big(\Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\Big\}\Big) \mathrm{d}\tau. \end{aligned}$$

This together with inequality (3.24) shows that

$$\liminf_{T\to\infty} \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) I\Big(\Big\{\tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}\Big) \mathrm{d}\tau > 0.$$

Now, we will prove the second part of the theorem. Consider the set

$$\hat{\mathcal{G}}_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\Big\}.$$

Since the boundary $\partial\hat{\mathcal{G}}_\varepsilon$ of the set $\hat{\mathcal{G}}_\varepsilon$ lies in the set

$$\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\},$$

we have that

$$\hat{\mathcal{G}}_{\varepsilon_1} \cap \hat{\mathcal{G}}_{\varepsilon_2} = \emptyset$$

for $\varepsilon_1 \neq \varepsilon_2$. From this, it follows that $P_\zeta(\hat{\mathcal{G}}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. This means that the set $\hat{\mathcal{G}}_\varepsilon$ is a continuity set of P_ζ for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 3.2 and Lemma 1.19, the limit

$$\lim_{T\to\infty} P_{T,w}(\hat{\mathcal{G}}_\varepsilon) = P_\zeta(\hat{\mathcal{G}}_\varepsilon)$$

exists for all but at most countably many $\varepsilon > 0$. Thus, by the definitions of $P_{T,w}$ and $\hat{\mathcal{G}}_\varepsilon$, the limit

$$\lim_{T\to\infty} \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) I\Big(\Big\{\tau \in [T_0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\Big\}\Big) \mathrm{d}\tau = P_\zeta(\hat{\mathcal{G}}_\varepsilon)$$

exists for all but at most countably many $\varepsilon > 0$. Therefore, it is sufficient to show that $P_\zeta(\hat{\mathcal{G}}_\varepsilon) > 0$. For this, we observe that $\mathcal{G}_\varepsilon \subset \hat{\mathcal{G}}_\varepsilon$, where $\mathcal{G}_\varepsilon$ was defined at the beginning of the proof. Actually, in virtue of (3.25), it is easily seen that if

$$\sup_{s\in K} |g(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}$$

then

$$\sup_{s\in K} |g(s) - f(s)| < \varepsilon.$$

This remark and the definitions of the sets $\mathcal{G}_\varepsilon$ and $\hat{\mathcal{G}}_\varepsilon$ imply the inclusion $\mathcal{G}_\varepsilon \subset \hat{\mathcal{G}}_\varepsilon$. Therefore, taking into account the inequality (3.23), we obtain that $P_\zeta(\hat{\mathcal{G}}_\varepsilon) > 0$. This together with (3.26) completes the proof of the theorem. $\square$

Chapter 4

Weighted discrete universality of the periodic zeta-function

In Chapter 3, a weighted universality theorem for periodic zeta-function $\zeta(s;\mathfrak{a})$ had been obtained. More precisely, it was proved that, for all $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and every $\varepsilon > 0$, the set of reals τ satisfying inequality

$$\sup_{s\in K} |\zeta(s + i\tau;\mathfrak{a}) - f(s)| < \varepsilon$$

has a positive weighted lower density. Theorem 3.1 can be called a weighted continuous universality theorem of the function $\zeta(s;\mathfrak{a})$ because τ in $\zeta(s + i\tau;\mathfrak{a})$ can take arbitrary real value. This chapter is devoted to weigthed discrete universality theorems for periodic zeta-function, i.e., to the approximation of analytic functions by shifts $\zeta(s+i\tau;\mathfrak{a})$ when τ takes values from a certain discrete set. First, we limit ourselves by a very simple discrete set, the arithmetic progression $\{kh : k \in \mathbb{N}\}$ with a certain fixed $h > 0$. Later, we will prove a theorem by using the set $\{k^\alpha h : k \in \mathbb{N}\}$ with fixed $0 < \alpha < 1$ and $h > 0$.

4.1 Statement of a weighted discrete universality theorem involving the arithmetic progression

Suppose that $w(t)$ is a non-increasing positive function for $t > 0$ having a continuous derivative such that, for h >0,

$$w(t) \ll_h w(ht) \text{ and } (w'(t))^2 \ll w(t).$$

Define

$$V(N, w) = \sum_{k=1}^{N} w(k),$$

where N runs over positive integers, and suppose that

$$\lim_{N\to\infty} V(N, w) = +\infty.$$

Denote the class of the above functions w by V_1. For example, $\frac{1}{t} \in V_1$. In this section, we will prove the following theorem.

Theorem 4.1. *Suppose that $w \in V_1$, the sequence $\mathfrak{a}$ is multiplicative, and the set*

$$L(\mathbb{P}, h, \pi) = \left\{(\log p : p \in \mathbb{P}), \frac{2\pi}{h}\right\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{V(N, w)} \sum_{k=1}^{N} w(k) I\left(\left\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh; \mathfrak{a}) - f(s)| < \varepsilon\right\}\right) > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

For example, in Theorem 4.1 we can take $h = \pi$ and $w(t) = \frac{1}{t}$, because, by the Lindemann theorem, the number e^k with $k \in \mathbb{R}\backslash\{0\}$ is transcendental.

4.2 Weighted discrete limit theorem involving the arithmetic progression

As in previous chapters, for the proof of Theorem 4.1, we will apply a limit theorem in the space of analytic functions. We preserve the notation of previous chapters.

Theorem 4.2. *Suppose that $w \in V_1$, the sequence $\mathfrak{a}$ is multiplicative and the set $L(\mathbb{P}, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$P_{N,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1 \leqslant k \leqslant N : \zeta(s+ikh; \mathfrak{a}) \in A\Big\}\Big), \qquad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ as $N \to \infty$.

As usual, we start the proof of Theorem 4.2 with a limit theorem on the torus Ω. Let, for $(\Omega, \mathcal{B}(\Omega))$

$$Q_{N,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1 \leqslant k \leqslant N : (p^{-ikh} : p \in \mathbb{P}) \in A\Big\}\Big).$$

Theorem 4.3. *Under hypotheses of Theorem 4.2, $Q_{N,w}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We consider the Fourier transform

$$g_{N,w}(\underline{k}) = \int\limits_{\Omega} \Big(\sideset{}{'}\prod_{p \in \mathbb{P}} w^{k p}(p)\Big) \mathrm{d}Q_{N,w},$$

$\underline{k} = (k_p : k_p \in \mathbb{R}, p \in \mathbb{P})$ of the measure $Q_{N,w}$. By the definition of $Q_{N,w}$.

$$\begin{aligned} g_{N,w}(\underline{k}) &= \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sideset{}{'}\prod_{p \in \mathbb{P}} p^{-ikhk_p} \\ &= \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \exp\Big\{ -ikh \sideset{}{'}\sum_{p \in \mathbb{P}} k_p \log p \Big\}, \end{aligned} \tag{4.1}$$

where sign "$'$" means that only a finite number of integers k_p are distinct from zero. We have by (4.1) that

$$g_{N,w}(\underline{0}) = 1. \tag{4.2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we observe that, for $\underline{k} \neq \underline{0}$,

$$\exp\left\{ -ih{\sum_{p\in\mathbb{P}}}' k_p \log p \right\} \neq 1. \tag{4.3}$$

Actually, if the latter inequality was not true, then we would have

$$\exp\left\{ -ih{\sum_{p\in\mathbb{P}}}' k_p \log p \right\} = \mathrm{e}^{2\pi i r}$$

with some $r \in \mathbb{Z}$. This leads to the equality

$${\sum_{p\in\mathbb{P}}}' k_p \log p + \frac{2\pi r_1}{h} = 0$$

with $r_1 \in \mathbb{Z}$ which contradicts the linear independence of the set $L(\mathbb{P}, h, \pi)$. Thus, inequality (4.3) is true, and, for $\underline{k} \neq \underline{0}$, we find that

$$\begin{aligned}
&\sum_{k\leqslant u} \exp\left\{ -ikh{\sum_{p\in\mathbb{P}}}' k_p \log p \right\} \\
&= \frac{\exp\left\{ -ih{\sum_{p\in\mathbb{P}}}' k_p \log p \right\} - \exp\left\{ -i([u]+1)h{\sum_{p\in\mathbb{P}}}' k_p \log p \right\}}{1 - \exp\left\{ -ih{\sum_{p\in\mathbb{P}}}' k_p \log p \right\}} \stackrel{\mathrm{def}}{=} r(u).
\end{aligned}$$

Then (4.1) and the summation by parts shows that, for $\underline{k} \neq \underline{0}$

$$\begin{aligned}
g_{N,w}(\underline{k}) &= \frac{W(N)}{V(N,w)} r(N) - \frac{1}{V(N,w)} \int_1^N r(u)\mathrm{d}w(u) + o(1) \\
&= \frac{W(N)}{V(N,w)} r(N) + O\left(\frac{V_1^N w}{V(N,w)} \right) + o(1) \\
&= O\left(\frac{V_1^N w}{V(N,w)} \right) + o(1) = o(1)
\end{aligned}$$

as $N \to \infty$, where $V_1^N w$ denotes the variation of w in the interval $[1, N]$. This together with (4.2) gives

$$\lim_{N\to\infty} g_{N,w}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Therefore, by a continuity theorem for probability measures on compact groups, we obtain that $Q_{n,w}$ converges weakly to the Haar measure m_H as $N \to \infty$. $\square$

Let $v_n(m)$ and $\zeta_n(a, \mathfrak{a})$ be the same as in Section 1.4, i.e.,

$$\zeta_n(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s}$$

where

$$v_n(m) = \exp\left\{ - \left(\frac{m}{n}\right)^{\theta} \right\}.$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,n,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1 \leqslant k \leqslant N : \zeta(s + ikh; \mathfrak{a}) \in A\Big\}\Big).$$

Moreover, let the function $u_n : \Omega \to H(D)$ be given by

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s} \overset{\text{def}}{=} \zeta_n(s, \omega; \mathfrak{a}),$$

and let $V_n = m_H u_n^{-1}$. Then we have the following limit lemma for $P_{N,n,w}$.

Lemma 4.4. *Under hypotheses of Theorem 4.2, $P_{N,n,w}$ converges weakly to the measure V_n as $N \to \infty$.*

Proof. From the definitions of u_n, $Q_{N,w}$ and $P_{N,n,w}$, we have that

$$u_n(p^{-ikh} : p \in \mathbb{P}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^{s+ikh}} = \zeta_n(s + ikh; \mathfrak{a}),$$

and, for $A \in \mathcal{B}(H(D)$,

$$P_{N,n,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1 \leqslant k \leqslant N : (p^{-ikh} : p \in \mathbb{P}) \in u_n^{-1} A\Big\}\Big)$$
$$= Q_{N,w}(u_n^{-1} A) = Q_{N,w} u_n^{-1}(A).$$

Thus, $P_{N,n,w} = Q_{N,w} u_n^{-1}$. Therefore, the assertion of the lemma follows from Lemmas 4.3, 1.9, and the continuity of the function u_n which was noted in the proof of Theorem 1.6. □

The next step of the proof of Theorem 4.2 is devoted to the approximation of $\zeta(s; \mathfrak{a})$ by $\zeta_n(s; \mathfrak{a})$. Let ρ be the metric on $H(D)$ defined in Section 1.5

Lemma 4.5. *Under hypotheses of Theorem 4.2, we have the equality*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \rho(\zeta(s+ikh; \mathfrak{a}), \zeta_n(s+ikh; \mathfrak{a})) = 0$$

For the proof of Lemma 4.5, we need the Gallagher lemma which connects the continuous and discrete mean squares of certain functions.

Lemma 4.6. *Suppose that T_0, $T_0 > \delta > 0$, $\mathcal{T}$ is a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$ and*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < \delta}} 1.$$

Let the complex-valued function S is continuous in $[T_0, T_0 + T]$ and have a continuous derivative on $(T_0, T_0 + T)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta(t) |S(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 \mathrm{d}x + \left(\int_{T_0}^{T_0+T} |S(x)|^2 \mathrm{d}x \int_{T_0}^{T_0+T} |S'(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Proof of the lemma is given in [40], Lemma 1.4.

Proof of Lemma 4.5. First we observe that

$$\int_1^N w(t) \mathrm{d}t \ll \sum_{k=1}^{N} w(k) \ll \int_1^N w(t) \mathrm{d}t.$$

Clearly, if $w \in V_1$, then $w \in W$. Therefore, in view of Lemma 3.7, for a fixed $\frac{1}{2} < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\int\limits_1^N w(t)|\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \ll V(N, w)(1 + |\tau|).$$

Hence, using Lemma 4.6 with $\delta = h$ and properties of the weight function $w(t)$, we find that

$$\begin{aligned}
&\sum_{k=1}^{N} w(k)|\zeta(\sigma + ikh + i\tau; \mathfrak{a})|^2 \ll \sum_{k=1}^{N} w(kh)|\zeta(\sigma + ikh + i\tau; \mathfrak{a})|^2 \\
&\ll \int\limits_1^{Nh} w(t)|\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \\
&+ \Bigg(\int\limits_1^{Nh} w(t)|\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \Bigg(\int\limits_1^{Nh} w(t)|\zeta'(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \\
&+ \int\limits_1^{Nh} \left(\left(\sqrt{w(k)}\right)'\right)^2 |\zeta(\sigma + it + i\tau; \mathfrak{a})|^2 \mathrm{d}t \Bigg)\Bigg)^{\frac{1}{2}} \ll V(N, w)(1 + |\tau|),
\end{aligned} \tag{4.4}$$

where we applied the estimate $\left(\left(\sqrt{w(t)}\right)'\right)^2 \ll w(t)$.

Now, we apply similar arguments as in the proof of Lemma 3.8. Let $K \subset D$ be an arbitrary compact subset, and let $\varepsilon > 0$ be such that $\frac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for $s \in K$. Then, as in the proof of Lemma 1.10, we obtain that

$$\frac{1}{V(N, w)} \sum_{k=1}^{N} w(k) \sup_{s \in K} |\zeta(s + ikh; \mathfrak{a}) - \zeta_n(s + ikh; \mathfrak{a})| \ll S_1 + S_2, \tag{4.5}$$

where

$$\begin{aligned}
S_1 = &\frac{1}{V(N, w)} \\
&\int\limits_{-\infty}^{+\infty} \left(\sum_{k=1}^{N} w(k) \left|\zeta\left(\frac{1}{2} + \varepsilon + i(t + kh); \mathfrak{a}\right)\right| \sup_{s \in K} \frac{|l_n(\frac{1}{2} + \varepsilon - s + it|)|}{|\frac{1}{2} + \varepsilon - s + it|}\right) \mathrm{d}t
\end{aligned}$$

and

$$S_2 = \frac{1}{V(T,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K} |R_n(s+ikh)|.$$

By the Cauchy inequality and (4.4),

$$\begin{aligned}
&\sum_{k=1}^{N} w(k) \left| \zeta\left(\frac{1}{2}+\varepsilon+i(t+kh);\mathfrak{a}\right)\right| \\
&\ll \left(\sum_{k=1}^{N} w(k) \sum_{k=1}^{N} w(k) \left| \zeta\left(\frac{1}{2}+\varepsilon+i(t+kh);\mathfrak{a}\right)\right|^2 \right)^{\frac{1}{2}} \\
&\ll \left(V(N,w)V(N,w)(1+|t|^2)\right)^{\frac{1}{2}} \ll V(N,w)(1+|t|).
\end{aligned}$$

Therefore, using (1.9), we obtain the bound

$$S_1 \ll_K n^{-\varepsilon} \int_{-\infty}^{+\infty} (1+|t|) \exp\{-c_1|t|\} \mathrm{d}t \ll_K n^{-\varepsilon}. \tag{4.6}$$

The estimate (1.10) yields

$$S_2 \ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{V(N,w)} \sum_{k=1}^{N} w(k) \exp\{-c_2 k\} \ll \frac{n^{\frac{1}{2}-2\varepsilon}}{V(N,w)}.$$

This, (4.6) and (4.5) show that

$$\begin{aligned}
&\frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - \zeta_n(s+ikh;\mathfrak{a})| \\
&\ll_K n^{-\varepsilon} + \frac{n^{\frac{1}{2}-2\varepsilon}}{V(N,w)}.
\end{aligned} \tag{4.7}$$

Thus, taking $N \to \infty$ and then $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - \zeta_n(s+ikh;\mathfrak{a})| = 0.$$

Therefore, this and the definition of the metric ρ prove the lemma. □

Proof of Theorem 4.2. Suppose that the random variable θ_N is defined on a certain probability space with measure μ by

$$\mu(\theta_N = kh) = \frac{w(k)}{V(N,w)}, k = 1, \dots, N.$$

Define

$$X_{N,n,w} = X_{N,n,w}(s) = \zeta_n(s + i\theta_N; \mathfrak{a}),$$

and let X_n be the $H(D)$-valued random element having distribution V_n, where V_n is defined in Lemma 4.4. Then, by Lemma 4.4, we have that

$$X_{N,n,w} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{4.8}$$

Next, we need to prove that the family of probability measures $\{V_n : n \in \mathbb{N}\}$ is tight.

By (4.4) with $\tau = 0$, we have that, for fixed σ, $\frac{1}{2} < \sigma < 1$,

$$\limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k)|\zeta(\sigma + ikh; \mathfrak{a})| \ll 1.$$

Let K_l be a compact set from the definition of the metric ρ. Then the later estimate together with the Cauchy integral formula implies

$$\limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta(s + ikh; \mathfrak{a})| \ll C_l < \infty. \tag{4.9}$$

Now, the estimates (4.7) and (4.9) give

$$\begin{aligned}
&\sup_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta_n(s + ikh; \mathfrak{a})| \\
\leqslant &\limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta(s + ikh; \mathfrak{a})| \\
&+ \sup_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta(s + ikh; \mathfrak{a}) - \zeta_n(s + ikh; \mathfrak{a})| \\
\leqslant &\hat{C}_l < \infty.
\end{aligned} \tag{4.10}$$

Now let $\varepsilon > 0$ be arbitrary fixed number, and $M_l = 2^l \hat{C}_l \varepsilon^{-1}$. Then

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \mu\Big(\sup_{s\in K_l} |X_{N,n,w}(s)| \geqslant M_l \Big) \\
&= \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{ 1\leqslant k\leqslant N : \sup_{s\in K_l} |\zeta_n(s+ikh;\mathfrak{a})| \geqslant M_l \Big\} \Big) \\
&\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M_l V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta_n(s+ikh;\mathfrak{a})| \leqslant \frac{\varepsilon}{2^l}
\end{aligned}$$

for all $l \in \mathbb{N}$. This and relation (4.8) yield the inequality

$$\mu\Big(\sup_{s\in K_l} |X_n(s)| > M_l \Big) \leqslant \frac{\varepsilon}{2^l} \tag{4.11}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$. Define the set

$$H_\varepsilon = \Big\{ g \in H(D) : \sup_{s\in K} |g(s)| \leqslant M_l, l \in \mathbb{N} \Big\}.$$

Then the set H_ε is compact in the space $H(D)$, and, in virtue of (4.11),

$$\mu(X_n(s) \in H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, by the definitions of X_n and V_n,

$$V_n(H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{V_n : n \in \mathbb{N}\}$ is tight.

Since the family $\{V_n : n \in \mathbb{N}\}$ is tight, by Lemma 1.16, it is relatively compact. This means that every sequence of $\{V_n\}$ contains a subsequence $\{V_{n_r}\}$ such that V_{n_r} converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$. Hence,

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{4.12}$$

Define one more $H(D)$-valued random element

$$X_{N,w} = X_{N,w}(s) = \zeta(s + i\theta_N; \mathfrak{a}).$$

Lemma 4.5 implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu(\rho(X_{N,w}, X_{N,n,w}) \geqslant \varepsilon) = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V(N,w)}$$

$$\sum_{k=1}^{N} w(k) I\big(\{1 \leqslant k \leqslant N : \rho(\zeta(s+ikh;\mathfrak{a}), \zeta_n(s+ikh;\mathfrak{a})) \geqslant \varepsilon\}\big)$$

$$\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon V(N,w)} \sum_{k=1}^{N} w(k) \rho(\zeta(s+ikh;\mathfrak{a}), \zeta_n(s+ikh;\mathfrak{a})) = 0.$$

This equality together with relations (4.8) and (4.12) and Lemma 1.17 leads to the relation

$$\zeta(s+i\theta_N;\mathfrak{a}) \xrightarrow[N\to\infty]{\mathcal{D}} P.$$

Thus, the measure $P_{N,w}$ converges weakly to P as $N \to \infty$. Moreover, from this it follows that the measure P is independent on the choice of the subsequence $\{V_{n_r}\}$. Since $\{V_n\}$ is relatively compact, this shows that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

and therefore, V_n converges weakly to P as $n \to \infty$. Thus, we obtain that the measure $P_{N,w}$ converges weakly to the limit measure of V_n as $N \to \infty$. However, by Theorem 1.6, the measure P_T as $T \to \infty$, also converges weakly to the limit measure P of V_n, and that P converges with P_ζ. Therefore, $P_{N,w}$ also converges weakly to P_ζ as $N \to \infty$. $\square$

4.3 Proof of the weighted universality

We note that, by Theorem 1.6, the set

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$$

is the support of the measure P_ζ.

Proof of Theorem 4.1. The first part. By Lemma 1.27 there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}. \tag{4.13}$$

Define the set

$$\mathcal{G}_\epsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\Big\}.$$

Since $\mathrm{e}^{p(s)} \neq 0$, the function $\mathrm{e}^{p(s)}$ is an element of the support of the measure P_ζ. Therefore

$$P_\zeta(\mathcal{G}_\epsilon) > 0. \tag{4.14}$$

This inequality, Theorem 4.2 and Lemma 1.26 imply the inequality

$$\liminf_{N\to\infty} P_{N,w}(\mathcal{G}_\varepsilon) \geqslant P_\zeta(\mathcal{G}_\varepsilon) > 0.$$

Thus, by the definition of $P_{N,w}$ and $\mathcal{G}_\varepsilon$,

$$\liminf_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\Big\}\Big) > 0.$$

This and inequality (4.13) show that

$$\liminf_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I(\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\}) > 0.$$

The second part. Define the set

$$\hat{\mathcal{G}}_\varepsilon = \{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\}.$$

Then the set $\hat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure P_ζ for all but at most countable many $\varepsilon > 0$. Therefore, in virtue of Theorem 4.2 and Lemma 1.19, the limit

$$\lim_{N\to\infty} P_{N,w}(\hat{\mathcal{G}}_\varepsilon) = P_\zeta(\hat{\mathcal{G}}_\varepsilon)$$

exists for all but at most countable many $\varepsilon > 0$. Hence, the definitions of $P_{N,w}$ and $\hat{\mathcal{G}}_\varepsilon$ imply that the limit

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I(\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\}) \\ &= P_\zeta(\hat{\mathcal{G}}_\varepsilon) \end{aligned} \tag{4.15}$$

exists for all but at most countable many $\varepsilon > 0$. Moreover, it is easily seen that, in view of (4.13),

$$\mathcal{G}_\varepsilon \subset \hat{\mathcal{G}}_\varepsilon.$$

Thus, by (4.14), we obtain that $P_\zeta(\hat{\mathcal{G}}_\varepsilon) > 0$. This and (4.15) prove the second part of the theorem. The theorem is proved. $\square$

4.4 Statement of a weighted discrete universality theorem involving the set $\{k^\alpha h\}$

We preserve the notation of Section 4.1

$$V(N, w) = \sum_{k=1}^{N} w(k),$$

and suppose that $\lim\limits_{N\to\infty} V(N, w) = +\infty$. Moreover, we suppose, that the function $w(t)$ has a continuous derivative $w'(t)$ such that for $t \geqslant 1$

$$\int_1^N t|w'(t)|\mathrm{d}t \ll V(N, w).$$

Denote by V_2 the class of the above functions. This section is devoted to a weighted discrete theorem for the function $\zeta(s; \mathfrak{a})$ with a weight function $w \in V_2$ and a discrete set $\{k^\alpha h : k \in \mathbb{N}\}$, where α, $0 < \alpha < 1$, and $h > 0$ are fixed numbers. The main result of this section is the following theorem.

Theorem 4.7. *Suppose that $w \in V_2$, the sequence $\mathfrak{a}$ is multiplicative, and $0 < \alpha < 1$ is fixed. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{V(N, w)} \sum_{k=1}^{N} w(k) I(\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + ik^\alpha h; \mathfrak{a}) - f(s)| < \varepsilon\}) > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

Let Ω be the same as above. We begin the proof of Theorem 4.7 with a limit theorem on the torus Ω. For $A \in \mathcal{B}(\Omega)$, define

$$Q_{N,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I(\{1 \leqslant k \leqslant N : (p^{-ik^\alpha h} : p \in \mathbb{P}) \in A\}).$$

Before the statement of a limit theorem on the torus, we recall some facts on the uniform distribution modulo 1 of sequences of real numbers.

Lemma 4.8 (Veyl criterion). *A sequence $\{x_k\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for all $m \in \mathbb{Z}\backslash\{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathrm{e}^{2\pi i x_k m} = 0.$$

Proof of the lemma can be found, for example, in [19].

Lemma 4.9. *The sequence $\{ak^\alpha\}$ with fixed $0 < \alpha < 1$ and every real $a \neq 0$ is uniformly distributed modulo 1.*

The assertion on the lemma is a well-known exercise, see [19].

Lemma 4.10. *Suppose that $w \in V_2$, and $0 < \alpha < 1$ is fixed. Then $Q_{N,w}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. As in the previous sections, we will apply the Fourier transform method, i.e., we will consider the Fourier transform $g_{N,w}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, where

$$g_{N,w}(k) = \int_{\Omega} {\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p) \mathrm{d}Q_{N,w}.$$

By the definition of $Q_{N,w}$, we find that

$$\begin{aligned} g_{N,w}(\underline{k}) &= \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) {\prod_{p\in\mathbb{P}}}' p^{-ik^\alpha h k_p} \\ &= \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \exp\left\{ -ik^\alpha h {\sum_{p\in\mathbb{P}}}' k_p \log p \right\}. \end{aligned} \tag{4.16}$$

We recall that the sign "′" means that only a finite number of integers k_p are distinct from zero. Clearly, by (4.16),

$$g_{N,w}(\underline{0}) = 1. \tag{4.17}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we have that

$$\sum_{p\in\mathbb{P}}{}' k_p \log p \neq 0$$

for $\underline{k} \neq \underline{0}$. Therefore, by Lemmas 4.8 and 4.9, for $\underline{k} \neq \underline{0}$,

$$R(u) \stackrel{\text{def}}{=} \sum_{k\leqslant u} \exp\left\{ -ik^\alpha h \sum_{p\in\mathbb{P}}{}' k_p \log p \right\} = o(u)$$

as $u \to \infty$. Hence, using (4.16) and summing by parts, we find that

$$\begin{aligned} g_{N,w}(\underline{k}) &= \frac{R(N)w(N)}{V(N,w)} - \frac{1}{V(N,w)} \int_1^N R(u)w'(u)\mathrm{d}u \\ &= o\left(\frac{Nw(N)}{V(N,w)}\right) + o\left(\frac{1}{V}\int_1^N u|w'(u)|\mathrm{d}u\right) = o(1) \end{aligned}$$

as $N \to \infty$, since

$$Nw(N) \ll V(N,w) + \int_1^N u|w'(u)|\mathrm{d}u \ll V(N,w).$$

This together with (4.17) gives

$$\lim_{N\to\infty} g_{N,w}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

This equality, as in previous sections, proves the lemma. $\square$

Next, we will prove a limit theorem for absolutely convergent Dirichlet series $\zeta_n(s;\mathfrak{a})$, where $\zeta_n(s;\mathfrak{a})$ is the same as in previous sections. We use the function $u_n : \Omega \to H(D)$ defined by the formula

$$u_n(\omega) = \zeta_n(s,\omega;\mathfrak{a}).$$

Then we have the following analogue of Lemma 4.4 for the measure

$$P_{T,n,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1 \leqslant k \leqslant N : \zeta(s+ik^\alpha h; \mathfrak{a}) \in A\Big\}\Big),$$
$$A \in \mathcal{B}(H(D)).$$

Lemma 4.11. *Suppose that $w \in V_2$, and $0 < \alpha < 1$ is fixed. Then $P_{T,n,w}$ converges weakly to the measure $V_n = m_H u_n^{-1}$ as $N \to \infty$.*

Proof. The lemma follows from Lemmas 4.10 and 1.9 by using the same arguments as in the proof of Lemma 4.4. □

The next lemma is devoted to the approximation of the function $\zeta(s;\mathfrak{a})$ by $\zeta_n(s;\mathfrak{a})$.

Lemma 4.12. *Suppose that $w \in V_2$, and $0 < \alpha < 1$ is fixed. Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \rho(\zeta(s+ik^\alpha h;\mathfrak{a}), \zeta_n(s+ik^\alpha h;\mathfrak{a})) = 0.$$

Proof. As in the proof of Lemma 4.5, we apply Lemma 4.6. It is easily seen that, for $2 \leqslant k \leqslant N$,

$$(k+1)^\alpha - k^\alpha \geqslant \frac{\alpha}{2N^{1-\alpha}}.$$

Therefore, an application of Lemma 4.6 with $\delta = \frac{h\alpha}{2N^{1-\alpha}}$ gives, for $\frac{1}{2} < \sigma < 1$ and $\tau \in \mathbb{R}$, the estimate

$$\sum_{k=1}^{N} |\zeta(\sigma+ik^\alpha h+i\tau;\mathfrak{a})|^2 \ll N^{1-\alpha} \int_1^{N^\alpha h} |\zeta(\sigma+it+i\tau;\mathfrak{a})|^2 \mathrm{d}t$$
$$+ \left(\int_1^{N^\alpha h} |\zeta(\sigma+it+i\tau;\mathfrak{a})|^2 \mathrm{d}t \int_1^{N^\alpha h} |\zeta'(\sigma+it+i\tau;\mathfrak{a})|^2 \mathrm{d}t \right)^{\frac{1}{2}} \ll N(1+|\tau|^2)$$

in view of the known estimate

$$\int_1^{T} |\zeta(\sigma+it+i\tau;\mathfrak{a})|^2 \mathrm{d}t \ll T(1+|\tau|^2)$$

and

$$\int_1^T |\zeta'(\sigma+it+i\tau;\mathfrak{a})|^2 \mathrm{d}t \ll T(1+|\tau|^2),$$

and the equality

$$N_\delta(t)=1.$$

Therefore, for the same σ and τ,

$$\begin{aligned}
&\sum_{k=1}^N w(k)|\zeta(\sigma+ik^\alpha h+i\tau;\mathfrak{a})|^2 \ll w(N)\sum_{k=1}^N |\zeta(\sigma+ik^\alpha h+i\tau;\mathfrak{a})|^2\\
&+\int_1^N \sum_{k\leqslant u}|\zeta(\sigma+k^\alpha h+i\tau;\mathfrak{a})|^2|w'(u)|\mathrm{d}u \qquad (4.18)\\
&\ll Nw(N)(1+|\tau|^2)+(1+|\tau|^2)\int_1^N u|w'(u)|\mathrm{d}u \ll V(N,w)(1+|\tau|)
\end{aligned}$$

because

$$Nw(N)\ll V(N,w).$$

The further proof is similar to that of Lemma 4.5. We take an arbitrary compact set $K\subset D$ and fix $\varepsilon>0$ such that $\frac{1}{2}+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$ for $s\in K$. Then

$$\frac{1}{V(N,w)}\sum_{k=1}^N w(k)\sup_{s\in K}|\zeta(s+ik^\alpha h;\mathfrak{a})-\zeta_n(s+ik^\alpha h;\mathfrak{a})|\ll S_1+S_2, \qquad (4.19)$$

where

$$S_1=\frac{1}{V(N,w)}$$

$$\int_{-\infty}^{+\infty}\left(\sum_{k=1}^N w(k)\left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+k^\alpha h);\mathfrak{a}\right)\right|\right)\sup_{s\in K}\frac{\left|l_n\left(\frac{1}{2}+\varepsilon-s+it\right)\right|}{\left|\frac{1}{2}+\varepsilon-s+it\right|}\mathrm{d}t$$

and

$$S_2=\frac{1}{V(N,w)}\sum_{k=1}^N w(k)\sup_{s\in K}|R_n(s+ik^\alpha h)|.$$

In view of (4.18),

$$\sum_{k=1}^{N} w(k)\left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+k^{\alpha}h);\mathfrak{a}\right)\right|$$
$$\ll \left(\sum_{k=1}^{N} w(k)\sum_{k=1}^{N} w(k)\left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+k^{\alpha}h);\mathfrak{a}\right)\right|^2\right)^{\frac{1}{2}} \ll V(N, w(1+|t|)).$$

Therefore,

$$S_1 \ll_K n^{-\varepsilon}\int_{-\infty}^{+\infty}(1+|t|)\exp\{-c_1|t|\} \ll_K n^{-\varepsilon}. \tag{4.20}$$

Similarly, as in Section 4.2, we find that

$$\begin{aligned} S_2 &\ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{V(N,w)}\sum_{k=1}^{N} w(k)\exp\{-c_2k^{\alpha}\} \\ &\ll_K \frac{n^{\frac{1}{2}-2\varepsilon}}{V(N,w)}\left(\exp\{-c_2N^{\alpha}\}V(N,w)+\int_1^N \exp\{-c_2u^{\alpha}\}|w'(u)|\mathrm{d}u\right). \end{aligned} \tag{4.21}$$

Let $N_1 \to \infty$ bet such that

$$\int_1^{N_1} u|w'(u)|\mathrm{d}u = o(V(N,w)) \tag{4.22}$$

as $N \to \infty$. Then

$$\int_1^N \exp\{-c_2u^{\alpha}\}|w'(u)|\mathrm{d}u = \left(\int_1^{N_1}+\int_{N_1}^{N}\right)\exp\{-c_2u^{\alpha}\}|w'(u)|\mathrm{d}u$$
$$\ll o(V(N,w)) + \exp\{-c_2N_1^{\alpha}\}\int_1^N u|w'(u)|\mathrm{d}u = o(V(N,w))$$

as $N \to \infty$. Therefore, by (4.21) and (4.22), $S_2 = o(1)$ as $N \to \infty$. This and (4.19), (4.20) together with the definition of the metric ρ prove the lemma. □

Now, we state a weighted discrete limit theorem. Let, in this section,

$$P_{N,w}(A) = \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I(\{1 \leqslant k \leqslant N : \zeta(s + ik^{\alpha}h; \mathfrak{a}) \in K\}),$$
$$A \in \mathcal{B}(H(D)).$$

We also preserve the notation of Section 3.2 for $\zeta(s, \omega; \mathfrak{a})$, P_ζ and the set S.

Theorem 4.13. *Suppose that $w \in V_2$, α, $0 < \alpha < 1$, is fixed, and the sequence $\mathfrak{a}$ is multiplicative. Then $P_{N,w}$ converges weakly to the measure P_ζ as $n \to \infty$.*

Proof. We apply similar arguments used in the proof of Theorem 4.2. On a certain probability space with measure μ, define the random variable θ_N by the formula

$$\mu(\theta_N = k^{\alpha}h) = \frac{w(k)}{V(N,w)}, k = 1, \dots, N.$$

Let

$$X_{N,n,w} = X_{N,n,w}(s) = \zeta_n(s + i\theta_N; \mathfrak{a}),$$

and let X_n be the $H(D)$-valued random element with the distribution V_n, where the measure V_n is defined in Lemma 4.11. Thus, by Lemma 4.11, we have the relation

$$X_{N,n,w} \xrightarrow[N \to \infty]{\mathcal{D}} X_n. \tag{4.23}$$

The series for $\zeta_n(s; \mathfrak{a})$ and $\zeta'_n(s; \mathfrak{a})$ are absolutely convergent for $\sigma > \frac{1}{2}$. Therefore

$$\limsup_{T \to \infty} \frac{1}{T} \int_1^T |\zeta_n(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)}{m^{2\sigma}} \leqslant \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} \leqslant C_1 < \infty$$

and

$$\limsup_{T \to \infty} \frac{1}{T} \int_1^T |\zeta'_n(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m) \log^2 m}{m^{2\sigma}}$$
$$\leqslant \sum_{m=1}^{\infty} \frac{|a_m|^2 \log^2 m}{m^{2\sigma}} \leqslant C_2 < \infty.$$

Hence, using Lemma 4.6, we find as above that, for $\frac{1}{2} < \sigma < 1$,

$$\sum_{k=1}^{N} |\zeta_n(\sigma + ik^\alpha h; \mathfrak{a})|^2 \ll N^{1-\alpha} \int\limits_1^{N^\alpha h} |\zeta_n(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t$$
$$+ \Bigg(\int\limits_1^{N^\alpha h} |\zeta_n(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t \int\limits_1^{N^\alpha h} |\zeta_n'(\sigma + it; \mathfrak{a})|^2 \mathrm{d}t \Bigg)^{\frac{1}{2}} \ll N.$$

Therefore, by properties of the weight function $w(k)$, we obtain that, for $\frac{1}{2} < \sigma < 1$,

$$\sup_{n \in \mathbb{N}} \limsup_{N \to \infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) |\zeta_n(\sigma + ik^\alpha h; \mathfrak{a})| \leqslant C < \infty.$$

This and the Cauchy integral formula imply

$$\sup_{n \in \mathbb{N}} \limsup_{N \to \infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s \in K_l} |\zeta_n(\sigma + ik^\alpha h; \mathfrak{a})| \leqslant C_l < \infty, \quad (4.24)$$

where $\{K_l\}$ is a sequence of compact sets from the definition of metric ρ.

Now, we fix $\varepsilon > 0$ and define $\mu_l = \mu_l(\varepsilon) = 2^l C_l \varepsilon^{-1}$. Then by the definition of $X_{N,n,w}$ and (4.24),

$$\limsup_{N \to \infty} \mu\Big(\sup_{s \in K} |X_{N,n,w}(s)| > M_l \Big)$$
$$= \limsup_{N \to \infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{ 1 \leqslant k \leqslant N : \sup_{s \in K_l} |\zeta_n(s + ik^\alpha h; \mathfrak{a})| > M_l \Big\} \Big)$$
$$\leqslant \sup_{n \in \mathbb{N}} \limsup_{N \to \infty} \frac{1}{M_l V(N,w)} \sum_{k=1}^{N} w(k) \sup_{s \in K_l} |\zeta_n(s + ik^\alpha h; \mathfrak{a})| \leqslant \frac{\varepsilon}{2^l}.$$

From this and (4.23), we deduce that, for all $n, l \in \mathbb{N}$,

$$\mu\Big(\sup_{s \in K_l} |X_n(s)| > M_l \Big) \leqslant \frac{\varepsilon}{2^l}. \quad (4.25)$$

The set $H_\varepsilon = \{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leqslant M_l, l \in \mathbb{N} \}$ is compact in the space

$H(D)$. Moreover, in view of (4.25),

$$\mu(X_n \in H_\varepsilon) \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} \geqslant 1 - \varepsilon.$$

Hence, by the definition of V_n, for all $n \in \mathbb{N}$,

$$V_n(H_\varepsilon) \geqslant 1 - \varepsilon.$$

This shows that the sequence $\{V_n : n \in \mathbb{N}\}$ is tight. Therefore, by Lemma 1.16, it is relatively compact. Thus, there exists a subsequence $\{V_{n_r}\} \subset \{V_n\}$ weakly convergent to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$. In other words,

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{4.26}$$

An application of Lemma 4.12 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V(N,w)} \\
&\sum_{k=1}^{N} w(k) I\big(\{1 \leqslant k \leqslant N : \rho(\zeta(s+ik^\alpha h;\mathfrak{a}), \zeta_n(s+ik^\alpha h;\mathfrak{a})) \geqslant \varepsilon\}\big) \\
&\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon V(N,w)} \sum_{k=1}^{N} w(k) \rho(\zeta(s+ik^\alpha h;\mathfrak{a}), \zeta_n(s+ik^\alpha h;\mathfrak{a})) = 0.
\end{aligned} \tag{4.27}$$

Now, in view of relations (4.23), (4.26) and (4.27), we can apply Lemma 1.18 which shows that

$$\zeta(s+i\theta_N;\mathfrak{a}) \xrightarrow[N\to\infty]{\mathcal{D}} P.$$

This means that $P_{N,w}$ converges weakly to P as $N \to \infty$. Moreover, this shows that the measure P is independent of the subsequence $\{V_{n_r}\}$. This remark together with relative compactness of $\{V_n\}$ implies the relation

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

Consequently, by the definition of X_n, we have that V_n converges weakly to P as $n \to \infty$, i.e., $P_{N,w}$, as $N \to \infty$, converges weakly to the limit measure of V_n as $n \to \infty$. Since, by the proof of Theorem 1.18, the measure P_T, as $T \to \infty$, also converges weakly to the limit measure P of V_n as $n \to \infty$, and

P coincides with P_ζ, we obtain that $P_{N,w}$ converges weakly to P_ζ as $N \to \infty$. The theorem is proved. $\square$

4.5 Proof of Theorem 4.7

As all previous universality theorems, Theorem 4.7 follows from a limit theorem in the space of analytic functions (Theorem 4.13) and the Mergelyan theorem (Lemma 1.27).

Proof of Theorem 4.7. The first part. We find a polynomial $p(s)$ such that inequality (4.13) would be satisfied, and consider the set

$$\mathcal{G}_\varepsilon = \{g \in H(D) : \sup_{s\in K} |g(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\}.$$

By Theorem 1.7, the support of the measure P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Thus, $\mathcal{G}_\varepsilon$ is an open neighborhood of $e^{p(s)} \in S$. Hence,

$$P_\zeta(\mathcal{G}_\varepsilon) > 0. \tag{4.28}$$

Therefore, Theorem 4.13 and Lemma 1.26 imply the inequality

$$\liminf_{N\to\infty} P_{N,w}(\mathcal{G}_\varepsilon) \geqslant P_\zeta(\mathcal{G}_\varepsilon) > 0.$$

This, and the definitions of $P_{N,w}$ and $\mathcal{G}_\varepsilon$ show that

$$\liminf_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\Big(\Big\{1 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ik^\alpha h; \mathfrak{a}) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}\Big\}\Big) > 0,$$

and using of inequality (4.13) proves the theorem.

The second part. The set

$$\hat{\mathcal{G}}_\varepsilon = \{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\}.$$

is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Thus, by Theorem 4.13 and Lemma 1.19,

$$\lim_{N\to\infty} P_{N,w}(\hat{\mathcal{G}}_\varepsilon) = P_\zeta(\hat{\mathcal{G}}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. Therefore, the definitions of $P_{N,w}$ and $\mathcal{G}_\varepsilon$ imply

$$\lim_{N\to\infty} \frac{1}{V(N,w)} \sum_{k=1}^{N} w(k) I\left(\left\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ik^\alpha h;\mathfrak{a})-f(s)| < \varepsilon\right\}\right) = P_\zeta(\hat{\mathcal{G}}_\varepsilon)$$

for all but at most countably many $\varepsilon > 0$. Since, in view of (4.13), $\mathcal{G}_\varepsilon \subset \hat{\mathcal{G}}_\varepsilon$, this together with (4.28) proves the theorem. $\square$

Chapter 5

Value distribution of a certain composition

In this chapter, together with the periodic zeta-function $\zeta(s; \mathfrak{a})$ we will consider the periodic Hurwitz zeta-function $\zeta(s; \alpha; \mathfrak{b})$, where α, $0 < \alpha \leqslant 1$, is a fixed parameter, and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $l \in \mathbb{N}_0$. Then the function $\zeta(s; \alpha; \mathfrak{b})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s},$$

and, using the equality

$$\zeta(s; \alpha; \mathfrak{b}) = \frac{1}{l} \sum_{m=1}^{l-1} b_m \zeta\Big(s, \frac{m + \alpha}{l}\Big), \sigma > 1$$

can be meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$. If $b_m \equiv 1$, then $\zeta(s; \alpha; \mathfrak{b})$ becomes the classical Hurwitz zeta-function $\zeta(s; \alpha)$.

In this chapter, we will prove a joint universality theorem for the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s; \alpha; \mathfrak{b})$, and will obtain some estimates for the number of zeros of certain compositions of the above periodic functions. The results obtained generalize and extend similar known theorems.

5.1 A joint universality theorem for periodic zeta-functions

Let $L(\mathbb{P};\alpha) = \{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\}$. Moreover, let $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K which are continuous in the interior of K. The main result of this section is the following joint universality theorem.

Theorem 5.1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P};\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Bigg\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon, \\ \sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon\Bigg\} > 0.$$

Moreover, the same inequality with "$\lim$" holds for all but at most countably many $\varepsilon > 0$.

We notice that earlier Theorem 5.1 was known under certain additional restrictions. For example, in [16], it was required the transcendence of the parameter α and the inequality

$$\sum_{m=1}^{\infty} \frac{|a_p m|}{p^{\frac{m}{s}}} \leqslant c < 1$$

for each prime p.

We will use the method of Chapter 1, therefore, we will omit some details.

As in Chapter 1, we will use some topological structure. Together with the torus

$$\Omega = \prod_p \gamma_p,$$

we define one more torus

$$\Omega_1 = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With the product topology ant pointwise multiplication, the torus Ω_1, as Ω, is a compact topological Abelian group.

Putting

$$\hat{\Omega} = \Omega \times \Omega_1$$

again gives a new compact topological Abelian group. Therefore, on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$, the probability Haar measure $\hat{m}_H$ exists, and we obtain the probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$. On this probability space, define the $H^2(D) = H(D) \times H(D)$-valued random element

$$\hat{\zeta}(s, \omega, \omega_1, \alpha; \mathfrak{a}, \mathfrak{b}) = (\zeta(s, \omega; \mathfrak{a}), \zeta(s, \omega_1, \alpha; \mathfrak{b})),$$

where

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}$$

and

$$\zeta(s, \omega, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_1(m)}{(m+\alpha)^s},$$

and $\omega_1(m)$ denotes the m-the component of an element $\omega_1 \in \Omega_1$, $m \in \mathbb{N}_0$, and by $\hat{\omega} = (\omega, \omega_1)$ the element of $\hat{\Omega}$. Denote by $P_{\hat{\zeta}}$ the distribution of the random element $\hat{\zeta}(s, \omega, \omega_1; \mathfrak{a}, \mathfrak{b})$, i.e.,

$$P_{\hat{\zeta}}(A) = \hat{m}_H\{\hat{\omega} \in \hat{\Omega} : \hat{\zeta}(s, \omega, \omega_1; \mathfrak{a}, \mathfrak{b}) \in A\}, A \in \mathcal{B}(H^2(D)).$$

Let, for brevity,

$$\hat{\zeta}(s, \alpha; \mathfrak{a}, \mathfrak{b}) = (\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b})).$$

The proof of Theorem 5.1 is based on a limit theorem for

$$P_{T,\hat{\zeta}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : \hat{\zeta}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b}) \in A\}, A \in \mathcal{B}(H^2(D)).$$

Theorem 5.2. *Suppose that the set $L(\mathbb{P}; \alpha)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,\hat{\zeta}}$ converges to the measure $P_{\hat{\zeta}}$ as $T \to \infty$.*

We start the proof of Theorem 5.2 with a limit theorem for probability measure on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$. Let, for $A \in \mathcal{B}(\hat{\Omega})$,

$$\hat{Q}_T(A) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : ((p^{-i\tau} : p \in \mathbb{P}), ((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0)) \in A\}.$$

Lemma 5.3. *Suppose that the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$. Then*

$\hat{Q}_T$ converges weakly to the Haar measure $\hat{m}_H$ as $T \to \infty$.

Proof. We consider the Fourier transform $\hat{g}_T(\underline{k}, \underline{l})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, $\underline{l} = (l_m : l_m \in \mathbb{Z}, m \in \mathbb{N}_0)$, of the measure $\hat{Q}_T$, i.e.,

$$\begin{aligned}
\hat{g}_T(\underline{k}, \underline{l}) &= \int\limits_{\hat{\Omega}} {\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p) {\prod_{m\in\mathbb{N}_0}}' \omega^{l_m}(m) \mathrm{d}\hat{Q}_T \\
&= \frac{1}{T} \int\limits_0^T {\prod_{p\in\mathbb{P}}}' p^{-ik_p\tau} {\prod_{m\in\mathbb{N}_0}}' (m+\alpha)^{-il_p\tau} \mathrm{d}\tau \qquad (5.1)\\
&= \frac{1}{T} \int\limits_0^T \exp\left\{ -i\tau \left({\sum_{p\in\mathbb{P}}}' k_p \log p + {\sum_{m\in\mathbb{N}_0}}' l_m \log(m+\alpha) \right) \right\} \mathrm{d}\tau,
\end{aligned}$$

where "$'$" means that only a finite number of integers k_p and l_m are distinct form zero. Clearly, in view of (5.1)

$$\hat{g}_T(\underline{0}, \underline{0}) = 1. \tag{5.2}$$

If $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$, then

$$A(\underline{k}, \underline{l}) \stackrel{\text{def}}{=} {\sum_{p\in\mathbb{P}}}' k_p \log p + {\sum_{m\in\mathbb{N}_0}}' l_m \log(m+\alpha) \neq 0$$

because the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$. Thus, integrating in (5.1), we find that

$$\hat{g}_T(\underline{k}, \underline{l}) = \frac{1 - \exp\{-iTA(\underline{k}, \underline{l})\}}{iTA(\underline{k}, \underline{l})}.$$

This and (5.2) show that

$$\lim_{T\to\infty} \hat{g}_T(\underline{k}, \underline{l}) = \begin{cases} 1 & \text{if} \quad (\underline{k}, \underline{l}) = (\underline{0}, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0}). \end{cases}$$

Therefore, the lemma follows by a continuity theorem for probability measures on compact groups. $\square$

Additionally to $v_n(m)$ defined in Section 1.4, we define

$$v_n(m, \alpha) = \exp\left\{ -\left(\frac{m+\alpha}{n+\alpha} \right)^{\theta} \right\}, m \in \mathbb{N}_0, n \in \mathbb{N},$$

and

$$\zeta_n(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_n(m,\alpha)}{(m+\alpha)^s}.$$

Let, for brevity,

$$\hat{\zeta}_n(s,\alpha;\mathfrak{a},\mathfrak{b}) = (\zeta_n(s;\mathfrak{a}),\zeta_n(s,\alpha;\mathfrak{b}))$$

and, for $(\omega,\omega_1) \in \hat{\Omega}$,

$$\hat{\zeta}_n(s,\omega,\omega_1,\alpha;\mathfrak{a},\mathfrak{b}) = (\zeta_n(s,\omega;\mathfrak{a}),\zeta_n(s,\omega_1,\alpha;\mathfrak{b})),$$

where

$$\zeta_n(s,\omega_1,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_1(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

The series for $\zeta_n(s,\alpha;\mathfrak{b})$ and $\zeta_n(s,\omega_1,\alpha;\mathfrak{b})$, as for $\zeta_n(s;\mathfrak{a})$ and $\zeta_n(s,\omega;\mathfrak{a})$, are absolutely convergent for $\sigma > \frac{1}{2}$ [11]. Define the function $u_{n,\alpha}(\hat{\omega}) : \hat{\Omega} \to H^2(D)$ by

$$u_{n,\alpha}(\omega,\omega_1) = (\zeta_n(s,\omega;\mathfrak{a}),\zeta_n(s,\omega_1,\alpha;\mathfrak{b})).$$

This function is continuous because of the absolute convergence of the above series. Let, for $A \in \mathcal{B}(H^2(D))$,

$$P_{T,n,\alpha}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \hat{\zeta}_n(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}) \in A\},$$

and let $V_{n,\alpha} = \hat{m}_H u_{n,\alpha}^{-1}$. Then, similarly to the proof of Theorem 1.7, using of Lemma 5.3 leads to the following assertion.

Lemma 5.4. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,n,\alpha}$ converges weakly to the measure $V_{n,\alpha}$ as $T \to \infty$.*

On the space $H^2(D)$, define the metric inducing its product topology. For $\underline{g}_1 = (g_{11},g_{12})$, $\underline{g}_2 = (g_{21},g_{22})$, we put

$$\rho_2(\underline{g}_1,\underline{g}_2) = \max(\rho(g_{11},g_{12}),\rho(g_{21},g_{22})),$$

where ρ is the metric in $H(D)$ defined in Section 1.5. Then we have

Lemma 5.5. *The equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T \rho_2(\hat{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}),\hat{\zeta}_n(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}))\mathrm{d}\tau=0$$

holds.

Proof. By the definition of the metric ρ, the equality of the lemma follows from the equalities

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T \rho(\zeta(s+i\tau;\mathfrak{a}),\zeta_n(s+i\tau;\mathfrak{a}))\mathrm{d}\tau=0$$

and

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T \rho(\zeta(s+i\tau,\alpha;\mathfrak{b}),\zeta_n(s+i\tau,\alpha;\mathfrak{b}))\mathrm{d}\tau=0.$$

However, the first of these equality is Lemma 1.10, and the second one is equality (4) of [11]. We note that the proof is independent on arithmetic of the parameter α. □

The next lemma repeats the proof of Lemma 1.19, and uses Lemmas 5.4, 5.5 and 1.16, see, also [16].

Lemma 5.6. *The sequence $\{V_{n,\alpha}\}$ is tight.*

Proof of Theorem 5.2. By Lemmas 5.6 and 1.16, the sequence $\{V_{n,\alpha}\}$ is relatively compact. Therefore, there exists a subsequence $\{V_{n_k,\alpha}\}\subset\{V_{n,\alpha}\}$ such that $\{V_{n_k,\alpha}\}$ converges weakly to a certain probability measure P_α on $(H^2(D),\mathcal{B}(H^2(D)))$ as $k\to\infty$. Hence,

$$X_{n_k,\alpha}\xrightarrow[k\to\infty]{\mathcal{D}} P_\alpha, \tag{5.3}$$

where $X_{n,\alpha}=X_{n,\alpha}(s)$ is the $H^2(D)$-valued random element having the distribution $V_{n,\alpha}$. Let ξ bet the same random element as in Section 1.6. Define the $H^2(D)$-valued random element $X_{T,n,\alpha}$ by

$$X_{T,n,\alpha}=X_{T,n,\alpha}(s)=\hat{\zeta}_n(s+i\xi T,\alpha;\mathfrak{a},\mathfrak{b}).$$

Then, in view of Lemma 5.4, we have the relation

$$X_{T,n,\alpha} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\alpha}. \tag{5.4}$$

Define one more $H^2(D)$-valued random element $Y_{T,\alpha}$ by

$$Y_{T,\alpha} = Y_{T,\alpha}(s) = \hat{\zeta}(s + i\xi T, \alpha; \mathfrak{a}, \mathfrak{b}).$$

Then Lemma 5.5 shows that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \mu\{\rho_2(Y_{T,\alpha}, X_{T,n,\alpha}) \geqslant \varepsilon\} = \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \\ &\operatorname{meas}\big\{\tau \in [0,T] : \rho_2(\hat{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}), \hat{\zeta}_n(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})) \geqslant \varepsilon\big\} \\ &\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \rho_2(\hat{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}), \hat{\zeta}_n(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}))\mathrm{d}\tau = 0. \end{aligned} \tag{5.5}$$

From (5.3)-(5.5), it follows that it is possible to apply Lemma 1.19. This application gives the relation

$$Y_{T,\alpha} \xrightarrow[T\to\infty]{\mathcal{D}} P_\alpha. \tag{5.6}$$

This means that the measure $P_{T,\hat{\zeta}}$ converges weakly to P_α as $T \to \infty$, and that the measure P_α is independent of the choice of the subsequence $\{X_{n_k,\alpha}\}$. Therefore, we have the relation

$$X_{n,\alpha} \xrightarrow[n\to\infty]{\mathcal{D}} P_\alpha,$$

which together with (5.6) shows that the measure $P_{T,\hat{\zeta}}$ converges weakly as $T \to \infty$ to the limit measure P_α of the measure $V_{n,\alpha}$ as $n \to \infty$.

The linear independence of the set $L(\mathbb{P}, \alpha)$ is used only in the proof of Lemma 5.3. Therefore, the further proof of Theorem 5.2 is independent of α. Since we have obtained that $P_{T,\hat{\zeta}}$ converges weakly to the limit measure of $V_{n,\alpha}$, this limit measure must be the same as in [11] in the case of transcendental α. Thus, P_α coincides with $P_{\hat{\zeta}}$. The theorem is proved. □

Proof of Theorem 5.1. It is known by [11] that the support of the measure $P_{\hat{\zeta}}$ is the set $S \times H(D)$, where, as in Chapter 1,

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

By Lemma 1.27, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} |f_1(s) - \mathrm{e}^{p_1(s)}| < \frac{\varepsilon}{2} \tag{5.7}$$

and

$$\sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{5.8}$$

Define the set

$$\mathcal{G}_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |f_1(s) - \mathrm{e}^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2} \right\}.$$

Then, by the above remark, the set $\mathcal{G}_\varepsilon$ is an open neighborhood of an element $(\mathrm{e}^{p_1(s)}, p_2(s))$ of the support of the measure $P_{\hat{\zeta}}$. Hence, by Lemma 1.26 and Theorem 5.2,

$$\liminf_{T\to\infty} P_{T,\hat{\zeta}}(\mathcal{G}_\varepsilon) \geqslant P_{\hat{\zeta}}(\mathcal{G}_\varepsilon) > 0.$$

This, the definitions of $P_{T,\hat{\zeta}}$ and $\mathcal{G}_\varepsilon$ together with (5.7) and (5.8) prove the first part of the theorem.

To prove the second assertion of the theorem, define the set

$$\hat{\mathcal{G}}_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\}.$$

Then the boundary lies in the set

$$\left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\}$$
$$\bigcup \left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| = \varepsilon \right\}$$
$$\bigcup \left\{ (g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| = \varepsilon \right\},$$

therefore, $\delta\hat{\mathcal{G}}_{\varepsilon_1} \cap \delta\hat{\mathcal{G}}_{\varepsilon_2} = \emptyset$ for different positive ε_1 and ε_2. Thus, the set $\hat{\mathcal{G}}_\varepsilon$ is a continuity set for all but at most countably many $\varepsilon > 0$. Hence, in view of Lemma 1.19 and Theorem 5.2,

$$\lim_{T\to\infty} P_{T,\hat{\zeta}}(\hat{\mathcal{G}}_\varepsilon) = P_{\hat{\zeta}}(\hat{\mathcal{G}}_\varepsilon) \tag{5.9}$$

for all but at most countably many $\varepsilon > 0$. Suppose that $(g_1, g_2) \in \mathcal{G}_\varepsilon$. Then taking into account (5.8) and (5.9), we find that

$$\sup_{s\in K_1}|g_1(s)-f_1(s)|\leqslant \sup_{s\in K_1}|g_1(s)-\mathrm{e}^{p_1(s)}|+\sup_{s\in K_1}|f_1(s)-\mathrm{e}^{p_1(s)}|<\varepsilon$$

and

$$\sup_{s\in K_2}|g_2(s)-f_2(s)|\leqslant \sup_{s\in K_2}|g_2(s)-p_2(s)|+\sup_{s\in K_2}|f_2(s)-p_2(s)|<\varepsilon.$$

Thus, $(g_1, g_2) \in \hat{\mathcal{G}}_\varepsilon$, and we have the inclusion $\mathcal{G}_\varepsilon \subset \hat{\mathcal{G}}_\varepsilon$. Since $P_{\hat{\zeta}}(\mathcal{G}_\varepsilon) > 0$, hence it follows that also $P_{\hat{\zeta}}(\hat{\mathcal{G}}_\varepsilon) > 0$. This, the definitions of $P_{T,\hat{\zeta}}$ and $\hat{\mathcal{G}}_\varepsilon$, and (5.10) prove the second part of the theorem. The theorem is proved. □

5.2 Universality of some compositions

First of all, we recall the classical Rouché Theorem.

Lemma 5.7. *Let the functions $g_1(s)$ and $g_1(s)$ be analytic in the interior of a closed simple contour L and on L, and let on L the inequalities $g_1(s) \neq 0$ and $|g_2(s)| < |g_1(s)|$ be satisfied, Then the functions $g_1(s)$ and $g_1(s)+g_2(s)$ have the same number of zeros in the interior of L.*

Proof of the lemma can be found, for example, in [51].

Define the function

$$\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b}) = c_1\zeta(s;\mathfrak{a}) + c_2\zeta(s,\alpha;\mathfrak{b}), c_1, c_2 \in \mathbb{C}\backslash\{0\}.$$

We will prove a lower bound for the number of zeros of the function $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$.

Theorem 5.8. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$, and the sequence $\mathfrak{a}$ is multiplicative. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, \mathfrak{b}) > 0$ such that, for sufficiently large T, the function $\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b})$ has more than cT zeros in the rectangle*

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

Proof. Let

$$\sigma_0 = \frac{\sigma_1+\sigma_2}{2}, r = \frac{\sigma_2-\sigma_1}{2},$$

and let the number $\varepsilon > 0$ satisfy the inequality

$$(|c_1| + |c_2|)\varepsilon < \frac{1}{10} \min_{|s-\sigma_0|=r} |s - \sigma_0| = \frac{r}{10}. \tag{5.10}$$

We take in Theorem 5.1

$$f_1(s) = \varepsilon, f_2(s) = \frac{1}{c_2}(s - \sigma_0).$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequalities

$$\sup_{|s-\sigma_0|\leqslant r} |\zeta(s + i\tau; \mathfrak{a}) - f_1(s)| < \varepsilon \tag{5.11}$$

and

$$\sup_{|s-\sigma_0|\leqslant r} |\zeta(s + i\tau, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon. \tag{5.12}$$

Then, for these τ, we have that

$$\sup_{|s-\sigma_0|\leqslant r} |c_1\zeta(s+i\tau; \mathfrak{a})+c_2\zeta(s+i\tau, \alpha; \mathfrak{b})-(c_1 f_1(s)+c_2 f_2(s))| < 2(|c_1|+|c_2|)\varepsilon.$$

Moreover, by the definitions of $f_1(s)$ and $f_2(s)$,

$$\sup_{|s-\sigma_0|\leqslant r} |c_1 f_1(s) + c_2 f_2(s) - (s - \sigma_0)| = |c_1|\varepsilon.$$

Therefore,

$$\sup_{|s-\sigma_0|=r} |c_1\zeta(s + i\tau; \mathfrak{a}) + c_2\zeta(s + i\tau, \alpha; \mathfrak{b}) - (s - \sigma_0)| < 3(|c_1| + |c_2|)\varepsilon.$$

This and (5.10) show that the functions $s - \sigma_0$ and

$$c_1\zeta(s + i\tau; \mathfrak{a}) + c_2\zeta(s + i\tau, \alpha; \mathfrak{b}) - (s - \sigma_0)$$

on the disc $|s - \sigma_0|\leqslant r$ satisfy the hypotheses of Lemma 5.7. Therefore, the function $c_1\zeta(s + i\tau; \mathfrak{a}) + c_2\zeta(s + i\tau, \alpha; \mathfrak{b})$ has a zero in the disc $|s - \sigma_0| < r$. However, by Theorem 5.1, the set of τ satisfying inequalities (5.11) and (5.12) has a positive lower density. Hence, there exists a constant $c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, \mathfrak{b}) > 0$ such that, for sufficiently large T, the function $\underline{\zeta}(s, \alpha; \mathfrak{a}, \mathfrak{b})$ has more than cT

zeros in the rectangle

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

$\square$

Now, we define one class of operators $F : H^2(D) \to H(D)$. Let $\beta_1 > 0$ and $\beta_2 > 0$. We say that the operator F belongs to the class $\mathrm{Lip}(\beta_1, \beta_2)$ if it satisfies the following conditions:

1° For each polynomial $p = p(s)$ and any set $K \in \mathcal{K}$, there exists an element $(g_1, g_2) \in F^{-1}\{p\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on K;

2° For any set $K \in \mathcal{K}$, there exists a positive constant c and sets $K_1, K_2 \in \mathcal{K}$ such that

$$\sup_{s\in K} |F(g_{11}(s), g_{12}(s)) - F(g_{21}(s), g_{22}(s))| \leqslant c \sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{j1}, g_{j2}) \in H^2(D)$, $j = 1, 2$.

We recall that $\mathcal{K}$ is the class of compact subsets of the strip D with connected complements.

Now, we state an universality theorem for the composition $F(\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b}))$ with $F \in \mathrm{Lip}(\beta_1, \beta_2)$.

Theorem 5.9. *Suppose that the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$, the sequence $\mathfrak{a}$ is multiplicative and $F \in \mathrm{Lip}(\beta_1, \beta_2)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\{\tau \in [0, T] : \sup_{s\in K} |F(\zeta(s+i\tau; \mathfrak{a}), \zeta(s+i\tau, \alpha; \mathfrak{b})) - f(s)| < \varepsilon\} > 0.$$

Proof. By Lemma 1.27, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{5.13}$$

Let $K_1, K_2 \in \mathcal{K}$ correspond the set K in condition 2° of the class $\mathrm{Lip}(\beta_1, \beta_2)$. By 1° of the class $\mathrm{Lip}(\beta_1, \beta_2)$, we find $(g_1, g_2) \in F^{-1}\{p\}$ such that $g_1(s) \neq 0$ on K. For simplicity, let $c_1 = \max(1, c)$, where $c > 0$ is from the condition

2° of the class $\mathrm{Lip}(\beta_1, \beta_2)$. Define the set

$$\mathcal{G}_{\varepsilon,T} = \Bigg\{ \tau \in [0,T] : \sup_{s \in K_1} |\zeta(s+i\tau;\mathfrak{a}) - g_1(s)| < c_1^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}, \\ \sup_{s \in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - g_2(s)| < c_1^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}} \Bigg\},$$

where $\beta = \max(\beta_1, \beta_2)$. Then, by Theorem 5.1

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \mathcal{G}_{\varepsilon,T} > 0. \tag{5.14}$$

Using 2° of the class $\mathrm{Lip}(\beta_1, \beta_2)$, we find that, for $\tau \in \mathcal{G}_{\varepsilon,T}$,

$$\begin{aligned} &\sup_{s\in K} |F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b})) - p(s)| \\ &\leqslant c \max\left(\sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - g_1(s)|^{\beta_1}, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - g_2(s)|^{\beta_2} \right) \\ &\leqslant c c_1^{-\frac{\beta}{2}} \left(\frac{\varepsilon}{2}\right)^{\frac{\beta}{2}} \leqslant \frac{\varepsilon}{2}. \end{aligned}$$

Hence, taking into account (5.14), we obtain that

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b})) - p(s)| < \frac{\varepsilon}{2} \right\} > 0,$$

and using of (5.13) completes the proof. □

Theorem 5.9 contains an information on the zeros of the composition $F(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$.

Theorem 5.10. *Suppose that the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$, the sequence $\mathfrak{a}$ is multiplicative, and $F \in \mathrm{Lip}(\beta_1, \beta_2)$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, \mathfrak{b}, F) > 0$ such that, for sufficiently large T, the function $F(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$ has more than cT zeros in the rectangle*

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

Proof. We apply arguments similar to those used in the proof of Theorem 5.8.

We use the same notation as above. Suppose that $\varepsilon > 0$ satisfies the inequality

$$\varepsilon < \frac{1}{10} \min_{|s-\sigma_0|=r} |s - \sigma_0| = \frac{r}{10},$$

and that reals τ satisfy

$$\sup_{|s-\sigma_0|\leqslant r} |F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau, \alpha; \mathfrak{b})) - (s-\sigma_0)| < \varepsilon. \tag{5.15}$$

Then the functions $s - \sigma_0$ and

$$F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau, \alpha; \mathfrak{b})) - (s-\sigma_0)$$

on the disc $|s - \sigma_0|\leqslant r$ satisfy the requirements of Lemma 5.7. Hence, the function

$$F(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau, \alpha; \mathfrak{b}))$$

has a zero in the disc $|s - \sigma_0| < r$. However, in virtue of Theorem 5.9, the set of τ satisfying (5.15) has a positive lower density. Therefore, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, \mathfrak{b}, F) > 0$ such that, the function $F(\zeta(s;\mathfrak{a}), \zeta(s, \alpha; \mathfrak{b}))$, for sufficiently large T, has more than cT zeros in the rectangle

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

$\square$

Conclusions

1. For the periodic zeta-function $\zeta(s; \mathfrak{a})$ with multiplicative periodic sequence $\mathfrak{a}$, the universality theorem on the approximation of analytic functions by shifts $\zeta(s + i\tau; \mathfrak{a})$, $\tau \in \mathbb{R}$, is valid.

2. For the periodic sequences $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ with minimal period q (q is a prime number) satisfying

$$a_q = \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l,$$

   where $\varphi(q)$ is Euler totient function, the function $\zeta(s; \mathfrak{a})$ is universal or strongly universal.

3. For the function $\zeta(s; \mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$, a weighted universality theorem is true.

4. For the function $\zeta(s; \mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$, a weighted discrete universality theorem is true.

5. The compositions $F(\zeta(s; \mathfrak{a}), \zeta(s, \alpha; \mathfrak{b}))$, where $\zeta(s, \alpha; \mathfrak{b})$ is the periodic Hurwitz zeta-function, for some classes of operators F in the space of analytic functions, are universal. Moreover, they have infinely many zeros in the critical strip.

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