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On joint approximation of analytic functions by nonlinear shifts of zeta-functions of certain cusp forms*

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Abstract. In the paper, joint discrete universality theorems on the simultaneous approximation of a collection of analytic functions by a collection of discrete shifts of zeta-functions attached to normalized Hecke-eigen cusp forms are obtained. These shifts are defined by means of nonlinear differentiable functions that satisfy certain growth conditions, and their combination on positive integers is uniformly distributed modulo 1.

Keywords: Hecke-eigen cusp form, joint universality, uniform distribution modulo 1, zeta-function of cusp form.

1 Introduction

It is known that some of zeta and L-functions are universal in the sense that their shifts approximate wide classes of analytic functions. This interesting phenomenon was discovered by S.M. Voronin. In [22], he proved the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + \mathrm{i}t$. More precisely, the Voronin theorem says that if $f(s)$ is a continuous nonvanishing function on the disc $|s| \leqslant r$, $0 < r < 1/4$, and analytic in the interior of that disc, then, for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s|\leqslant r}\left|\zeta\left(s + \frac{3}{4} + \mathrm{i}\tau\right) - f(s)\right| < \varepsilon.$$

Later, various authors improved the Voronin theorem and generalized it for other zeta-functions. One of universal classes contains the zeta-functions attached to certain cusp forms.

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Let
$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \colon\ a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$$
be the full modular group. The function $F(z)$ analytic in the half-plane $\operatorname{Im} z > 0$ is called a cusp form of weight $\kappa \in 2\mathbb{N}$ (κ is even because if κ is odd, then F is identically zero) for the full modular group if, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, it satisfies the functional equation
$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z)$$
and, at infinity, has the following Fourier series expansion:
$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi \mathrm{i} m z}.$$
Then the zeta-function $\zeta(s,F)$ of the cusp form $F(z)$ is defined for $\sigma > (\kappa+1)/2$ by the Dirichlet series
$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$
and is analytically continued to an entire function. It is convenient to require additionally that $F(z)$ would be the Hecke-eigen cusp form, i.e., that $F(z)$ would be the eigenfunction of all Hecke operators
$$T_m F(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^{\kappa}} \sum_{b\,(\mathrm{mod}\,d)} F\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N}.$$
In this case, the form can be normalized, and the function $\zeta(s,F)$ has Euler's product representation over primes
$$\zeta(s,F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$
where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $\alpha(p)+\beta(p)=c(p)$. The classical theory of cusp forms and related zeta-functions can be found, for example, in [1].

Denote $D = D_\kappa = \{s \in \mathbb{C}\colon\ \kappa/2 < \sigma < (\kappa+1)/2\}$. Let $\mathcal{K} = \mathcal{K}_\kappa$ be the class of compact subsets of the strip D_κ with connected complements, and $H_0(K) = H_{0\kappa}(K)$ with $K \in \mathcal{K}_\kappa$ be the class of continuous nonvanishing functions on K that are analytic in the interior of K. Then, in [10], the following universality theorem has been obtained.

Theorem 1. *Suppose that $F(z)$ is a normalized Hecke-eigen cusp form of weight κ. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*
$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0,T]\colon\ \sup_{s\in K} \left|\zeta(s+\mathrm{i}\tau,F) - f(s)\right| < \varepsilon \right\} > 0.$$

Here $\operatorname{meas} A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

In [20], Theorem 1 was generalized for approximation of analytic functions by the shifts $\zeta(s+\mathrm{i}\varphi(\tau), F)$, where $\varphi(\tau)$ is a real differentiable function for $\tau \geqslant \tau_0$ such that the derivative $\varphi'(\tau)$ is positive, monotonic, $1/\varphi'(\tau) = o(\tau)$ and

$$\varphi(2\tau) \max_{\tau \leqslant t \leqslant 2\tau} \frac{1}{\varphi'(t)} \ll \tau$$

as $\tau \to \infty$.

In shifts $\zeta(s+\mathrm{i}\tau, F)$, τ can take arbitrary real values, therefore Theorem 1 is of continuous type. Also, a discrete version of Theorem 1 is known, see [12]. Denote by $\#A$ the cardinality of the set A.

Theorem 2. *Suppose that $F(z)$ is a normalized Hecke-eigen cusp form of weight κ. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Bigl\{0 \leqslant k \leqslant N\colon \sup_{s\in K} \bigl|\zeta(s+\mathrm{i}kh, F) - f(s)\bigr| < \varepsilon\Bigr\} > 0.$$

In [13], Theorem 2 was generalized for more general than $\{kh\}$ nonlinear sets $\{\varphi(k)\colon k \in \mathbb{N}\}$.

Some collections of zeta and L-functions have a joint universality property. In this case, a collection of analytic functions is simultaneously approximated by a collection consisting of shifts of zeta or L-functions. The first joint universality theorem also was obtained by Voronin. In [21], he proved a joint universality theorem for Dirichlet L-functions $L(s,\chi_1), \dots, L(s,\chi_r)$ with nonequivalent Dirichlet characters, see also [2, 4, 7]. A discrete version of the joint universality theorem by using the linear set $\{kh\}$ was proposed by Bagchi [2]. Later, many results on the joint universality of zeta and L-functions were obtained, see, for example, [6, 9, 11] and a survey paper [15].

In joint universality theorems, the zeta-functions approximating a collection of analytic functions must be independent in a certain sense. For example, in the case of Dirichlet L-functions, this independence is described by nonequivalence of characters. In the case of periodic zeta-functions, some rank conditions are applied. However, if the coefficients of Dirichlet series defining zeta-functions are nonperiodic, then the problem of joint universality for those zeta-functions becomes very complicated. This remark also concerns the zeta-functions of cusp forms. The first result for a pair of zeta-functions of cusp forms belongs to Mishou [17]. Let F_1 and F_2 be two different normalized Hecke-eigen cusp forms for the full modular group of weight κ_1 and κ_2 and Fourier coefficients $c_1(m)$ and $c_2(m)$, respectively,

$$\hat{c}_j(m) = c_j(m) m^{-(\kappa_j - 1)/2}, \quad j = 1, 2,$$

and

$$\hat{\zeta}(s, F_j) = \sum_{m=1}^{\infty} \frac{\hat{c}_j(m)}{m^s}, \quad \sigma > 1,\ j = 1, 2.$$

Let $\hat{D} = \{s \in \mathbb{C}\colon 1/2 < \sigma < 1\}$. Then the Mishou theorem is the following statement.

Theorem 3. *For $j = 1, 2$, let K_j be a compact subset of $\hat{D}$ with connected complement, and $f_j(s)$ be a continuous nonvanishing functions on K_j, that is, analytic in the interior of K_j. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Bigl\{\tau \in [0, T]\colon \sup_{1\leqslant j\leqslant 2} \sup_{s\in K_j} \bigl|\hat{\zeta}(s + \mathrm{i}\tau, F_j) - f_j(s)\bigr| < \varepsilon\Bigr\} > 0.$$

Theorem 3 remains valid [17] also for pairs consisting of the Riemann zeta-function, Rankin–Selberg L-functions and symmetric square L-functions.

Lee, Nakamura and Pańkowski proved in [14] the joint universality theorem for arbitrary number of automorphic zeta-functions.

In [8], joint discrete universality theorems for zeta-functions of cusp forms were obtained. Let $F_1, \dots, F_r$ be different normalized Hecke-eigen cusp forms of weight $\kappa_1, \dots, \kappa_r$ with Fourier coefficients $c_1(m), \dots, c_r(m)$, respectively, and let

$$\zeta(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)}{m^s}, \quad \sigma > \frac{\kappa_j + 1}{2}, \quad j = 1, \dots, r,$$

be the corresponding zeta-functions. For positive numbers h_j, $j = 1, \dots, r$, define

$$L(\mathbb{P}; h_1, \dots, h_r; \pi) = \bigl\{(h_1 \log p\colon\ p \in \mathbb{P}),\ \dots,\ (h_r \log p\colon\ p \in \mathbb{P}),\ 2\pi\bigr\},$$

where $\mathbb{P}$ is the set of all prime numbers. Let $D_j = \{s \in \mathbb{C}\colon\ \kappa_j/2 < \sigma < (\kappa_j + 1)/2\}$, $\mathcal{K}_j$ be the class of compact subset of the strip D_j with connected complements, and let $H_0(K_j)$, $K_j \in \mathcal{K}_j$, denote the class of continuous nonvanishing functions on K_j that are analytic in the interior of K_j, $j = 1, \dots, r$. The set $L(\mathbb{P}; h_1, \dots, h_r; \pi)$ is used for the definition of a certain independence of the functions $\zeta(s, F_1), \dots, \zeta(s, F_r)$. The following theorem is proved in [8].

Theorem 4. *Suppose that the set $L(\mathbb{P}; h_1, \dots, h_r; \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}_j$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Bigl\{0 \leqslant k \leqslant N\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \bigl|\zeta(s + \mathrm{i}kh_j, F_j) - f_j(s)\bigr| < \varepsilon\Bigr\} > 0.$$

In shifts $\zeta(s + \mathrm{i}kh_j, F_j)$ of Theorem 4, the linear sets $\{kh_j\}$, $j = 1, \dots, r$, are used. The aim of this paper is to obtain a version of Theorem 4 by using more complicated nonlinear sets in place of $\{kh_j\}$. We remind that a sequence $\{x_k\colon\ k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $[a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[a,b)}\bigl(\{x_k\}\bigr) = b - a,$$

where $\chi_{[a,b)}$ is the indicator function of $[a, b)$, and $\{x_k\}$ denotes the fractional part of x_k.

Suppose that $k_0 \in \mathbb{N}$, and $\varphi_1(t), \dots, \varphi_r(t)$ are continuously differentiable functions on $[k_0 - 1/2, \infty)$ such that their derivatives $\varphi_j'(t)$ satisfy the estimate

$$\varphi_j(2t)\left(\max_{t\leqslant u\leqslant 2t} \frac{1}{\varphi_j'(u)} + \max_{t\leqslant u\leqslant 2t} \varphi_j'(u)\right) \ll t, \tag{1}$$

and the sequence $\{\varphi_1(k)a_1 + \dots + \varphi_r(k)a_r\colon k \geqslant k_0\}$ is uniformly distributed modulo 1 with real numbers $a_1, \dots, a_r$ not all zeros. Denote the class of the above functions by $U_r(k_0)$. Then the following theorem is true. The forms $F_1, \dots, F_r$ are not necessarily different.

Theorem 5. *Suppose that* $(\varphi_1, \dots, \varphi_r) \in U_r(k_0)$. *For* $j = 1, \dots, r$, *let* $K_j \in \mathcal{K}_j$ *and* $f_j(s) \in H_0(K_j)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \big|\zeta(s + \mathrm{i}\varphi_j(k), F_j) - f_j(s)\big| < \varepsilon\Big\} > 0.$$

Theorem 5 has the following modification.

Theorem 6. *Suppose that* $(\varphi_1, \dots, \varphi_r) \in U_r(k_0)$. *For* $j = 1, \dots, r$, *let* $K_j \in \mathcal{K}_j$ *and* $f_j(s) \in H_0(K_j)$. *Then the limit*

$$\lim_{N\to\infty} \frac{1}{N-k_0+1} \#\Big\{k_0 \leqslant k \leqslant N\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \big|\zeta(s + \mathrm{i}\varphi_j(k), F_j) - f_j(s)\big| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

We give an example of the functions φ_j. Let $\alpha > 0$, and let $g(x)$ be a nonconstant linear combination of arbitrary real powers of x. Then the sequence $\{k^\alpha g(\log k)\colon k \geqslant 2\}$ is uniformly distributed modulo 1, see [5], Exercise 3.15. Therefore, we can take, for example, $\varphi_j(t) = t^\alpha \log^j t$ with $0 < \alpha < 1$ and $t \geqslant 2$. Then estimate (1) is satisfied, and the sequence $\{\varphi_1(k)a_1 + \dots + \varphi_r(k)a_r\colon k \geqslant k_0\}$ with real numbers $a_1, \dots, a_r$ not all zeros is uniformly distributed modulo 1.

Theorems 5 and 6 in a certain sense are joint generalizations of the corresponding one-dimensional theorems of [13].

Involving the uniform distribution modulo 1 makes the probabilistic method very convenient for the proof of universality theorems.

2 Probabilistic model

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We recall that P_n converges weakly to P as $n \to \infty$ if, for every real continuous bounded function g on $\mathbb{X}$,

$$\lim_{n\to\infty}\int\limits_{\mathbb{X}} g\,\mathrm{d}P_n = \int\limits_{\mathbb{X}} g\,\mathrm{d}P.$$

Let γ be the unit circle on the complex plane, and

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. The torus Ω, with the product topology and pointwise multiplication, is a compact topological Abelian group. Define

$$\underline{\Omega} = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1,\dots,r$. Then again, $\underline{\Omega}$ is a compact topological Abelian group. Therefore, on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$, the probability Haar measure m_H exists, and we obtain the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$. Denote the elements of Ω by $\omega = (\omega_1,\dots,\omega_r)$, where $\omega_j \in \Omega_j$, $j = 1,\dots,r$. We start with a limit theorem for probability measures on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. For $A \in \mathcal{B}(\underline{\Omega})$, define

$$Q_N(A) = \frac{1}{N-k_0+1}\#\bigl\{k_0 \leqslant k \leqslant N\colon \bigl(p^{-\mathrm{i}\varphi_1(k)}\colon p\in\mathbb{P}\bigr),\ \dots,\ \bigl(p^{-\mathrm{i}\varphi_r(k)}\colon p\in\mathbb{P}\bigr) \in A\bigr\}.$$

For the proof of weak convergence for Q_N, we will use the Weyl criterion.

Lemma 1. *The sequence $\{x_k\colon k\in\mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for all $m \in \mathbb{Z}\setminus\{0\}$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{2\pi\mathrm{i}mx_k} = 0.$$

Proof of the lemma is given, for example, in [5].

Lemma 2. *Suppose that the sequence $\{\varphi_1(k)a_1 + \cdots + \varphi_r(k)a_r\colon k \geqslant k_0\}$ is distributed modulo 1, where $a_1,\dots,a_r$ are real numbers not all zeros. Then Q_N converges weakly to the Haar measure m_H as $N\to\infty$.*

Proof. We apply the uniform distribution modulo 1 for the investigation of the Fourier transform $g_N(\underline{k}_1,\dots,\underline{k}_r)$, $\underline{k}_j = (k_{jp}\colon k_{jp}\in\mathbb{Z},\ p\in\mathbb{P})$, $j = 1,\dots,r$. We have that the dual group of $\underline{\Omega}$ is isomorphic to

$$\bigoplus_{j=1}^{r}\bigoplus_{p\in\mathbb{P}}\mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for all $p\in\mathbb{P}$ and $j = 1,\dots,r$. Therefore,

$$g_N(\underline{k}_1,\dots,\underline{k}_r) = \int\limits_{\underline{\Omega}}\Biggl(\prod_{j=1}^{r}\prod_{p\in\mathbb{P}}{}' \omega_j^{k_{jp}}(p)\Biggr)\mathrm{d}Q_N,$$

where "$'$" means that only a finite number of integers k_{jp} are distinct from zero. Thus, by the definition of Q_N,

$$g_N(\underline{k}_1,\dots,\underline{k}_r) = \frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\prod_{j=1}^{r}{\prod_{p\in\mathbb{P}}}' p^{-\mathrm{i}k_{jp}\varphi_j(k)}$$
$$= \frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\exp\biggl\{-\mathrm{i}\sum_{j=1}^{r}\varphi_j(k){\sum_{p\in\mathbb{P}}}' k_{jp}\log p\biggr\}. \qquad (2)$$

If $(\underline{k}_1,\dots,\underline{k}_r) = (\underline{0},\dots,\underline{0})$, then, clearly,

$$g_N(\underline{k}_1,\dots,\underline{k}_r) = 1. \qquad (3)$$

Since the set $\{\log p\colon\ p\in\mathbb{P}\}$ is linearly independent over the field of rational numbers,

$${\sum_{p\in\mathbb{P}}}' k_{jp}\log p \neq 0 \quad \text{for } \underline{k}\neq\underline{0},\ j=1,\dots,r.$$

Therefore, by hypothesis of the lemma, Lemma 1 and (2),

$$\lim_{N\to\infty} g_N(\underline{k}_1,\dots,\underline{k}_r) = 0$$

for $(\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0})$. This and (3) give

$$\lim_{N\to\infty} g_N(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) = (\underline{0},\dots,\underline{0}),\\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0}),\end{cases}$$

and the assertion of the lemma follows by the continuity theorem for probability measures on compact groups. $\square$

Lemma 2 implies a limit theorem for probability measures on the space of analytic functions defined by means of absolutely convergent Dirichlet series. This theorem is quite standard but plays an important role in the sequel. Denote by $H(D_j)$ the space of analytic functions on D_j endowed with the topology of uniform convergence on compacta, $j=1,\dots,r$, and let $H(D_1,\dots,D_r) = H(D_1)\times\dots\times H(D_r)$. For fixed $\theta > 1/2$ and $m,n\in\mathbb{N}$, we set

$$v_n(m) = \exp\biggl\{-\biggl(\frac{m}{n}\biggr)^{\theta}\biggr\}$$

and define

$$\zeta_n(s,F_j) = \sum_{m=1}^{\infty}\frac{c_j(m)v_n(m)}{m^s}, \quad j=1,\dots,r,$$

and

$$\zeta_n(s,\omega_j,F_j) = \sum_{m=1}^{\infty}\frac{c_j(m)\omega_j(m)v_n(m)}{m^s}, \quad j=1,\dots,r,$$

where
$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N}.$$

Then the latter series are absolutely convergent for $\sigma > \kappa_j/2$ (see [10]). For brevity, let $\underline{s} = (s_1, \ldots, s_r)$, $\underline{F} = (F_1, \ldots, F_r)$, $\underline{\varphi}(k) = (\varphi_1(k), \ldots, \varphi_r(k))$, and
$$\underline{\zeta}_n(\underline{s} + \mathrm{i}\underline{\varphi}(k), \underline{F}) = \bigl(\zeta_n\bigl(s_1 + \mathrm{i}\varphi_1(k), F_1\bigr), \ldots, \zeta_n\bigl(s_r + \mathrm{i}\varphi_r(k), F_r\bigr)\bigr)$$
and
$$\underline{\zeta}_n(\underline{s}, \omega, \underline{F}) = \bigl(\zeta_n(s_1, \omega_1, F_1), \ldots, \zeta_n(s_r, \omega_r, F_r)\bigr).$$

Now, for $A \in \mathcal{B}(H(D_1, \ldots, D_r))$, define
$$P_{N,n}(A) = \frac{1}{N - k_0 + 1} \#\bigl\{k_0 \leqslant k \leqslant N\colon \underline{\zeta}_n(\underline{s} + \mathrm{i}\underline{\varphi}(k), \underline{F}) \in A\bigr\}.$$

Lemma 3. *Suppose that the sequence $\{\varphi_1(k)a_1 + \cdots + \varphi_r(k)a_r\colon k \geqslant k_0\}$ is distributed modulo 1, where $a_1, \ldots, a_r$ are real numbers not all zeros. Then, on $(H(D_1, \ldots, D_r), \mathcal{B}(H(D_1, \ldots, D_r)))$, there exists a probability measure $\hat{P}_n$ such that $P_{N,n}$ converges weakly to $\hat{P}_n$ as $N \to \infty$.*

Proof. Define the function $u_n : \underline{\Omega} \to H(D_1, \ldots, D_r)$ by the formula
$$u_n(\omega) = \underline{\zeta}_n(\underline{s}, \omega, \underline{F}).$$

Since the series $\zeta_n(s_j, \omega_j, F_j)$ are absolutely convergent for $\sigma_j > \kappa_j/2$, $j = 1, \ldots, r$, the function u_n is continuous. Moreover, for $A \in \mathcal{B}(H(D_1, \ldots, D_r))$,
$$\begin{aligned} P_{N,n}(A) &= \frac{1}{N - k_0 + 1} \#\bigl\{k_0 \leqslant k \leqslant N\colon \bigl(\bigl(p^{-\mathrm{i}\varphi_1(k)}\colon p \in \mathbb{P}\bigr), \ldots, \\ &\qquad \bigl(p^{-\mathrm{i}\varphi_r(k)}\colon p \in \mathbb{P}\bigr)\bigr) \in u_n^{-1}A\bigr\} \\ &= Q_N(u_n^{-1}A) \end{aligned}$$
because
$$u_n\bigl(\bigl(p^{-\mathrm{i}\varphi_1(k)}\colon p \in \mathbb{P}\bigr), \ldots, \bigl(p^{-\mathrm{i}\varphi_r(k)}\colon p \in \mathbb{P}\bigr)\bigr) = \underline{\zeta}_n\bigl(\underline{s} + \mathrm{i}\underline{\varphi}(k), \omega, \underline{F}\bigr).$$

Thus, we have that $P_{N,n} = Q_N u_n^{-1}$, where
$$Q_N u_n^{-1}(A) = Q_N\bigl(u_n^{-1}A\bigr), \quad A \in \mathcal{B}\bigl(H(D_1, \ldots, D_r)\bigr).$$

This equality, Lemma 2, the continuity of u_n and Theorem 5.1 of [3] imply the weak convergence of $P_{N,n}$ to $\hat{P}_n = m_H u_n^{-1}$ as $N \to \infty$. □

For further consideration, we recall the metric in $H(D_1,\dots,D_r)$. For $j=1,\dots,r$ and $g_1,g_2\in H(D_j)$, let

$$\rho_j(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_{jl}}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_{jl}}|g_1(s)-g_2(s)|},$$

where $\{K_{jl}\colon l\in\mathbb{N}\}$ is a sequence of compact subsets of D_j such that

$$\bigcup_{l=1}^{\infty}K_{jl}=D_j,$$

$K_{jl}\subset K_{j(l+1)}$, and if $K\subset D_j$ is a compact set, then $K\subset K_{jl}$ with some $l\in\mathbb{N}$. Then ρ_j is a metric in $H(D_j)$, $j=1,\dots,r$, inducing its topology of uniform convergence on compacta. Putting, for $\underline{g}_1=(g_{11},\dots,g_{1r})$, $\underline{g}_2(g_{21},\dots,g_{2r})\in H(D_1,\dots,D_r)$,

$$\underline{\rho}(\underline{g}_1,\underline{g}_2)=\max_{1\leqslant j\leqslant r}\rho_j(g_{1j},g_{2j})$$

gives the metric in the space $H(D_1,\dots,D_r)$ that induces the product topology.

Now, we are able to approximate the collection

$$\underline{\zeta}\bigl(\underline{s}+\mathrm{i}\underline{\varphi}(k),\underline{F}\bigr)=\bigl(\zeta\bigl(s_1+\mathrm{i}\varphi_1(k),F_1\bigr),\ \dots,\ \zeta\bigl(s_r+\mathrm{i}\varphi_r(k),F_r\bigr)\bigr)$$

by $\underline{\zeta}_n(\underline{s}+\mathrm{i}\underline{\varphi}(k),\underline{F})$. For this, the Gallagher lemma that connects the continuous and discrete mean squares of certain functions is useful.

Lemma 4. *Suppose that T_0, $T\geqslant\delta>0$ are real numbers, and $\mathcal{T}\neq\emptyset$ is a finite set in the interval $[T_0+\delta/2,\,T_0+T-\delta/2]$. Define*

$$N_\delta(x)=\sum_{\substack{t\in\mathcal{T}\\ |t-x|<\delta}}1.$$

Let $S(x)$ be a complex-valued continuous function on $[T_0,\,T+T_0]$ having a continuous derivative on $(T_0,\,T+T_0)$. Then

$$\sum_{t\in\mathcal{T}}N_\delta^{-1}(t)\bigl|S(t)\bigr|^2 \leqslant\frac{1}{\delta}\int_{T_0}^{T_0+T}\bigl|S(x)\bigr|^2\,\mathrm{d}x+\left(\int_{T_0}^{T_0+T}\bigl|S(x)\bigr|^2\,\mathrm{d}x\int_{T_0}^{T_0+T}\bigl|S'(x)\bigr|^2\,\mathrm{d}x\right)^{1/2}.$$

Proof of the lemma can be found in [18, Lemma 1.4].

Lemma 5. *We have*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\underline{\rho}\bigl(\underline{\zeta}\bigl(\underline{s}+\mathrm{i}\underline{\varphi}(k),\underline{F}\bigr),\ \underline{\zeta}_n\bigl(\underline{s}+\mathrm{i}\underline{\varphi}(k),\underline{F}\bigr)\bigr)=0.$$

Proof. From the definition of the metrics ρ_j and $\underline{\rho}$ it follows that it suffices to prove that, for every compact sets $K_j \subset D_j$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K_j} \bigl|\zeta(s+\mathrm{i}\varphi_j(k), F_j) - \zeta_n\bigl(s+\mathrm{i}\varphi_j(k), F_j\bigr)\bigr| = 0,$$

$j = 1, \dots, r$. Thus, let F be a normalized Hecke-eigen cusp form of weight κ, $\zeta(s, F)$ the corresponding zeta-function, and let $\varphi(t)$ have properties of the class $U_r(k_0)$.

It is well known that, for fixed σ, $\kappa/2 < \sigma < (\kappa+1)/2$,

$$\int_0^T \bigl|\zeta(\sigma+\mathrm{i}t, F)\bigr|^2 \,\mathrm{d}t \ll_\sigma T. \tag{4}$$

Hence, for $\tau \in \mathbb{R}$,

$$\int_0^{|\tau|+\varphi(t)} \bigl|\zeta(\sigma+\mathrm{i}u, F)\bigr|^2 \,\mathrm{d}u \ll_\sigma \bigl(|\tau| + \varphi(t)\bigr).$$

Therefore, for $X > 0$,

$$\begin{aligned}
&\int_X^{2X} \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t), F\bigr)\bigr|^2 \,\mathrm{d}t \\
&\quad= \int_X^{2X} \frac{1}{\varphi'(t)} \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t), F\bigr)\bigr|^2 \,\mathrm{d}\varphi(t) \\
&\quad\ll \max_{X\leqslant t\leqslant 2X} \frac{1}{\varphi'(t)} \int_X^{2X} \mathrm{d}\Biggl(\int_0^{\tau+\varphi(t)} \bigl|\zeta(\sigma+\mathrm{i}u, F)\bigr|^2 \,\mathrm{d}u\Biggr) \\
&\quad\ll_\sigma \bigl(|\tau| + \varphi(2X)\bigr) \max_{X\leqslant t\leqslant 2X} \frac{1}{\varphi'(t)} \ll_\sigma X\bigl(1+|\tau|\bigr).
\end{aligned}$$

Thus, taking $X = 2^{k-1}T$ and summing over k give the estimate

$$\int_{k_0-1/2}^{T} \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t), F\bigr)\bigr|^2 \,\mathrm{d}t \ll_\sigma T\bigl(1+|\tau|\bigr) \tag{5}$$

for all real τ and $\kappa/2 < \sigma < (\kappa+1)/2$.

Estimate (4) and the Cauchy integral formula imply, for fixed σ, $\kappa/2 < \sigma < (\kappa+1)/2$, the bound

$$\int_0^T \bigl|\zeta'\bigl(\sigma+\mathrm{i}t, F\bigr)\bigr|^2 \,\mathrm{d}t \ll_\sigma T.$$

Therefore, similarly as above, we obtain

$$\begin{aligned}
&\int\limits_X^{2X} \bigl(\varphi'(t)\bigr)^2 \bigl|\zeta'\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t),F\bigr)\bigr|^2 \,\mathrm{d}t \\
&\quad= \int\limits_X^{2X} \varphi'(t) \bigl|\zeta'\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t),F\bigr)\bigr|^2 \,\mathrm{d}\varphi(t) \\
&\quad\ll \max_{X\leqslant t\leqslant 2X} \varphi'(t) \int\limits_X^{2X} \mathrm{d}\Biggl(\int\limits_0^{\tau+\varphi(t)} \bigl|\zeta(\sigma+\mathrm{i}u,F)\bigr|^2 \,\mathrm{d}u \Biggr) \\
&\quad\ll_\sigma \bigl(|\tau|+\varphi(2X)\bigr) \max_{X\leqslant t\leqslant 2X} \varphi'(t) \ll_\sigma X\bigl(1+|\tau|\bigr).
\end{aligned}$$

Thus, we have that, for all real τ and $\kappa/2<\sigma<(\kappa+1)/2$,

$$\int\limits_{k_0-1/2}^{T} \bigl(\varphi'(t)\bigr)^2 \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t),F\bigr)\bigr|^2 \,\mathrm{d}t \ll_\sigma T\bigl(1+|\tau|\bigr). \tag{6}$$

Now, we apply Lemma 4 with $\mathcal{T}=\{k\colon k\in\mathbb{N}, k_0\leqslant k\leqslant N\}$, $T_0=k_0-1/2$, $T=N-k_0+1$, $\delta=1$, and $S(x)=\zeta(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(x),F)$. This, together with (5) and (6), gives

$$\begin{aligned}
&\sum_{k=k_0}^{N} \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(k),F\bigr)\bigr|^2 \\
&\quad\ll \int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t),F\bigr)\bigr|^2 \,\mathrm{d}t + \Biggl(\int\limits_{k_0-1/2}^{N+1/2} \bigl|\zeta\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t),F\bigr)\bigr|^2 \,\mathrm{d}t \\
&\qquad\times \int\limits_{k_0-1/2}^{N+1/2} \bigl(\varphi'(t)\bigr)^2 \bigl|\zeta'\bigl(\sigma+\mathrm{i}\tau+\mathrm{i}\varphi(t),F\bigr)\bigr|^2 \,\mathrm{d}t \Biggr)^{1/2} \\
&\quad\ll_\sigma N\bigl(1+|\tau|\bigr)
\end{aligned} \tag{7}$$

for all real τ and $\kappa/2<\sigma<(\kappa+1)/2$.

Let θ be the same as definition of $v_n(m)$, and $\Gamma(s)$ denote the Euler gamma-function. Then it is known [10] that, for $\sigma>\kappa/2$,

$$\zeta_n(s,F) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty} \zeta(s+z,F) l_n(z) \,\frac{\mathrm{d}z}{z}, \tag{8}$$

where

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s.$$

Now, let K be an arbitrary fixed compact set of the strip $\kappa/2 < \sigma < (\kappa+1)/2$. We take $\varepsilon > 0$ such that $\kappa/2 + 2\varepsilon \leqslant \sigma \leqslant (\kappa+1)/2 - \varepsilon$ for point $s \in K$. Replace θ in (8) by $-\hat{\theta}$, where $\hat{\theta} > 0$. This gives

$$\zeta_n(s,F) - \zeta(s,F) = \frac{1}{2\pi\mathrm{i}} \int\limits_{-\hat{\theta}-\mathrm{i}\infty}^{-\hat{\theta}+\mathrm{i}\infty} \zeta(s+z,F) l_n(z) \frac{\mathrm{d}z}{z}. \tag{9}$$

Denote points of the set K by $s = \sigma + \mathrm{i}v$ and take

$$\hat{\theta} = \sigma - \varepsilon - \frac{\kappa}{2}, \qquad \theta = \frac{1}{2} + \varepsilon.$$

Then, in view of (9),

$$\begin{aligned}
&\bigl|\zeta_n\bigl(s+\mathrm{i}\varphi(k),F\bigr) - \zeta\bigl(s+\mathrm{i}\varphi(k),F\bigr)\bigr| \\
&\quad\leqslant \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \zeta\bigl(s+\mathrm{i}\varphi(k)-\hat{\theta}+\mathrm{i}t,F\bigr) \frac{|l_n(-\hat{\theta}+\mathrm{i}t)|}{|-\hat{\theta}+\mathrm{i}t|}\,\mathrm{d}t.
\end{aligned}$$

Hence, the shift $t + v \to t$ implies

$$\begin{aligned}
&\bigl|\zeta_n\bigl(s+\mathrm{i}\varphi(k),F\bigr) - \zeta\bigl(s+\mathrm{i}\varphi(k),F\bigr)\bigr| \\
&\quad\leqslant \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \zeta\biggl(\frac{\kappa}{2}+\varepsilon+\mathrm{i}\bigl(t+\varphi(k)\bigr),F\biggr) \frac{|l_n(\frac{\kappa}{2}+\varepsilon-s+\mathrm{i}t)|}{|\frac{\kappa}{2}+\varepsilon-s+\mathrm{i}t|}\,\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{N-k_0+1} \sum_{k=k_0}^{N} \sup_{s\in K} \bigl|\zeta\bigl(s+\mathrm{i}\varphi(k),F\bigr) - \zeta_n\bigl(s+\mathrm{i}\varphi(k),F\bigr)\bigr| \\
&\quad\leqslant \frac{1}{2\pi(N-k_0+1)} \int\limits_{-\infty}^{\infty} \Biggl(\sum_{k=k_0}^{N} \biggl|\zeta\biggl(\frac{\kappa}{2}+\varepsilon+\mathrm{i}\bigl(t+\varphi(k)\bigr),F\biggr)\biggr|\Biggr) \\
&\qquad\times \sup_{s\in K} \frac{|l_n(\frac{\kappa}{2}+\varepsilon-s+\mathrm{i}t)|}{|\frac{\kappa}{2}+\varepsilon-s+\mathrm{i}t|}\,\mathrm{d}t \\
&\quad\stackrel{\mathrm{def}}{=} J.
\end{aligned} \tag{10}$$

Using the well-known estimate

$$\Gamma(\sigma+\mathrm{i}t) \ll \exp\bigl\{-c|t|\bigr\}, \quad c > 0,$$

which is uniform for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, we find by the definition of $l_n(s)$ that

$$\frac{l_n(\frac{\kappa}{2}+\varepsilon-s+\mathrm{i}t)}{\frac{\kappa}{2}+\varepsilon-s+\mathrm{i}t} \ll \frac{n^{\kappa/2+\varepsilon-\sigma}}{\theta}\exp\biggl\{-\frac{c}{\theta}|t-v|\biggr\} \ll_K n^{-\varepsilon}\exp\bigl\{-c|t|\bigr\}.$$

Thus, in view of (7),

$$J \ll_K n^{-\varepsilon}\int\limits_{-\infty}^{\infty}\bigl(1+|t|\bigr)^{1/2}\exp\bigl\{-c|t|\bigr\}\,\mathrm{d}t \ll_K n^{-\varepsilon}.$$

This and (10) show that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-k_0+1}\sum_{k=k_0}^{N}\sup_{s\in K}\bigl|\zeta\bigl(s+\mathrm{i}\varphi(k),F\bigr)-\zeta_n\bigl(s+\mathrm{i}\varphi(k),F\bigr)\bigr| = 0,$$

and the lemma is proved. □

Now we are in position to prove a discrete limit theorem for the collection $\underline{\zeta}(\underline{s}+\mathrm{i}\underline{\varphi}(k),\underline{F})$. For $A \in \mathcal{B}(H(D_1,\dots,D_r))$, define

$$P_N(A) = \frac{1}{N-k_0+1}\#\bigl\{k_0 \leqslant k \leqslant N\colon\ \underline{\zeta}\bigl(\underline{s}+\mathrm{i}\underline{\varphi}(k),\underline{F}\bigr) \in A\bigr\}.$$

Moreover, on the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),m_H)$, define the $H(D_1,\dots,D_r)$-valued random element

$$\underline{\zeta}(\underline{s},\omega,\underline{F}) = \bigl(\zeta(s_1,\omega_1,F_1),\dots,\zeta(s_r,\omega_r,F_r)\bigr),$$

where

$$\zeta(s_j,\omega_j,F_j) = \sum_{m=1}^{\infty}\frac{c_j(m)\omega_j(m)}{m^s}, \qquad j=1,\dots,r,$$

and denote by $P_{\underline{\zeta}}$ its distribution, i.e.,

$$P_{\underline{\zeta}}(A) = m_H\bigl\{\omega \in \underline{\Omega}\colon\ \underline{\zeta}(\underline{s},\omega,\underline{F}) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H(D_1,\dots,D_r)\bigr).$$

Theorem 7. *Suppose that $(\varphi_1,\dots,\varphi_r) \in U_r(k_0)$. Then P_N converges weakly to $P_{\underline{\zeta}}$ as $N\to\infty$. Moreover, the support of the measure $P_{\underline{\zeta}}$ is the set $S = S_1\times\dots\times S_r$, where*

$$S_j = \bigl\{g \in H(D_j)\colon\ g(s) \neq 0 \text{ or } g(s) \equiv 0\bigr\}, \quad j=1,\dots,r.$$

Proof. Let $\hat{P}_n$, as above, be the limit measure in Lemma 3. We observe that the sequence $\{\hat{P}_n\colon\ n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D_1,\dots,D_r)$ such that

$$\hat{P}_n(K) > 1-\varepsilon$$

for all $n \in \mathbb{N}$. Actually, let $\hat{P}_{nj}$, $j = 1, \dots, r$, be the marginal measures of $\hat{P}_n$. Then it is well known that the sequences $\{\hat{P}_{nj}\colon n \in \mathbb{N}\}$ are tight, $j = 1, \dots, r$, see, for example, the proof of Lemma 4.11 from [19] for more general functions. This is also used in [10, 12]. Therefore, for every $\varepsilon > 0$, there exists a compact set $K_j \subset H(D_j)$ such that

$$\hat{P}_{nj}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r, \tag{11}$$

for all $n \in \mathbb{N}$. The set $K = K_1 \times \cdots \times K_r$ is compact in the space $H(D_1, \dots, D_r)$, and, by (11),

$$\hat{P}_n\bigl(H(D_1, \dots, D_r) \setminus K\bigr) \leqslant \sum_{j=1}^{r} \hat{P}_{nj}\bigl(H(D_j) \setminus K_j\bigr) < \varepsilon$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{\hat{P}_n\colon n \in \mathbb{N}\}$ is tight.

The Prokhorov theorem [3, Thm. 6.1] implies the relative compactness of $\{\hat{P}_n\}$. Thus, there exists a subsequence $\{\hat{P}_{n_l}\}$ weakly convergent to a certain probability measure P on $H(D_1, \dots, D_r)$, $\mathcal{B}(H(D_1, \dots, D_r))$ as $l \to \infty$.

In what follows, we will use the convergence in distribution $\xrightarrow{\mathcal{D}}$. Denote by $\underline{\hat{X}}_n = \underline{\hat{X}}_n(\underline{s})$ the $H(D_1, \dots, D_r)$-valued random element with the distribution $\hat{P}_n$. Then the weak convergence of $\hat{P}_{n_l}$ can be written as

$$\underline{\hat{X}}_{n_l} \xrightarrow[l \to \infty]{\mathcal{D}} P. \tag{12}$$

On a certain probability space with a measure μ, define a discrete random variable θ_N by

$$\mu\bigl(\theta_N = \underline{\varphi}(k)\bigr) = \frac{1}{N - k_0 + 1}, \quad k = k_0, \dots, N,$$

and define the $H(D_1, \dots, D_r)$-valued random element

$$\underline{X}_{N,n} = \underline{X}_{N,n}(\underline{s}) = \underline{\zeta}_n(\underline{s} + \mathrm{i}\theta_N, \underline{F}).$$

Then the assertion of Lemma 3 can be written in the form

$$\underline{X}_{N,n} \xrightarrow[N \to \infty]{\mathcal{D}} \underline{\hat{X}}_n. \tag{13}$$

Moreover, let

$$\underline{X}_N = \underline{X}_N(\underline{s}) = \underline{\zeta}(\underline{s} + \mathrm{i}\theta_N, \underline{F}).$$

The application of Lemma 5 shows that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n \to \infty} \limsup_{N \to \infty} \mu\bigl(\underline{\rho}\bigl(\underline{X}_N(\underline{s}), \underline{X}_{N,n}(\underline{s})\bigr) \geqslant \varepsilon\bigr) \\ &\quad \leqslant \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{(N - k_0 + 1)\varepsilon} \sum_{k=k_0}^{N} \underline{\rho}\bigl(\underline{\zeta}\bigl(\underline{s} + \mathrm{i}\underline{\varphi}(k), \underline{F}\bigr), \underline{\zeta}_n\bigl(\underline{s} + \mathrm{i}\underline{\varphi}(k), \underline{F}\bigr)\bigr) = 0. \end{aligned}$$

This equality together with relations (12) and (13) shows that all hypotheses of Theorem 4.2 from [3] are fulfilled. Thus,

$$\underline{X}_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{14}$$

or P_N converges weakly to the limit measure P as $N \to \infty$. Moreover, by (14), we have that the measure P is independent of the choice of the subsequence of $\{\hat{P}_{n_l}\}$. Since the sequence $\{\hat{P}_n\}$ is relatively compact, this implies the relation

$$\underline{\hat{X}}_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

This means that P_N converges weakly to the limit measure P of $\hat{P}_n$. In [8], it was obtained that P coincides with $P_{\underline{\zeta}}$. Moreover, the support of $P_{\underline{\zeta}}$ is the set S. For the proof of this, in [8], simple observations that $\mathcal{B}(H(D_1,\dots,D_r)) = \mathcal{B}(H(D_1)) \times \dots \times \mathcal{B}(H(D_r))$, and that the Haar measure m_H is the product of the Haar measures on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1,\dots,r$, are applied. $\square$

3 Proof of universality theorems

We recall the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 6. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $f(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} \bigl|f(s) - p(s)\bigr| < \varepsilon.$$

Proof of the lemma can be found in [16], see also [23].

We also recall two equivalents of weak convergence of probability measures.

Lemma 7. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following statements are equivalent*:

(i) *P_n converges weakly to P as $n \to \infty$;*

(ii) *For every open set $G \subset \mathbb{X}$,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

(iii) *For every continuity set A of the measure P (A is a continuity set of P if $P(\partial A) = 0$, where ∂A is the boundary of A),*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 of [3].

Proof of Theorem 5. Lemma 6 implies the existence of polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|f_j(s)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}. \tag{15}$$

Let

$$\underline{G}_\varepsilon=\left\{(g_1,\dots,g_r)\in H(D_1,\dots,D_r)\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|g_j(s)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}\right\}.$$

By the second assertion of Theorem 7, the collection $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ is an element of the support of the measure $P_{\underline{\zeta}}$. Thus, the set $\underline{G}_\varepsilon$ is an open neighbourhood of an element of the support of $P_{\underline{\zeta}}$. Hence,

$$P_{\underline{\zeta}}(\underline{G}_\varepsilon)>0. \tag{16}$$

This and the first assertion of Theorem 7, together with Lemma 7(ii) give the inequality

$$\liminf_{N\to\infty}P_N(\underline{G}_\varepsilon)>0,$$

or, by the definitions of P_N and $\underline{G}_\varepsilon$,

$$\liminf_{N\to\infty}\frac{1}{N-k_0+1}\#\left\{k_0\leqslant k\leqslant N\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|\zeta\left(s_j+\mathrm{i}\varphi_j(k),F_j\right)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}\right\}>0.$$

The latter inequality together with (15) proves the theorem. □

Proof of Theorem 6. Let

$$\underline{\hat{G}}_\varepsilon=\left\{(g_1,\dots,g_r)\in H(D_1,\dots,D_r)\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|g_j(s)-f_j(s)\right|<\varepsilon\right\}.$$

Since the boundary $\partial\underline{\hat{G}}_\varepsilon$ lies in the set

$$\left\{(g_1,\dots,g_r)\in H(D_1,\dots,D_r)\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|g_j(s)-f_j(s)\right|=\varepsilon\right\}$$

we have that the boundaries $\partial\underline{\hat{G}}_{\varepsilon_1}$ and $\partial\underline{\hat{G}}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Hence, the set $\underline{\hat{G}}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon>0$. Therefore, Theorem 7 and Lemma 7(iii) show that

$$\lim_{N\to\infty}P_n(\underline{\hat{G}}_\varepsilon)=P_{\underline{\zeta}}(\underline{\hat{G}}_\varepsilon) \tag{17}$$

for all but at most countably many $\varepsilon>0$. On the other hand, the definitions of $\underline{G}_\varepsilon$ and $\underline{\hat{G}}_\varepsilon$, together with inequality (15), imply the inclusion $\underline{G}_\varepsilon\subset\underline{\hat{G}}_\varepsilon$. Thus, in view of (16), we obtain that $P_{\underline{\zeta}}(\underline{\hat{G}}_\varepsilon)>0$, and (17) proves the theorem. □

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