

A transformation formula with primitive character

Aidas Balčiūnas

Institute of Mathematics and Informatics, Vilnius University
Akademijos 4, LT-08663 Vilnius
E-mail: aidas.balciunas@mii.stud.vu.lt

Abstract. We prove a transformation formula for the exponential sum involving the divisor function, and primitive character modulo q . This formula can be applied to obtain meromorphic continuation for the Mellin transform of the square of Dirichlet L -function with primitive character.

Keywords: Dirichlet L -function, Estermann zeta-function, Mellin transform.

1 Introduction

Let $s = \sigma + it$ be a complex variable, χ be a Dirichlet character modulo q , $q > 1$, and let

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

denote the corresponding Dirichlet L -function. For the investigation of the mean square of L -functions, the modified Mellin transform

$$\mathcal{Z}_1(s, \chi) = \int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 x^{-s} dx, \quad \sigma > 1,$$

can be applied.

For a meromorphic continuation of the transform $\mathcal{Z}_1(s, \chi)$, a certain transformation formula is needed. Similar transformation formulae are also used in the case of the Riemann zeta-function see [3, 4]. A transformation formula for $\mathcal{Z}_1(s, \chi)$ with principal character has been obtained in [1]. The aim of this note is to obtain a transformation formula with primitive character.

Let $G(\chi)$ denote the Gauss sum, i.e.,

$$G(\chi) = \sum_{a=1}^q \chi(a) e^{2\pi ia/q}.$$

As usual, denote by $d(m)$ the divisor function

$$d(m) = \sum_{d|m} 1,$$

and let γ be the Euler constant. For $\text{Re } z > 0, \text{Im } z \neq 0$, define

$$\begin{aligned} \Phi(z; \chi, q) &= \sum_{m=1}^{\infty} d(m)\chi(m)e^{2\pi im/q}e^{-mz} \\ &\quad - \frac{1}{G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \frac{(q, a+1)}{qz} \left(\gamma - 2 \log \frac{q}{(q, a+1)} - \log z \right). \end{aligned}$$

In this paper, we prove a formula for $\Phi(z^{-1}; \chi, q)$.

Let a_{0a}^+ and a_{0a}^- be the constant terms in the Laurent series expansion at $s = 1$ for the Estermann zeta-functions $E(s; \frac{\overline{(a+1)}}{(q, a+1)} / \frac{q}{(q, a+1)}, 0)$ and $E(s; -\frac{\overline{(a+1)}}{(q, a+1)} / \frac{q}{(q, a+1)}, 0)$, respectively, where $\frac{\overline{(a+1)}}{(q, a+1)} / \frac{q}{(q, a+1)}$ is connected to $\frac{a+1}{(q, a+1)} / \frac{q}{(q, a+1)}$ by the congruence $\frac{a+1}{(q, a+1)} \frac{\overline{(a+1)}}{(q, a+1)} \equiv 1 \pmod{\frac{q}{(q, a+1)}}$, $a = 1, \dots, q$. Moreover, denote

$$\delta = \begin{cases} 1 & \text{if } \text{Im } z > 0, \\ -1 & \text{if } \text{Im } z < 0. \end{cases}$$

and, for $1 < b < 2$, define

$$\begin{aligned} I(z, b) &= \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{b-i\infty}^{b+i\infty} \left(\frac{2\pi}{q}\right)^{1-2w} \Gamma(w) \left\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{(a+1)}}{(q, a+1)}, 0\right) \right. \\ &\quad \times (q, a+1)^{1-2w} + \cot(\pi w) \left(w; -\frac{\overline{(a+1)}}{(q, a+1)} / \frac{q}{(q, a+1)}, 0\right) (q, a+1)^{1-2w} \\ &\quad \left. + \delta i E\left(w; -\frac{\overline{(a+1)}}{(q, a+1)} / \frac{q}{(q, a+1)}, 0\right) \right\} z^{1-w} dw. \end{aligned}$$

Theorem 1. *Suppose that $\text{Re } z > 0$ and $\text{Im } z \neq 0$. Then*

$$\begin{aligned} \Phi(z^{-1}; \chi, q) &= -\frac{2\pi i \delta z}{q} \sum_{m=1}^{\infty} d(m)\chi(m)e^{-2\pi im/q}e^{-\frac{4\pi^2 mz}{q^2}} \\ &\quad + \frac{q}{2\pi^2 G(\bar{\chi})} \sum_{a=1}^q \frac{\bar{\chi}(a)}{(q, a+1)} (a_{0a}^+ - a_{0a}^-) + I(z, b). \end{aligned}$$

2 Proof of the transformation formula

Firstly, similarly as in [4], we express $\sum_{m=1}^{\infty} d(m)\chi(m) \exp\{2\pi im/q\}m^{-s}$ by the Gaussian sum and Estermann zeta-function:

$$\sum_{m=1}^{\infty} d(m)\chi(m) \exp\{2\pi im/q\}m^{-s} = \frac{1}{G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) E\left(s; \frac{\overline{(a+1)}}{(q, a+1)} / \frac{q}{(q, a+1)}, 0\right).$$

Then, using the Mellin transformation formula, we find

$$\begin{aligned}
& \sum_{m=1}^{\infty} d(m)\chi(m) \exp\{2\pi im/q\} e^{-mz} \\
&= \sum_{m=1}^{\infty} d(m)\chi(m) \exp\{2\pi im/q\} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w)(mz)^{-w} dw \\
&= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w)z^{-w} \sum_{m=1}^{\infty} d(m)\chi(m) \exp\{2\pi im/q\} m^{-w} dw \\
&= \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{2-i\infty}^{2+i\infty} \Gamma(w) E\left(w; \frac{a+1}{q}, \frac{a+1}{(q, a+1)}, 0\right) z^{-w} dw. \tag{1}
\end{aligned}$$

From the Laurent series expansion for Estermann zeta-function at the point $w = 1$ (see in [2]), we have

$$E\left(w; \frac{a+1}{q}, \frac{a+1}{(q, a+1)}, 0\right) = \frac{(q, a+1)}{q} \left(\frac{1}{(w-1)^2} + \frac{2\gamma - 2 \log \frac{q}{(q, a+1)}}{w-1} \right) + a_{0a} + \dots \tag{2}$$

From this, using the expansions

$$\Gamma(w) = 1 - \gamma(w-1) + \frac{\Gamma''(1)(w-1)^2}{2} + \dots, \tag{3}$$

$$z^{-w} = z^{-1} e^{-(w-1)\log z} = z^{-1} \left(1 - (w-1) \log z + \frac{(w-1)^2 \log^2 z}{2} + \dots \right), \tag{4}$$

we find

$$\operatorname{Res}_{w=1} \Gamma(w) E\left(w; \frac{a+1}{q}, \frac{a+1}{(q, a+1)}, 0\right) z^{-w} = \frac{(q, a+1)}{qz} \left(\gamma - 2 \log \frac{q}{(q, a+1)} - \log z \right).$$

Now, moving the line of integration in (1) to the left and having in mind that $0 < c < 1$, from the definition of $\Phi(z; \chi, q)$ we obtain that

$$\Phi(z; \chi, q) = \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{c-i\infty}^{c+i\infty} \Gamma(w) E\left(w; \frac{a+1}{q}, \frac{a+1}{(q, a+1)}, 0\right) z^{-w} dw. \tag{5}$$

We substitute z^{-1} instead of z in (5). This gives

$$\begin{aligned}
\Phi(z^{-1}; \chi, q) &= \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{c-i\infty}^{c+i\infty} \Gamma(w) E\left(w; \frac{a+1}{q}, \frac{a+1}{(q, a+1)}, 0\right) z^w dw \\
&= \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{1-c-i\infty}^{1-c+i\infty} \Gamma(1-w) E\left(1-w; \frac{a+1}{q}, \frac{a+1}{(q, a+1)}, 0\right) z^{1-w} dw. \tag{6}
\end{aligned}$$

Now we apply the functional equation for the Estermann zeta-function (see [2])

$$E\left(1-w; \frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}}, 0\right) = \frac{1}{\pi} \left(\frac{2\pi(q, a+1)}{q}\right)^{1-2w} \Gamma^2(w) \left(E\left(w; \frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right) + \cos(\pi w) E\left(w; -\frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right)\right)$$

and

$$\Gamma(1-w) = \frac{\pi}{\Gamma(w) \sin \pi w}.$$

Thus, from (6), we get

$$\begin{aligned} \Phi(z^{-1}; \chi, q) &= \frac{1}{2\pi i G(\overline{\chi})} \sum_{a=1}^q \overline{\chi}(a) \int_{1-c-i\infty}^{1-c+i\infty} \left(\frac{2\pi(q, a+1)}{q}\right)^{1-2w} \Gamma(w) \left\{ \sin^{-1}(\pi w) \right. \\ &\quad \times E\left(w; \frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right) + \cot(\pi w) E\left(w; -\frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right) \left. \right\} z^{1-w} dw \\ &= -\frac{2\pi i \delta z}{q} \left(\frac{4\pi^2 z}{q^2}; \chi, -q\right) + \frac{1}{2\pi i G(\overline{\chi})} \sum_{a=1}^q \overline{\chi}(a) \int_{1-c-i\infty}^{1-c+i\infty} \left(\frac{2\pi}{q}\right)^{1-2w} \\ &\quad \times \Gamma(w) \left\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right) (q, a+1)^{1-2w} + \cot(\pi w) \right. \\ &\quad \times E\left(w; -\frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right) (q, a+1)^{1-2w} + \delta i E\left(w; -\frac{\overline{\frac{a+1}{(q,a+1)}}}{\frac{q}{(q,a+1)}}, 0\right) \left. \right\} z^{1-w} dw, \end{aligned} \tag{7}$$

since $0 < 1 - c < 1$. Now we transform the last integral. From formulae (2)–(4), using the expansions

$$\begin{aligned} \left(\frac{2\pi(q, a+1)}{q}\right)^{1-2w} &= \frac{q}{2\pi(q, a+1)} - \frac{q}{\pi(q, a+1)} \log \frac{2\pi(q, a+1)}{q} (w-1) \\ &\quad + \frac{q}{\pi(q, a+1)} (\dots) (w-1)^2 + \dots, \end{aligned}$$

$$\sin^{-1}(\pi w) = -\frac{1}{\pi(w-1)} - \frac{\pi}{6}(w-1) + \dots$$

and

$$\cot(\pi w) = \frac{1}{\pi(w-1)} - \frac{\pi}{3}(w-1) + \dots,$$

we find that

$$\begin{aligned} & \operatorname{Res}_{w=1} \left(\frac{2\pi}{q} \right)^{1-2w} \Gamma(w) \left\{ \sin^{-1}(\pi w) E \left(w; \frac{\overline{\left(\frac{a+1}{(q,a+1)} \right)}}{\frac{q}{(q,a+1)}}, 0 \right) (q, a+1)^{1-2w} \right. \\ & \quad \left. + \cot(\pi w) E \left(w; -\frac{\overline{\left(\frac{a+1}{(q,a+1)} \right)}}{\frac{q}{(q,a+1)}}, 0 \right) (q, a+1)^{1-2w} + \delta i E \left(w; -\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}}, 0 \right) \right\} z^{1-w} \\ & = -\frac{1}{4} + \frac{\delta i}{2\pi} (q, a+1) \left(\gamma - \log \frac{4\pi^2 z}{(q, a+1)^2} \right) - \frac{q}{2\pi^2 (q, a+1)} (a_{0a}^+ - a_{0a}^-). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{1-c-i\infty}^{1-c+i\infty} \left(\frac{2\pi}{q} \right)^{1-2w} \Gamma(w) \left\{ \sin^{-1}(\pi w) E \left(w; \frac{\overline{\left(\frac{a+1}{(q,a+1)} \right)}}{\frac{q}{(q,a+1)}}, 0 \right) \right. \\ & \quad \times (q, a+1)^{1-2w} + \cot(\pi w) E \left(w; -\frac{\overline{\left(\frac{a+1}{(q,a+1)} \right)}}{\frac{q}{(q,a+1)}}, 0 \right) (q, a+1)^{1-2w} \\ & \quad \left. + \delta i E \left(w; -\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}}, 0 \right) \right\} z^{1-w} dw \\ & = \frac{1}{2\pi i G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) \int_{b-i\infty}^{b+i\infty} \left(\frac{2\pi}{q} \right)^{1-2w} \Gamma(w) \\ & \quad \times \left\{ \sin^{-1}(\pi w) E \left(w; \frac{\overline{\left(\frac{a+1}{(q,a+1)} \right)}}{\frac{q}{(q,a+1)}}, 0 \right) (q, a+1)^{1-2w} + \cot(\pi w) E \left(w; -\frac{\overline{\left(\frac{a+1}{(q,a+1)} \right)}}{\frac{q}{(q,a+1)}}, 0 \right) \right. \\ & \quad \times (q, a+1)^{1-2w} + \delta i E \left(w; -\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}}, 0 \right) \left. \right\} z^{1-w} dw \\ & \quad + \frac{1}{4G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) - \frac{\delta i}{2\pi G(\bar{\chi})} \sum_{a=1}^q \bar{\chi}(a) (q, a+1) \left(\gamma - \log \frac{4\pi^2 z}{(q, a+1)^2} \right) \\ & \quad + \frac{q}{2\pi^2 G(\bar{\chi})} \sum_{a=1}^q \frac{\bar{\chi}(a)}{(q, a+1)} (a_{0a}^+ - a_{0a}^-), \end{aligned}$$

where $1 < b < 2$. From this and (7), the theorem follows.

References

- [1] A. Balčiūnas. A transformation formula related to Dirichlet l -functions with principal character. *Liet. Matem. Rink., Ser. A*, **53**:13–18, 2012.
- [2] M. Jutila. *Lectures on a Method in the Theory of Exponential Sums*. Tata Institute of Fundamental Research, Bombay, 1987, 8–12.
- [3] A. Laurinćikas. One transformation formula related to the riemann's zeta-function. *Int. Trans. Spec. Funct.*, **19**(8):577–583, 2008.
- [4] M. Lukkarinen. The Mellin transform of the square of Riemann's zeta-function and Atkinson's formula. *Ann. Acad. Sci. Fenn., Math. Diss.*, **140**, 2005.

REZIUMĒ

Transformācijas formulē su primityvuoju charakteriu

A. Balčivūnas

Straipsnyje gauta formulē eksponentinei eilutei su primityvuoju charakteriu. Ši formulē gali būti pritaikyta L -funkcijas su primityvuoju charakteriu kvadrato Melino transformācijas meromorfiniam tēsiniui gauti.

Raktiniai žodžiai: Estermano dzeta-funkcija, Gauso suma, Melino transformacija.