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# Two complete finitary sequent calculi for reflexive common knowledge

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**Abstract.** This paper discusses the use of complete sequent calculi for reflexive common knowledge logic. Description of language and complete infinitary calculus for RCL is presented. Then finitary calculi $RCL_I$ and $RCL_L$ are introduced and completeness of finitary calculi $RCL_I$ and $RCL_L$ is proven.
**Keywords:** common knowledge logic, reflexive common knowledge logic, sequent calculi.

## 1 Introduction

A reflexive common knowledge logic (RCL) containing individual knowledge operators, reflexive "common knowledge" and "everyone knows" operators is considered. Complete sequent calculi for reflexive common knowledge logic is discucced ,finitary calculi $RCL_I$ and $RCL_L$ are introduced and completeness of these calculi is obtained using completeness of the infinitary calculus for RCL.

## 2 Descpription of language and complete infinitary calculus for $RCL$

The language of considered $RCL$ contains a set of propositional symbols $P, P_1, P_2, \dots,$ $Q, Q_1, Q_2, \dots$ the set of logical connectives $\supset, \wedge, \vee, \neg$; finite set of agent constants $i, i_1, i_2, \dots$; multiple knowledge modality $K(i)$, where $i$ is an agent constant; everyone knows operator $E$; common knowledge operator $C$.

A formula of $RCL$ is defined inductively as follows: every propositional symbol is a formula; if $A, B$ are formulas, then $(A \supset B)$, $(A \wedge B)$, $(A \vee B)$, $\neg(A)$ are formulas; if $i$ is an agent, $A$ is a formula, then $K(i)A$ is a formula; if $A$ is a formula, then $E(A)$ and $C(A)$ are formulas. The operator $K(i)$ behaves as modality of multi-modal logic $K_n$ [1].

The formula $K(i)A$ means "agent $i$ knows $A$". The formula $E(A)$ means "every agent knows $A$", i.e. $E(A) = \wedge_{i=1}^{n} K(i)A$ ($n$ is a number of agents). The formula $C(A)$ means "$A$ is common knowledge of all agents"; it is assumed that there is perfect communication between agents. The operator $C$ and $E$ behave as modalities of modal logic $S5$. In addition these operators satisfy the following powerful properties: $C(A) = A \wedge E(C(A))$ (fixed point) and $A \wedge C(A \supset E(A)) \supset C(A)$ (induction). Formal semantics of the formulas $K(i)$, $E(A)$, $C(A)$ are defined as in the reflexive common knowledge logic [3].

Bellow we consider calculi based on sequents, i.e., formal expressions $A_1, \dots, A_k \to B_1, \dots, B_m$, where $A_1, \dots, A_k$ $(B_1, \dots, B_m)$ is a multiset of arbitrary formulas. The infinatary calculus, denoted by $RCL_\omega$, for $RCL$ is defined by following postulates [3].

Axiom: $\Gamma, A \to \Delta, A$.

Rules consist of logical rules and modal ones. Logical rules consist of traditional invertible rules for logical symbols.

Modal rules:

$$\frac{\Gamma \to A}{\Pi, K_i\Gamma \to \Delta, K_i(A)}(K_i),$$

where $K_i\Gamma = K_iA_1, \dots, K_iA_n$ $(n \geqslant 0)$; $\Pi, \Delta$ consist of multisets of arbitrary formulas.

$$\frac{\wedge_{i=1}^m K_i(A), \Gamma \to \Delta}{E(A), \Gamma \to \Delta}(E \to),$$

$$\frac{\Gamma \to \Delta, \wedge_{i=1}^m K_i(A)}{\Gamma \to \Delta, E(A)}(\to E),$$

where $m$ is number of agents.

$$\frac{A, E(C(A)), \Gamma \to \Delta}{C(A), \Gamma \to \Delta}(C \to),$$

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, E(A); \dots; \Gamma \to \Delta, E^k(A) \dots}{\Gamma \to \Delta, C(A)}(\to C_\omega),$$

where $E^0(A) = A$; $E^k(A) = E(E^{k-1}(A))$, $k \geqslant 1$.

It is known (see e.g. [3]) that calculus $RCL_\omega$ is sound and complete.

$$\frac{\Gamma \to A}{\Pi, E(\Gamma) \to \Delta, E(A)}(E).$$

The rule is derivable in RCL* where $RCL^*$ is obtained from $RCL_\omega$ by dropping the rule $(\to C_\omega)$.

Let $\Gamma = A_1, \dots, A_n$ then derivability of $(E)$ is carried out in the following way:

$$\cfrac{\cfrac{A_1, \dots, A_n \to A}{\dots, K_i(A_1), \dots, K_i(A_n) \dots \to \dots K_i(A) \dots}(K_i)}{\cfrac{\Pi, E(A_i), \dots, \wedge_{i=1}^m K_i(A_i), \dots, E(A_n) \to \wedge_{i=1}^m K_i(A)}{\Pi, E(A_1), \dots, E(A_n) \to E(A)}(E \to), (\to E).}(\wedge \to)(\to \vee)$$

Derivations in $RCL_\omega$ are built in the form of the infinite tree, each branch of this tree is finitary. The height of a derivation $D$ (denoted by $O(D)$) is evaluated in ordinals.

A derivation $D$ in $RCL_\omega$ is called atomic if all axioms occurring in $D$ are the form $\Gamma, P \to \Delta, P$.

**Lemma 1.** *An arbitrary derivation in $RCL_\omega$ may be transformed into an atomic one.*

*Proof.* Let us denote by $g(A)$ the complexity of $A$ defined by the number of occurrences of logical and knowledge operators $C, E, K_i$ in $A$. The lemma is moved by induction on $g(A)$.

It is easy to see that all rules of $RCL_\omega$, except $(K_i)$ are invertible. Let us present a specialization of the rule $(K_i)$ which is existential invertible.

A sequent $S$ is a primary if $S = \Sigma_1, K(\Gamma_1) \to \Sigma_2, K(\Gamma_2)$, where $\Sigma_i$ $(i \in \{1,2\})$ is empty or consists of propositional symbols, $K(\Gamma_i)$ $(i \in \{1,2\})$ is empty or consists of formulas of the shape $K_l(A)$.

**Lemma 2.** *By backward applications of rules, except $(K_i)$ of $RCL_\omega$ any sequent $S$ can be reduced to a set of primary sequents $S_1, \ldots, S_n$ $(n \geqslant 1)$ such that if $RCL_\omega \vdash S_l$ then $\forall l$ $(l \geqslant 1)$ $RCL_\omega \vdash S_l$.*

*Proof.* Follows from invertibility of rules $RCL_\omega$ except $(K_i)$.

Let $RCL'_\omega$ be the calculus obtained from $RCL_\omega$ replacing the rule $(K_i)$ by the following one:

$$\frac{\Gamma_p \to A}{\Sigma_1, K_1^a(\Gamma_1), \ldots, K_n^a(\Gamma_n) \to \Sigma_2, K_1^S(\Delta_1), \ldots, K_l^s(A), \ldots, K_m^s(\Delta_m)} (K'_i),$$

$n \geqslant 0$, $m \geqslant 0$; $K_l^a = K_p^s$, $\Sigma_1 \cap \Sigma_2 = \emptyset$.

**Lemma 3 [Existential invertability of the rule $(K'_l)$ in $RCL_\omega$].** *Let $S = \Sigma_1$, $K_1^a(\Gamma_1), \ldots, K_n^a(\Gamma_n) \to \Sigma_2, K_1^s(\Delta_1), \ldots, K_m^s(\Delta_m)$ be a primary sequent satisfying the condition of the conclusion of $(K'_l)$ and let $RCL_\omega \vdash {}^D S$, then there exists a formula $K_l^s(A)$ such that $RCL_\omega \vdash \Gamma_p \to A$.*

*Proof.* From Lemma 1 it follows that all axioms in $D$ are atomic ones. Another hand, $\Sigma_1 \cap \Sigma_2 = \emptyset$ therefore $h(D) > 1$. Therefore from the scope of the rule $(K'_i)$ it follows that there exists a formula $K_l^s(A)$ from the succedent of $S$ such that $RCL_\omega \vdash \Gamma_p \to A$.

## 3 Finitary calculi $RCL_I$ and $RCL_L$

Infinitary calculus $RCL_\omega$ possesses the following beautiful property: it allows to present simple and evident completeness proof (see e.g. [3]). Despite of this property: all derivation containing infinitary rule $(\to C_\omega)$ are informal. To avoid this bad property several finitary complete sequent calculi for RCL can be presented [2].

(1) Calculus containing invariant-like rule.

The finitary calculus $RCL_I$ is obtained from the calclus $RCL_\omega$ replacing infinitary rule $(\to C_\omega)$ by following (cut) – like rule:

$$\frac{\Gamma \to \Delta, I; I \to E(I); I \to A}{\Gamma \to \Delta, C(A)} \to (C_I)$$

where the formula $I$ (called an invariant formula) is constructed from subformulas of formulas in the conclusion of the rule. There are some works in which constructive methods for finding invariant formulas in sequent calculi of epistemic logic are presented, e.g. [4, 5]. Using these methods we can find invariant formulas and for the rule $(\to C_I)$.

(2) Calculus containing weak-induction like rule and loop axiom.

The finitary calculus $RCL_L$ is obtained from the calculus $RCL_I$ in the following way:

(a) replacing the invariant rule $(\to C_I)$ by the following rule:

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, E(C(A))}{\Gamma \to \Delta, C(A)} (\to C_L).$$

This rule corresponds to the so-called weak-induction axiom: $A \wedge E(C(A)) \supset C(A)$.

(b) Adding loop-type axioms as follows: a sequent $S'$ is a loop type axiom if (1) $S'$ is above a sequent $S$ on a branch of derivation tree, (2) $S'$ is such that it subsumes $S'$ ($S \succcurlyeq'$ in notation), i.e. we can get $S'$ from $S$ using structural rules of weakening and contraction, in separate case $S = S'$.

(c) There is right premise of $(\to C_L)$ between $S$ and $S'$.

The completeness of finitary calculi $RCL_I$ and $RCL_L$ is obtained proving that the calculi $RCL_\omega$, $RCL_I$ and $RCL_L$ are equivalent to each other. The completeness of $RCL_\omega$ is used.

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REZIUMĖ

**Du pilni finitariniai skaičiavimai refleksyviai bendrojo žinojimo logikai**

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Straipsnyje pateikiami du pilni sekvenciniai skaičiavimai bendrojo žinojimo logikai. Pristatyta kalba ir pilnas begalinis skaičiavimas skirtas RCL. Straipsnyje pristatyti baigtiniai skaičiavimai $RCL_I$ ir $RCL_L$, ir įrodytas tų skaičiavimų pilnumas remiantis baigtinio skaičiavimo $RCL_\omega$ pilnumu.

*Raktiniai žodžiai*: bendrojo žinojimo logika, refleksyvi bendrojo žinojimo logika, sekvencinis skaičiavimas