Lietuvos matematikos rinkinys
Proc. of the Lithuanian Mathematical Society, Ser. A
Vol. **54**, 2013, 43–48

ISSN 0132-2818
www.mii.lt/LMR/

Geometry and topology of groups*

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Abstract. This short note is a slightly expanded version of the talk given by the author at the 54th LMD Conference, devoted to present a very informal and brief introduction to the main ideas of the asymptotic geometry of groups.

Keywords: Discrete groups, group presentations, Cayley graph and complex, quasi-isometry, ends of groups.

Introduction

We are interested in the *asymptotic topology* and *geometric invariants* of infinite discrete groups: these are notions that help the understanding of the topological and geometrical behavior at infinity of groups, or, more precisely, of universal covers of compact spaces having a given fundamental group.

M. Dehn was the first who solved algebraic problems using topological methods (in particular considering only groups that are fundamental groups of 2-dimensional surfaces). Then people started generalizing these kind of methods to more general groups acting 'in a nice way' on non-Euclidean geometry, culminating in a genuinely new class of groups: *Gromov-hyperbolic groups* [3].

Actually, in the 80's, M. Gromov [4] highlighted that the *large-scale* (asymptotic) geometrical and topological 'shape' of spaces endowed with a good group action, only depends, in some sense, JUST on the group itself and practically not on the space. The underlying idea one may follow is that ALL the various possible spaces associated to a given group should share some global geometrical and topological properties (at infinity), which are called *asymptotic properties.*

Other applications of geometrical methods to infinite group theory can be also find in 3-dimensional topology, starting with the celebrated studies of W.P. Thurston [8], where one understands that the main invariant of 3-manifolds is the fundamental group, whose nature is influenced by the geometry associated.

Nowadays, the Geometric Group Theory is a very rich and active research field connecting several branches of (modern) mathematics: algebra, analysis, combinatorics, discrete mathematics, dynamics, geometry, informatics, logic and topology (for more information see e.g. [1, 2, 5, 6, 7]).

* The author wishes to thank the Institute of Mathematics and Informatics of Vilnius University for the worm hospitality, as well as Prof. E. Mazėtis for the invitation.

1 Groups and associated spaces

If one wants to do some geometry/topology with a group, the first things needed are a metric and/or some spaces associated, somehow geometrically, to the group studied. Is it possible to say something algebraic (on the group) starting from geometrical properties of the metric and/or of the spaces? This is one of the key problem in geometric group theory: relating geometric conditions of a group (i.e. of the spaces associated to it), with its algebraic structure.

Let's start with the basic definitions:

Definition 1. A discrete group is a countable group with the discrete topology.

- A (discrete) group Γ is **finitely generated** if there exists a finite set S of generators (i.e. every element of Γ can be written as a product of powers of some of these generators $s_i \in S$). [The set S will always be assumed to satisfy: $S = S^{-1}$ and $e \notin S$.]
- The group is **finitely presented**, $\Gamma = \langle S \mid \mathcal{R} \rangle$, if, in addition, it has a finite number of relations $1 = r_j \in \mathcal{R}$ (words in terms of the generators).

To any finitely generated group Γ with generating set $\langle S \rangle$ one can associate a very 'natural' metric on it, called **the word metric**.

Definition 2. The length $l_S(g)$ of any element g of Γ is the smallest integer n such that there exists a sequence $(s_1, s_2, \ldots, s_n)$ of generators in S for which $g = s_1 s_2 \cdots s_n$. The distance (with respect to S) of two elements a, b of Γ is

$$d_S(a,b) = l_S\big(a^{-1}b\big).$$

With this distance the space (Γ, d_S) becomes a (0-dimensional) metric space, even if this metric is discrete. One way to overcome this problem is to associate to Γ a 1-dimensional metric space: a graph, called the **Cayley graph** of Γ.

Definition 3. The vertex set of the Cayley graph is identified with the elements of the group Γ. For any $g \in \Gamma$ and $s \in S$, the vertices corresponding to the elements g and gs are joined by an edge (labelled by s). Equivalently, two group elements a, b are joined by an edge if and only if their distance $d_S(a,b)$ is 1.

Considering any edge of the Cayley graph as a metric space isometric to the interval $[0,1] \subset \mathbb{R}$, the Cayley graph of a finitely generated group Γ becomes a nice metric space, containing (Γ, d_S) isometrically, and acted upon by Γ.

Remark 1. Note that both the metric d_S and the Cayley graph are constructed with respect to S (the generating set of Γ). Moreover:

- The generating set is needed to have an arc-connected (Cayley) graph, and one needs it to be finite in order to obtain a locally-finite graph.
- The Cayley graph is infinite if and only if the group is infinite.
- An edge-loop in the Cayley graph corresponds to a word equals to the identity in the group (and vice-versa).

Already the Cayley graph carries on several information on the group (as we will see), but before this we want to present other spaces associated to groups.

Whenever the group Γ is finitely presented, one can associate to it also a nice 2-dimensional space. Let $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ be a finite presentation for Γ.

Definition 4. The **Cayley 2-complex** of $\Gamma = \langle S \mid \mathcal{R} \rangle$ is the 2-dimensional complex obtained by gluing a disk on all paths of the Cayley graph labelled by a relator $r \in R$. [Being S and $\mathcal{R}$ finite, the Cayley 2-complex is locally-finite.]

Remark 2. Since closed paths in the Cayley graph label words equal to 1 in Γ, and, by definition, relators generate all the relations, then the Cayley complex is simply connected.

Actually, there is another (topological) way to define and construct both the Cayley graph and complex, starting from a finite presentation of the group Γ.

Definition 5. The **standard 2-complex** $X_{\mathcal{P}}$ associated to the presentation $\mathcal{P}$ of Γ is constructed as follows: starts with a bouquet of loops, i.e. a vertex v and one edge loop at v for any $s \in S$ oriented and labelled by s.

Then, if $l(g)$ denotes the length of $g \in \Gamma$, for each relator $r \in \mathcal{R}$, attach an $l(r)$-sides 2-cell to the bouquet using r to describe the attaching map. The resulting space is a finite 2-complex $X_{\mathcal{P}}(\Gamma)$.

By construction, $\pi_1(X_{\mathcal{P}}) = \Gamma$ (Van Kampen Theorem), and its universal covering space $\tilde{X}_{\mathcal{P}}$ is precisely the Cayley 2-complex of Γ (constructed topologically), while the 1-skeleton of $\tilde{X}_{\mathcal{P}}$ is exactly the Cayley graph of Γ.

But often a (finitely presented) group comes as the fundamental group of a compact n-manifold M^n, and hence naturally acts on $\tilde{M}^n$ (by covering transformations); or sometimes one may just observe that a given group acts on some (nice) space in a (nice) way (as it acts nicely on the universal cover whenever it is a fundamental group). This motivates the following definition:

Definition 6. A **geometry** is a topological space endowed with a path-metric (i.e. a metric such that the distance between any two points is realized as the length of some path joining them) which is proper (meaning that closed metric balls of finite radius are compact).

An action of a group Γ on a geometry X is said **geometric** if it is isometric $\big($i.e. $d(g \cdot x_1, g \cdot x_2) = d(x_1, x_2)\big)$, co-compact (i.e. with a compact quotient X/Γ) and properly discontinuous (i.e. such that for any compact subset $K \subset X$, the set of the elements $\{g \in \Gamma$ for which $K \cap gK \neq \emptyset\}$ is finite).

Wherever a group acts geometrically on a geometry these notions reflect the following information: firstly, when the group acts, the metric of the space is not changed by it; secondly, the group, though discrete, is, in some sense, ‘proportionally comparable’ with the size of the space (in the sense that it has ‘enough’ elements). In this way one can look at finitely generated/presented groups as ‘discrete models’ of the spaces on which they act (and approximate).

1.1 Examples

The easiest (but very instructive) example of a finitely presented group is the group of integers $\mathbb{Z}$. It is obviously generated by the set $S_1 = \{1, -1\}$, but another set of generators may be $S_2 = \{2, -2, 3, -3\}$. In this last case, there are, obviously, also some relations, and in particular, $\mathbb{Z}$ can be written as the (abstract) group $Z_1 = \langle a, b \mid aaab^{-1}b^{-1}, aba^{-1}b^{-1}\rangle$. Another way of presenting $\mathbb{Z}$ is the following one: $Z_2 = \langle a, b \mid baba^{-1}b^{-1}\rangle$.

It is easy to show that these two groups are isomorphic to $\mathbb{Z}$. Actually, for Z_1, the relation $aba^{-1}b^{-1}$ implies that a and b commute, and hence, the first relation may be written as: $aaab^{-1}b^{-1} = aab^{-1}ab^{-1} = a[(ba^{-1})(ba^{-1})]^{-1}$. From the second relation $aba^{-1}b^{-1} = 1$ one obtains that $bab^{-1}a^{-1} = 1$, hence $b[(ba^{-1})^{-1}a^{-1}] = 1$, and so $b[a(ba^{-1})]^{-1} = 1$.

In this way, the presentation $\langle a, b \mid aaab^{-1}b^{-1}, aba^{-1}b^{-1}\rangle$ is equal to $\langle a, b \mid a[(ba^{-1})(ba^{-1})]^{-1}, b[a(ba^{-1})]^{-1}\rangle$, which is equal to the presentation $\langle a, b, ba^{-1} \mid a[(ba^{-1})(ba^{-1})]^{-1}, b[a(ba^{-1})]^{-1}\rangle$ (by an application of a Tietze operation of type (T_1), see [5]). To the presentation written in this way, one can apply a Tietze movement (T_2), cancelling the generator b with the second relation, and obtaining $\langle a, ba^{-1} \mid a[(ba^{-1})(ba^{-1})]^{-1}\rangle$. Another application of such a Tietze operation (T_2) leads to the presentation $Z_1 = \langle ba^{-1} \mid \emptyset\rangle$, namely $Z_1 \simeq \mathbb{Z}$.

For the group $Z_2 = \langle a, b \mid baba^{-1}b^{-1}\rangle$, the relation $baba^{-1}b^{-1} = 1$ implies that $baba^{-1} = b$, and then $aba^{-1} = 1$, i.e. $ab = a$ and hence $b = 1$. Thus the group Z_2 is just the group generated by a, namely the integers $\mathbb{Z}$.

Having all these different presentations for the same group $\mathbb{Z}$, one can start to draw the various Cayley graphs and 2-complexes (see Figs. 1, 2, 3).

Of course, the group $\mathbb{Z} = \langle a\rangle$, is the fundamental group of the circle S^1, whose universal cover is the real line $\mathbb{R}$; but $\mathbb{Z}$ may also be considered as the fundamental group of a compact cylinder $S^1 \times [0, n]$, whose universal covering is the strip $\mathbb{R} \times [0, n]$; or as the fundamental group of $S^1 \times S^2$, and so on.

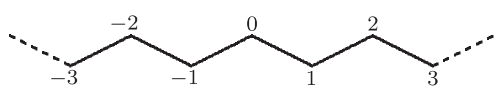

Fig. 1. The Cayley graph of $\mathbb{Z}$ for the standard presentation.

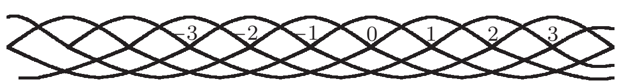

Fig. 2. The Cayley graph of Z_1.

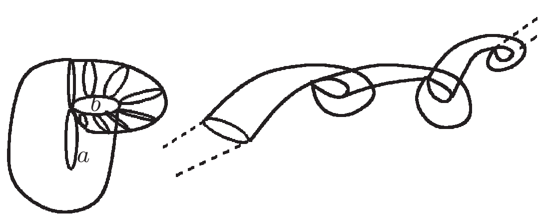

Fig. 3. The presentation 2-complex for the group Z_2, with its universal covering space.

This is just to say that, to a given finitely presented group (in our case $\mathbb{Z}$), one can associate several different spaces: itself with a word metric (one for any choice of the generating set S), the Cayley graph and complex (each time different for any choice of the presentation); or the group may be given as the π_1 of some compact manifold or as a group acting on some geometry.

Question: *do these spaces have something in common*?

The answer is, surprisingly, YES! Even if, locally, all these spaces are very different from each others, a group Γ with any of its word metric, all the Cayley graphs and 2-complexes associated to distinct presentations of it, and any geometry on which it acts geometrically, are ALL 'similar': i.e. **quasi-isometric** (see [1]).

For example, looking at the spaces associated to $\mathbb{Z}$, one realizes that, globally, all of them are alikes: just 'thin and long' in two different 'directions'. A quasi-isometry catches exactly these features: it is a map that 'ignores' the small-scale local details of the spaces, but preserving the coarse global structure.

Definition 7. A **quasi-isometry** between two metric spaces (X, d_X) and (Y, d_Y) is a map $f : X \to Y$ s. t., for some fixed positive constants C and λ:

$$\lambda^{-1} d_X(x_1, x_2) - C \le d_Y\big(f(x_1), f(x_2)\big) \le \lambda d_X(x_1, x_2) + C, \quad \text{and}$$
$$\forall y \in Y,\ \exists\, x \in X \text{ such that } d_Y\big(y, f(x)\big) \le C.$$

Since an algebraic classification of finitely presented groups is not possible (the word problem is undecidable), Gromov's idea was to try to classify them 'geometrically', i.e. up to quasi-isometries. Hence, from this viewpoint, interesting properties of groups are those that are invariants under quasi-isometries (and such a property is called *geometric* or *asymptotic*, see [4]).

Remark 3.

- A quasi-isometry may not be continuous: real numbers $\mathbb{R}$ and integers $\mathbb{Z}$ are quasi-isometric by the map $r \in \mathbb{R} \to [r] \in \mathbb{Z}$.
- More generally, a group G (with a finite generating set S) is quasi-isometric to its Cayley graph associated to S. Moreover, if S_1 and S_2 are two generating sets for the group G, then (G, d_{S_1}) and (G, d_{S_2}) are quasi-isometric.

  Thus, THE word metric and THE Cayley graph are (uniquely) well-defined up to quasi-isometries.
- A metric space is quasi-isometric to a point iff its diameter is finite. In particular, the Cayley graph of a group G is quasi-isometric to a point iff G is finite. Hence, the quasi-isometry class of the trivial group is the set of finite groups (then, from this viewpoint, finite groups are not relevant).

2 Topology at infinity: ends

The first asymptotic property of topological nature is the condition of being **one-ended**, which means that, outside very large compacts, there is only one 'way to go to infinity'. More generally, for a metric space X, the **number of ends** of X, $e(X)$, is the supremum of the number of unbounded connected component of complements of compacts. [The real line $\mathbb{R}$ has 2 ends.]

Definition 8. The number of ends of a finitely generated group is the number of ends of its Cayley graph (or, equivalently, of its Cayley 2-complex).

Theorem 1. *(H. Hopf, [7].) If K is a finite simplicial complex, then the set of ends of its universal covering only depends on $\pi_1 K$.*

Furthermore, the number of ends of a group belongs to the set $\{0, 1, 2, \infty\}$.

Thus, the number of ends of a group is well-defined (and then independent on the presentation), may take very few values (because the existence of a group action puts some constraints on the space), and is also a quasi-isometry invariant [2]. Now, is it possible to say something algebraic starting from the geometrical notion of the number of ends? One of the first results in geometric group theory relating algebraic properties with the topology at infinity is the following one:

Theorem 2. *(J.R. Stallings, [7].) A finitely generated group G has more than one end if and only if it admits a nontrivial decomposition as an amalgamated free product or an HNN-extension over a finite subgroup.*

This theorem also implies that the (algebraic) property of having a nontrivial splitting over a finite subgroup is a quasi-isometry (geometric) invariant of groups. For more recent developments in Geometric Group Theory see [1, 2, 4].

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REZIUMĖ

Grupių geometrija ir topologija
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Šis darbas yra trumpas įvadas į asimptotinę geometrinę grupių teoriją.

Raktiniai žodžiai: Diskrečios grupės, Cayley grafas, kvazi-izometrijos, galai.