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Investigation of matrix nullity for the second order discrete nonlocal boundary value problem

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Abstract. In this paper we investigate the relation between the matrix nullity of the second order discrete boundary value problem and nonlocal boundary conditions. The obtained classification and examples are also presented.

Keywords: discrete boundary value problem, kernel, nullity, nonlocal boundary conditions.

1 Introduction

Let us investigate the second order differential equation with nonlocal boundary conditions (NBC)

$$-u''(x) = f(x), \quad x \in (0,1), \tag{1}$$

$$\langle L_j, u\rangle := \langle \kappa_j, u\rangle - \gamma_j \langle \varkappa_j, u\rangle = 0, \quad j = 1,2, \tag{2}$$

where L_1, L_2 are linear functionals, $\langle \kappa_j, u\rangle$, $j = 1,2$, are classical parts and $\langle \varkappa_j, u\rangle$, $j = 1,2$, are nonlocal parts of boundary conditions (BC). We introduce the mesh $\overline{\omega}^h := \{x_i = ih\colon i \in X_n,\ nh = 1\}$, where $X_n := \{0,1,2,\dots,n\}$. Then the problem (1)–(2) can be approximated by a discrete problem

$$\mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i h^2, \quad i \in X_{n-2}, \tag{3}$$

$$\left\langle L_j^k, u_k\right\rangle := \sum_{k=0}^{n} L_j^k u_k = 0, \quad j = 1,2, \tag{4}$$

where $f_i = f(x_{i+1})$, $i \in X_{n-2}$. In [1], S. Roman presented the necessary and sufficient existence condition of the unique solution for the discrete problem (3)–(4), which is given by

$$D(\boldsymbol{L})[\boldsymbol{u}] := \begin{vmatrix} \langle L_1, 1\rangle & \langle L_2, 1\rangle \\ \langle L_1, x\rangle & \langle L_2, x\rangle \end{vmatrix} \neq 0.$$

In this paper we investigate the matrix nullity of the discrete problem (3)–(4) and its dependence on NBC when $D(\boldsymbol{L})[\boldsymbol{u}] = 0$.

2 Investigation of nullity

The problem (3)–(4) is equivalent to the linear system of equations $\mathbf{Au} = \mathbf{f}$. Then the solution of the kernel $\ker \mathbf{A} = \{\mathbf{u} \in \mathbb{R}^{n+1} \colon \mathbf{Au} = \mathbf{0}\}$ is equivalent to the homogeneous problem

$$\mathbf{Au} = \mathbf{0}. \tag{5}$$

Let $u_i^1 = 1$ and $u_i^2 = x_i$ be the fundamental system of the homogeneous equation (3). This system of solutions satisfies the first $n-1$ equations of (5), that correspond to the operator $\mathcal{L}$. Then the linear combination $u_i = c_1 u_i^1 + c_2 u_i^2$, $c_1, c_2 \in \mathbb{R}$, also satisfies the first $n-1$ equations of (5) and the last two equations of (5) satisfy equalities

$$\langle L_1, 1\rangle c_1 + \langle L_1, x\rangle c_2 = 0, \qquad \langle L_2, 1\rangle c_1 + \langle L_2, x\rangle c_2 = 0,$$

or

$$\begin{pmatrix} \langle L_1, 1\rangle & \langle L_1, x\rangle \\ \langle L_2, 1\rangle & \langle L_2, x\rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case, the condition $\det \mathbf{A} = 0$ is equivalent to the condition

$$D(\boldsymbol{L})[\boldsymbol{u}] = \begin{vmatrix} \langle L_1, 1\rangle & \langle L_2, 1\rangle \\ \langle L_1, x\rangle & \langle L_2, x\rangle \end{vmatrix} = 0. \tag{6}$$

If $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$, then the problem (5) has a unique solution $\mathbf{u} = \mathbf{0}$ and $\dim\ker \mathbf{A} = 0$. Moreover, the first $n-1$ equations of (5), that correspond to the operator $\mathcal{L}$, are linearly independent. Therefore, $\dim\ker \mathbf{A} \in \{0, 1, 2\}$.

Thus, generally the classification can be given:

1. $\dim\ker \mathbf{A} = 0$ if and only if $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$.
2. $\dim\ker \mathbf{A} = 1$. In this respect, two cases are possible:

   2.1. The only one row of matrix $\mathbf{A}$ that corresponds to the functional L_j is a linear combination only of the first $n-1$ rows, that correspond to the operator $\mathcal{L}$; the row of $\mathbf{A}$, that corresponds to the functional L_{3-j}, and the first $n-1$ rows of $\mathbf{A}$ are linearly independent if and only if

   $$\langle L_j, 1\rangle = \langle L_j, x\rangle = 0, \qquad \bigl|\langle L_{3-j}, 1\rangle\bigr| + \bigl|\langle L_{3-j}, x\rangle\bigr| \neq 0, \tag{7}$$

   where $j = 1, 2$.

   2.2. The row that corresponds to the functional L_j, $j = 1, 2$, is a linear combination of the row, that corresponds to the functional L_{3-j}, necessarily, and the first $n-1$ rows; the row, that corresponds to the functional L_{3-j}, and the first $n-1$ rows are linearly independent if and only if

   $$\bigl|\langle L_1, 1\rangle\bigr| + \bigl|\langle L_1, x\rangle\bigr| \neq 0, \qquad \bigl|\langle L_2, 1\rangle\bigr| + \bigl|\langle L_2, x\rangle\bigr| \neq 0, \qquad D(\boldsymbol{L})[\boldsymbol{u}] = 0. \tag{8}$$

3. $\dim\ker \mathbf{A} = 2$ if and only if

$$\langle L_1, 1\rangle = \langle L_1, x\rangle = \langle L_2, 1\rangle = \langle L_2, x\rangle = 0. \tag{9}$$

Remark 1. Property 2.2 is obtained for both rows that correspond to the first and the second boundary conditions, respectively, because the condition

$$\mathbf{v}_{n-1+j} = c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \dots + c_{n-2}\mathbf{v}_{n-2} + c_{n-1+k}\mathbf{v}_{n-1+k}, \quad c_{n-1+k} \neq 0,$$

implies

$$\mathbf{v}_{n-1+k} = \frac{1}{c_{n-1+k}}\mathbf{v}_{n-1+j} - \frac{c_0}{c_{n-1+k}}\mathbf{v}_0 - \dots - \frac{c_{n-2}}{c_{n-1+k}}\mathbf{v}_{n-2}, \quad k = 3 - j.$$

Here $\mathbf{v}_i$, $i \in X_n$, corresponds to the i-th row of $\mathbf{A}$. Therefore, in this respect, we can choose the row, which will be considered a linear combination of other rows of the matrix $\mathbf{A}$. The row, that corresponds to the other functional, and the first $n-1$ rows are linearly independent.

Remark 2. Property 3 means the both rows of $\mathbf{A}$ that correspond to boundary conditions are linear combinations of the first $n-1$ rows of $\mathbf{A}$.

Remark 3. The investigation of matrix nullity for any second order discrete nonlocal problem

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in X_{n-2},$$
$$\langle L_j^k, u_k \rangle := \sum_{k=0}^{n} L_j^k u_k = 0, \quad j = 1, 2,$$

where $\dim \operatorname{im} \mathcal{L} = n - 1$, is absolutely analogous.

Example 1. Let us consider Eq. (1) with NBC $u(0) = 0$, $u(1) = \gamma u(\xi)$, $0 < \xi < 1$. It can be approximated by a discrete problem

$$\begin{gathered} \mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i h^2, \quad i \in X_{n-2}, \\ \langle L_1, u \rangle := u_0 = 0, \qquad \langle L_2, u \rangle := u_n - \gamma u_s = 0, \quad \text{where } \xi = sh. \end{gathered} \tag{10}$$

According to (6), we get $\gamma\xi = 1$. Then we observe that

$$\left|\langle L_1, 1 \rangle\right| + \left|\langle L_1, x \rangle\right| \neq 0, \qquad \left|\langle L_2, 1 \rangle\right| + \left|\langle L_2, x \rangle\right| \neq 0,$$

because

$$\langle L_1, 1 \rangle = 1, \qquad \langle L_2, 1 \rangle = 1 - \gamma \neq 0, \quad \text{since } \ 0 < \xi < 1 \text{ and } \gamma = \frac{1}{\xi} > 1,$$
$$\langle L_1, x \rangle = 0, \qquad \langle L_2, x \rangle = 1 - \gamma\xi = 0, \quad \text{since } \gamma\xi = 1.$$

Corollary 1. *For the problem* (10) $\dim \ker \mathbf{A} = 1 \Leftrightarrow D(\boldsymbol{L})[\boldsymbol{u}] = 0$.

Corollary 2. *Either row that corresponds to a boundary condition can be considered a linear combination of all the other rows of* $\mathbf{A}$ *if and only if* $D(\boldsymbol{L})[\boldsymbol{u}] = 0$.

If $n = 4$, $h = 1/4$, $\xi = 1/2$, $\gamma = 1/\xi = 2$ and $s = 2$, then the problem (10) has the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence we observe that $\mathbf{v}_4 = -\mathbf{v}_0 - 2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$ and $\dim\ker\mathbf{A} = 1$.

Example 2. Let us consider Eq. (1) with NBC $u(0) = \gamma_0 u(\xi_0)$, $u(1) = \gamma_1 u(\xi_1)$, where $0 < \xi_j < 1$, $j = 1, 2$. It can be approximated by a discrete problem

$$\begin{gathered} \mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i h^2, \quad i \in X_{n-2}, \\ \langle L_1, u\rangle := u_0 - \gamma_0 u_{s_0} = 0, \qquad \langle L_2, u\rangle := u_n - \gamma_1 u_{s_1} = 0, \end{gathered} \tag{11}$$

where $\xi_j = s_j h$, $j = 1, 2$. Then from (6) follows

$$\gamma_0(1 - \xi_0) + \gamma_1\xi_1 - \gamma_0\gamma_1(\xi_1 - \xi_0) = 1. \tag{12}$$

Moreover, we have

$$\langle L_1, 1\rangle = 1 - \gamma_0, \qquad \langle L_1, x\rangle = -\gamma_0\xi_0, \qquad \langle L_2, 1\rangle = 1 - \gamma_1, \qquad \langle L_2, x\rangle = 1 - \gamma_1\xi_1.$$

Thus, four cases are possible:

(1) $\langle L_1, 1\rangle = 0 \Leftrightarrow \gamma_0 = 1$. Then $\langle L_1, x\rangle \neq 0$, since $0 < \xi_0 < 1$.
(2) $\langle L_1, x\rangle = 0 \Leftrightarrow \gamma_0 = 0$. Then $\langle L_1, 1\rangle \neq 0$, since $0 < \xi_0 < 1$.
(3) $\langle L_2, 1\rangle = 0 \Leftrightarrow \gamma_1 = 1$. Then $\langle L_2, x\rangle \neq 0$, since $0 < \xi_1 < 1$.
(4) $\langle L_2, x\rangle = 0 \Leftrightarrow \gamma_1 \neq 0$ and $\gamma_1\xi_1 = 1$. Then $\langle L_2, 1\rangle \neq 0$, since $0 < \xi_1 < 1$.

Therefrom, we observe that

$$\bigl|\langle L_1, 1\rangle\bigr| + \bigl|\langle L_1, x\rangle\bigr| \neq 0, \qquad \bigl|\langle L_2, 1\rangle\bigr| + \bigl|\langle L_2, x\rangle\bigr| \neq 0, \quad \forall\gamma_0, \gamma_1 \in \mathbb{R}.$$

Corollary 3. *For the problem* (11) $\dim\ker\mathbf{A} = 1 \Leftrightarrow D(\boldsymbol{L})[\boldsymbol{u}] = 0$.

Corollary 4. *Either row that corresponds to a boundary condition can be considered a linear combination of all the other rows of* $\mathbf{A}$ *if and only if* $D(\boldsymbol{L})[\boldsymbol{u}] = 0$.

If $n = 4$, $h = 1/4$, $\xi_0 = 1/4$, $\xi_1 = 1/2$, $\gamma_0 = \gamma_1 = 1$, then the problem (11) has the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence we observe that $\mathbf{v}_4 = -2\mathbf{v}_0 - 2\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3$ and $\dim\ker\mathbf{A} = 1$.

Example 3. Let us consider Eq. (1) with NBC $u(0) = \sum_{j=0}^{l-1} \gamma_j u(\xi_j)$, $u(1) = \sum_{j=l}^{m} \gamma_j u(\xi_j)$, where $0 < \xi_j < 1$, $j \in X_m$. It can be approximated by a discrete problem

$$\mathcal{L}u := -u_{i+2} + 2u_{i+1} - u_i = f_i h^2, \quad i \in X_{n-2},$$
$$\langle L_1, u\rangle := u_0 - \sum_{j=0}^{l-1} \gamma_j u_{s_j} = 0, \qquad \langle L_2, u\rangle := u_n - \sum_{j=1}^{m} \gamma_j u_{s_j} = 0, \tag{13}$$

where $\xi_j = s_j h$, $i \in X_m$. Then from (6) follows

$$\sum_{j=0}^{l-1} \gamma_j(1-\xi_j) + \sum_{j=l}^{m} \gamma_j \xi_j - \sum_{j=0}^{l-1}\sum_{j=l}^{m} \gamma_j\gamma_k(\xi_k - \xi_j) = 1. \tag{14}$$

Furthermore,

$$\langle L_1, 1\rangle = 1 - \sum_{j=0}^{l-1} \gamma_j, \qquad \langle L_2, 1\rangle = 1 - \sum_{j=l}^{m} \gamma_j,$$
$$\langle L_1, x\rangle = -\sum_{j=0}^{l-1} \gamma_j \xi_j, \qquad \langle L_2, x\rangle = 1 - \sum_{j=l}^{m} \gamma_j \xi_j.$$

We observe

$$\langle L_1, 1\rangle = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{l-1} \gamma_j = 1, \qquad \langle L_1, x\rangle = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{l-1} \gamma_j \xi_j = 0,$$
$$\langle L_2, 1\rangle = 0 \quad \Leftrightarrow \quad \sum_{j=l}^{m} \gamma_j = 1, \qquad \langle L_2, x\rangle = 0 \quad \Leftrightarrow \quad \sum_{j=l}^{m} \gamma_j \xi_j = 1.$$

Corollary 5.

1. *For the problem* (13) $\dim\ker \mathbf{A} = 2$ *if and only if the equalities are satisfied*

$$\sum_{j=0}^{l-1} \gamma_j = \sum_{j=l}^{m} \gamma_j = \sum_{j=l}^{m} \gamma_j \xi_j = 1, \qquad \sum_{j=0}^{l-1} \gamma_j \xi_j = 0. \tag{15}$$

2. *For the problem* (13) $\dim\ker \mathbf{A} = 1$ *if and only if the equality* (14) *is satisfied and at least one equality* (15) *is not satisfied.*

Corollary 6.

1. *The row of matrix* $\mathbf{A}$ *that corresponds to the functional* L_1 *is a linear combination only of the first* $n-1$ *rows*; *the row of* $\mathbf{A}$, *that corresponds to the functional* L_2, *and the first* $n-1$ *rows of* $\mathbf{A}$ *are linearly independent if and only if the equalities*

$$\sum_{j=0}^{l-1} \gamma_j = 1, \qquad \sum_{j=0}^{l-1} \gamma_j \xi_j = 0$$

and at least one inequality

$$\sum_{j=l}^{m}\gamma_j \neq 1, \qquad \sum_{j=l}^{m}\gamma_j\xi_j \neq 1 \tag{16}$$

are satisfied.

2. *The row of matrix* $\mathbf{A}$ *that corresponds to the functional* L_2 *is a linear combination only of the first* $n-1$ *rows*; *the row of* $\mathbf{A}$, *that corresponds to the functional* L_1, *and the first* $n-1$ *rows of* $\mathbf{A}$ *are linearly independent if and only if the equalities*

$$\sum_{j=l}^{m}\gamma_j = \sum_{j=l}^{m}\gamma_j\xi_j = 1$$

and at least one inequality

$$\sum_{j=0}^{l-1}\gamma_j \neq 1, \qquad \sum_{j=0}^{l-1}\gamma_j\xi_j \neq 0 \tag{17}$$

are satisfied.

3. *Either row that corresponds to a boundary condition can be considered a linear combination of all the other rows of* $\mathbf{A}$ *if and only if the equality* (14) *is satisfied and at least one inequality* (16) *is satisfied, and at least one inequality* (17) *is satisfied.*
4. *The both rows of* $\mathbf{A}$ *that correspond to functionals* L_1, L_2 *are linear combinations of the rows of* $\mathbf{A}$, *that correspond to the operator* $\mathcal{L}$, *if and only if the equalities* (15) *are satisfied.*

Remark 4. In the cases 1–3 $\dim\ker\mathbf{A}=1$. In the case 4 $\dim\ker\mathbf{A}=2$.

References

[1] S. Roman. *Green's functions for boundary-value problems with nonlocal boundary conditions.* Doctoral dissertation, Vilnius University, 2011.

REZIUMĖ

Antrosios eilės diskrečiojo uždavinio su nelokaliosiomis kraštinėmis sąlygomis matricos branduolio dimensijos tyrimas

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Šiame darbe yra nagrinėjamas ryšys tarp antrosios eilės diskrečiojo uždavinio matricos defekto ir nelokaliųjų kraštinių sąlygų. Darbe taip pat pateikta gauta klasifikacija bei pavyzdžiai.

Raktiniai žodžiai: diskretusis kraštinis uždavinys, branduolys, defektas, nelokaliosios kraštinės sąlygos.