Joint universality of the Riemann zeta-function and Lerch zeta-functions

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Abstract. In the paper, we prove a joint universality theorem for the Riemann zeta-function and a collection of Lerch zeta-functions with parameters algebraically independent over the field of rational numbers.

Keywords: Lerch zeta-function, Riemann zeta-function, limit theorem, universality.

1 Introduction

Let $\lambda \in \mathbb{R}$ and α, $0 < \alpha \leqslant 1$, be fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + \mathrm{i}t$, is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i}\lambda m}}{(m+\alpha)^s}.$$

For $\lambda \in \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$ which is a meromorphic function with a unique simple pole at the point $s = 1$ with residue 1. If $\lambda \notin \mathbb{Z}$, then the Lerch zeta-function has analytic continuation to an entire function. In view of the periodicity of $\mathrm{e}^{2\pi \mathrm{i}\lambda m}$, we can suppose that $0 < \lambda \leqslant 1$.

It is well known that the Lerch zeta-function $L(\lambda, \alpha, s)$ with transcendental parameter α is universal (see [1], also [2]). Let $D = \{s \in \mathbb{C}\colon\ 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and, for $K \in \mathcal{K}$, denote by $H(K)$ the set of continuous functions on K which are analytic in the interior of K.

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Moreover, we use the notation $\mathrm{meas}\{A\}$ for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the universality of $L(\lambda, \alpha, s)$ is contained in the following theorem.

Theorem 1. *Suppose that α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T]\colon \sup_{s\in K} \big|L(\lambda, \alpha, s + \mathrm{i}\tau) - f(s)\big| < \epsilon\Big\} > 0.$$

Thus, the universality of $L(\lambda, \alpha, s)$ means that the shifts $L(\lambda, \alpha, s + \mathrm{i}\tau)$ approximate with a given accuracy a wide class of analytic functions.

The functions $\zeta(s, \alpha)$, $\alpha \neq 1, 1/2$, and $L(\lambda, \alpha, s)$ with rational λ are also universal in the above sense with rational parameter α. The case of $\zeta(s, \alpha)$ has been examined in [3]. The universality of $L(\lambda, \alpha, s)$ follows from its expression by a linear combination of Hurwitz zeta-functions.

Also, in [4–6] and [7], the joint universality of Lerch zeta-functions has been considered. We state a general result from [7].

Theorem 2. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $\lambda_j \in (0, 1]$, $K_j \in \mathcal{K}$, and $f_j(s) \in H(K_j)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \big|L(\lambda_j, \alpha_j, s + \mathrm{i}\tau) - f_j(s)\big| < \epsilon\Big\} > 0.$$

We note that the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ can be replaced by a more general hypothesis that the set

$$\mathrm{L}(\alpha_1, \dots, \alpha_r) = \big\{\log(m + \alpha_j)\colon m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ j = 1, \dots, r\big\}$$

is linearly independent over $\mathbb{Q}$. In the case $\lambda_j \in \mathbb{Z}$, $j = 1, \dots, r$, this was done in [8].

In [9], a joint universality theorem for the Riemann zeta-function $\zeta(s)$ and periodic Hurwitz zeta-functions has been obtained. Let $\mathfrak{A} = \{a_m\colon m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. We remind that the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$ with parameter α, $0 < \alpha \leqslant 1$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},$$

and is meromorphically continued to the whole complex plane with a unique possible pole at the point $s = 1$ with residue

$$a \stackrel{\mathrm{def}}{=} \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

If $a = 0$, then $\zeta(s, \alpha; \mathfrak{A})$ is an entire function.

For $j = 1, \dots, r$, let $l_j \in \mathbb{N}$. In [9], the joint universality for the functions

$$\zeta(s),\ \zeta(s, \alpha_1; \mathfrak{A}_{11}), \dots,\ \zeta(s, \alpha_1; \mathfrak{A}_{1l_1}), \dots,\ \zeta(s, \alpha_r; \mathfrak{A}_{r1}), \dots,\ \zeta(s, \alpha_r; \mathfrak{A}_{rl_r}) \qquad (1)$$

has been proved. Here a collection of periodic sequences $\mathfrak{A}_{jl}$, $\mathfrak{A}_{jl} = \{a_{mjl}\colon m \in \mathbb{N}_0\}$, with minimal period $k_{jl} \in \mathbb{N}$, $l = 1, \dots, l_j$, corresponds the parameter α_j, $0 < \alpha_j \leqslant 1$, $j = 1, \dots, r$. For $K \in \mathcal{K}$, denote by $H_0(K)$ the class of continuous non-vanishing functions on K which are analytic in the interior of K. Let k_j be the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, and

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j j1} & a_{k_j j2} & \dots & a_{k_j jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Then the main result of [9] is of the form.

Theorem 3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(A_j) = l_j$, $j = 1, \dots, r$. *For $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl} \in H(K_{jl})$. Moreover, let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0; T]\colon \sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} \big|\zeta(s + \mathrm{i}\tau, \alpha_j; \mathfrak{A}_{jl}) - f_{jl}(s)\big| < \epsilon,$$
$$\sup_{s\in K} \big|\zeta(s + \mathrm{i}\tau) - f(s)\big| < \epsilon\Big\} > 0.$$

We call the approximation property of the functions (1) in Theorem 3 a mixed joint universality because the function $\zeta(s)$ and the functions $\zeta(s, \alpha_j; \mathfrak{A}_{jl})$ are of different types: the function $\zeta(s)$ has Euler product, while the functions $\zeta(s, \alpha_j; \mathfrak{A}_{jl})$ with transcendental α_j do not have Euler product over primes. This is reflected in the approximated functions: the function $f(s)$ must be non-vanishing on K, while the functions f_{jl} are arbitrary continuous functions on K_{jl}.

The first mixed joint universality theorem has been obtained by Mishou [10] for the Riemann zeta-function and Hurwitz zeta-function $\zeta(s, \alpha)$ with transcendental parameter α. This result in [11] has been generalized for a periodic zeta-function and a periodic Hurwitz zeta-function. In [12], the latter mixed joint universality theorem has been extended for several periodic zeta-functions and periodic Hurwitz zeta-functions.

Universality theorems for zeta-functions have a series of interesting applications. From them, for example, various denseness results of Bohr's type for values of zeta-functions follow. The universality implies the functional independence of zeta-functions. This property of zeta-functions is applied to the zero-distribution of those zeta-functions. In [13], the universality has been applied to the famous class number problem. Universality theorems find applications even in solving some problems of physics [14]. For the above mentioned and other facts related to universality and references, we refer to [2, 15–20].

Thus, the universality of zeta-functions is a very interesting and useful property which motivates to continue investigations in the field.

The aim of this paper is to replace the zeta-functions $\zeta(s, \alpha_j; \mathfrak{A}_{jl})$ with periodic coefficients in Theorem 3 by Lerch zeta-functions $L(\lambda_j, \alpha_j, s)$ with arbitrary $\lambda_j \in (0, 1]$ whose coefficients, in general, are not periodic. This is the novelty of the paper.

Theorem 4. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $\lambda_j \in (0, 1]$, $K_j \in \mathcal{K}$ and $f_j \in H(K_j)$. Moreover, let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0;T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \big|L(\lambda_j, \alpha_j, s + \mathrm{i}\tau) - f_j(s)\big| < \epsilon, \\ \sup_{s\in K} \big|\zeta(s + \mathrm{i}\tau) - f(s)\big| < \epsilon\Big\} > 0.$$

We note that the linear independence of the set $\mathrm{L}(\alpha_1, \ldots, \alpha_r)$ is not sufficient for the proof of Theorem 4 because we need the linear independence of the set

$$\mathrm{L} \stackrel{\text{def}}{=} \big\{(\log p\colon p \in \mathcal{P}), \mathrm{L}(\alpha_1, \ldots, \alpha_r)\big\},$$

where $\mathcal{P}$ is the set of all prime numbers. This set consists of logarithms of all prime numbers and of all logarithms $\log(m + \alpha_j)$, $m \in \mathbb{N}$, $j = 1, \ldots, r$. Really, L is a multiset. For example, if L has two identical elements, then it is linearly dependent over $\mathbb{Q}$. The proof of Theorem 4 is based on a joint limit theorem on weakly convergent probability measures in the space of analytic functions.

2 Joint limit theorem

Denote by $\mathcal{B}(S)$ the σ-field of Borel sets of the space S, and by γ the unit circle on the complex plane. Define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathcal{P}$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication the tori $\hat{\Omega}$ and Ω are compact topological Abelian groups. Moreover, let

$$\underline{\Omega} = \hat{\Omega} \times \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$. Then $\underline{\Omega}$ again is a compact topological Abelian group. This gives the probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$, $(\Omega_j, \mathcal{B}(\Omega_j), m_{jH})$ and $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where $\hat{m}_H$, m_{jH} and $\underline{m}_H$ are the probability Haar measures on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$, $(\Omega_j, \mathcal{B}(\Omega_j))$ and $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$, respectively, $j = 1, \ldots, r$. We note that the measure $\underline{m}_H$ is the product of the measures $\hat{m}_H, m_{1H}, \dots, m_{rH}$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to γ_p, $p \in \mathcal{P}$, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to γ_m, $m \in \mathbb{N}_0$. For brevity, we set $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ and $\underline{\omega} = (\hat{\omega}, \omega_1, \dots, \omega_r) \in \underline{\Omega}$.

Let $H(D)$ be the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and let $r_1 = r + 1$. On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^{r_1}(D)$-valued random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$ by the formula

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) = \big(\zeta(s, \hat{\omega}), L(\lambda_1, \alpha_1, s, \omega_1), \dots, L(\lambda_r, \alpha_r, s, \omega_r)\big),$$

where

$$\zeta(s, \hat{\omega}) = \prod_p \left(1 - \frac{\hat{\omega}(p)}{p^s}\right)^{-1}$$

and

$$L(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i} \lambda_j m} \omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

Let $P_{\underline{\zeta}}$ stand for the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$, i.e., $P_{\underline{\zeta}}$ is the probability measure on $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$ given by

$$P_{\underline{\zeta}}(A) = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega}\colon \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\big).$$

We set

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}) = \big(\zeta(s), L(\lambda_1, \alpha_1, s), \dots, L(\lambda_r, \alpha_r, s)\big).$$

Now we state a limit theorem on the space $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$.

Theorem 5. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $\lambda_j \in (0, 1]$, $j = 1, \dots, r$. Then*

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas}\big\{\tau \in [0, T]\colon \underline{\zeta}(s + \mathrm{i}\tau, \underline{\alpha}; \underline{\lambda}) \in A\big\}, \quad A \in \mathcal{B}\big(H^{r_1}(D)\big),$$

converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.

We divide the proof of Theorem 5 into lemmas. The first lemma is a limit theorem on the torus $\underline{\Omega}$. For $A \in \mathcal{B}(\underline{\Omega})$, define

$$Q(A) = \frac{1}{T} \operatorname{meas}\big\{\big(\big(p^{-\mathrm{i}\tau}\colon p \in \mathcal{P}\big), \big((m + \alpha_j)^{-\mathrm{i}\tau}\colon m \in \mathbb{N}_0,\ j = 1, \dots, r\big)\big) \in A\big\}.$$

Lemma 1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then Q_T converges weakly to the Haar measure $\underline{m}_H$ as $T \to \infty$.*

Proof. The proof of the lemma is given in [9, Lemma 1]. □

Let $\sigma_1 > 1/2$ be a fixed number, and

$$u_n(m) = \exp\biggl\{-\biggl(\frac{m}{n}\biggr)^{\sigma_1}\biggr\}, \quad m, n \in \mathbb{N},$$

$$u_n(m, \alpha_j) = \exp\biggl\{-\biggl(\frac{m+\alpha_j}{n+\alpha_j}\biggr)^{\sigma_1}\biggr\}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

Define the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{u_n(m)}{m^s},$$

and

$$L_n(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i}\lambda_j m} u_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,$$

and, for $\underline{\omega} \in \underline{\Omega}$,

$$\zeta_n(s, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{\hat{\omega}(m) u_n(m)}{m^s},$$

$$L_n(\lambda_j, \alpha_j, \omega_j, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i}\lambda_j m} \omega_j(m) u_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r.$$

It is known, see, for example, [2, 16], that all above series converge absolutely for $\sigma > 1/2$. Let

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\lambda}) = \bigl(\zeta_n(s), L_n(\lambda_1, \alpha_1, s), \dots, L_n(\lambda_r, \alpha_r, s)\bigr)$$

and

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) = \bigl(\zeta_n(s, \hat{\omega}), L_n(\lambda_1, \alpha_1, \omega_1, s), \dots, L_n(\lambda_r, \alpha_r, \omega_r, s)\bigr).$$

Lemma 2. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $\underline{\omega} \in \underline{\Omega}$. Then*

$$\frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0, T]\colon \underline{\zeta}_n(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H^{r_1}(D)\bigr),$$

and

$$\frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0, T]\colon \underline{\zeta}_n(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H^{r_1}(D)\bigr),$$

converges weakly to the same probability measure P_n on $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$ as $T \to \infty$.

Proof. The proof uses Lemma 1 and does not depend on the coefficients of the functions $L_n(\lambda_j, \alpha_j, s)$, $j = 1, \dots, r$. Therefore, it coincides with the proof of [9, Lemma 2]. □

Now we define a metric on $H^{r_1}(D)$ which induces the topology of uniform convergence on compacta. For $g_1, g_2 \in H(D)$, we define

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s\in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_m} |g_1(s) - g_2(s)|},$$

where $\{K_m\colon m \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that

$$D = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, if $K \subset D$ is a compact set, then $K \subset K_m$ for some $m \in \mathbb{N}$. The existence of the sequence $\{K_m\}$ follows from a general theorem, see, for example, [21], however, in the case of the region D, it is easily seen that we can take closed rectangles. Clearly, ρ is a metric on $H(D)$ inducing its topology. For $\underline{g}_j = (g_j, g_{j1}, \dots, g_{jr}) \in H^{r_1}(D)$, $j = 1, 2$, we put

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max\Bigl(\rho(g_1, g_2), \max_{1\leqslant j\leqslant r} \rho(g_{1j}, g_{2j})\Bigr).$$

Then we have that $\underline{\rho}$ is a desired metric on $H^{r_1}(D)$. Using this metric, we approximate $\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda})$ and $\underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$ by $\underline{\zeta}_n(s, \underline{\alpha}, \underline{\lambda})$ and $\underline{\zeta}_n(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$, respectively.

Lemma 3. *We have*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}\bigl(\underline{\zeta}(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}), \underline{\zeta}_n(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda})\bigr)\, \mathrm{d}\tau = 0.$$

Moreover, suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}\bigl(\underline{\zeta}(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}), \underline{\zeta}_n(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega})\bigr)\, \mathrm{d}\tau = 0.$$

Proof. In [16], it is proved that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\bigl(\zeta(s + \mathrm{i}\tau), \zeta_n(s + \mathrm{i}\tau)\bigr)\, \mathrm{d}\tau = 0,$$

and, for almost all $\hat{\omega} \in \hat{\Omega}$

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\bigl(\zeta(s + \mathrm{i}\tau, \hat{\omega}), \zeta_n(s + \mathrm{i}\tau, \hat{\omega})\bigr)\, \mathrm{d}\tau = 0.$$

Since the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, each number α_j is transcendental. Therefore, in [2], it was obtained that, for $j = 1, \dots, r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\bigl(L(\lambda_j, \alpha_j, s + \mathrm{i}\tau), L_n(\lambda_j, \alpha_j, s + \mathrm{i}\tau)\bigr)\, \mathrm{d}\tau = 0,$$

and, for almost all $\omega_j \in \Omega_j$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\bigl(L(\lambda_j, \alpha_j, \omega_j, s + \mathrm{i}\tau), L_n(\lambda_j, \alpha_j, \omega_j, s + \mathrm{i}\tau)\bigr)\, \mathrm{d}\tau = 0.$$

All these equalities together with the definition of the metric $\underline{\rho}$ prove the lemma. □

On $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$, define one more probability measure

$$\hat{P}_T(A) = \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0, T]\colon \underline{\zeta}(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\bigr\}.$$

Lemma 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then P_T and $\hat{P}_T$ both converge weakly for almost all $\underline{\omega} \in \underline{\Omega}$ to the same probability measure P on $(H^{r_1}(D), \mathcal{B}(H^{r_1}(D)))$ as $T \to \infty$.*

Proof. We give a shortened proof because we apply similar arguments as in [9]. Let θ be a random variable defined on a certain probability space $(\Omega_0, \mathcal{A}, \mathbf{P})$ and uniformly distributed on $[0, 1]$. Let

$$\underline{X}_{T,n}(s) = \underline{\zeta}_n(s + \mathrm{i}\theta T, \underline{\alpha}, \underline{\lambda}). \tag{2}$$

Then, in view of Lemma 2, $\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n$, where $\underline{X}_n$ is the random element with the distribution P_n (P_n is the limit measure in Lemma 2), and $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. Using the absolute convergence of series for $\zeta_n(s)$ and $L_n(\lambda_j, \alpha_j, s)$, $j = 1, \dots, r$, we prove without difficulties that the family of probability measures $\{P_n\colon n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem, this family is relatively compact. Thus, we have a subsequence $\{P_{n_k}\}$ such that P_{n_k} converges weakly to some probability measure P as $k \to \infty$. Hence,

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P.$$

Define

$$\underline{X}_T(s) = \underline{\zeta}(s + \mathrm{i}\theta T, \underline{\alpha}, \underline{\lambda}). \tag{3}$$

Then Lemma 3 implies that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbf{P}\bigl(\underline{\rho}\bigl(\underline{X}_T(s), \underline{X}_{T,n}(s)\bigr) \geqslant \epsilon\bigr) = 0.$$

This, (2), (3) and Theorem 4.2 of [22] show that

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P,$$

and this is equivalent to the weak convergence of P_T to P as $T \to \infty$.

Repeating the above arguments for the random elements

$$\underline{\hat{X}}_{T,n}(s) = \underline{\zeta}_n(s + \mathrm{i}\theta T, \underline{\alpha}, \underline{\lambda}, \underline{\omega})$$

and

$$\underline{\hat{X}}_T(s) = \underline{\zeta}(s + \mathrm{i}\theta T, \underline{\alpha}, \underline{\lambda}, \underline{\omega}),$$

and using Lemmas 2 and 3, we find that the measure $\hat{P}_T$ also converges weakly to P as $T \to \infty$ for almost all $\underline{\omega} \in \underline{\Omega}$. □

Proof of Theorem 5. In virtue of Lemma 4, it suffices to check that the measure P in Lemma 4 coincides with $P_{\underline{\zeta}}$.

Let, for $\tau \in \mathbb{R}$,

$$a_\tau = \big((p^{-\mathrm{i}\tau}\colon\ p \in \mathcal{P}), \big((m+\alpha_j)^{-\mathrm{i}\tau}\colon\ m \in \mathbb{N}_0,\ j = 1, \dots, r\big)\big),$$

and

$$\Phi_\tau(\underline{\omega}) = a_\tau \underline{\omega}, \quad \underline{\omega} \in \underline{\Omega}.$$

Then $\{\Phi_\tau\colon\ \tau \in \mathbb{R}\}$ is an ergodic group of measurable measure preserving transformations on $\underline{\Omega}$ (see [12]).

Let ξ be a random variable on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$ given by

$$\xi(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A, \\ 0 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \notin A, \end{cases}$$

where A is a fixed continuity set of the measure P.

By Lemma 4, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\big\{\tau \in [0,T]\colon\ \underline{\zeta}(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\big\} = P(A). \tag{4}$$

The ergodicity of the group $\{\Phi_\tau\colon\ \tau \in \mathbb{R}\}$ implies that of the process $\xi(\Phi_\tau(\underline{\omega}))$. Therefore, the classical Birkhoff–Khintchine theorem shows that, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi\big(\Phi_\tau(\underline{\omega})\big)\,\mathrm{d}\tau = \mathbf{E}\xi, \tag{5}$$

where $\mathbf{E}\xi$ denotes the expectation of ξ. The definitions of ξ and of Φ_τ give the equalities

$$\mathbf{E}\xi = \int_{\underline{\Omega}} \xi\,\mathrm{d}\underline{m}_H = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega}\colon\ \underline{\zeta}(s, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\big) = P_{\underline{\zeta}}(A), \tag{6}$$

$$\frac{1}{T} \int_0^T \xi\big(\Phi_\tau(\underline{\omega})\big)\,\mathrm{d}\tau = \frac{1}{T} \operatorname{meas}\big\{\tau \in [0,T]\colon\ \underline{\zeta}(s + \mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}, \underline{\omega}) \in A\big\}.$$

Thus, by (5) and (6),

$$\lim_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\bigl\{\tau\in[0,T]\colon\ \underline{\zeta}(s+\mathrm{i}\tau,\underline{\alpha},\underline{\lambda},\underline{\omega})\in A\bigr\}=P_{\underline{\zeta}}(A).$$

This and (4) show that $P(A)=P_{\underline{\zeta}}(A)$ for all continuity sets of P. Hence, $P=P_{\underline{\zeta}}$. The theorem is proved. □

3 Support

A proof of Theorem 4 is based on Theorem 5 and the support of the limit measure $P_{\underline{\zeta}}$ in it. We remind that the support of $P_{\underline{\zeta}}$ is a minimal closed set $S_{P_{\underline{\zeta}}}\subset H^{r_1}(D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}})=1$. The set $S_{P_{\underline{\zeta}}}$ consists of all elements $\underline{g}\in H^{r_1}(D)$ such that, for every open neighbourhood G of $\underline{g}$, the inequality $P_{\underline{\zeta}}(G)>0$ is satisfied.

Define

$$S=\bigl\{g\in H(D)\colon\ g(s)\neq 0 \text{ or } g(s)\equiv 0\bigr\}.$$

Theorem 6. *The support of the measure $P_{\underline{\zeta}}$ is the set $\underline{S}=S\times H^r(D)$.*

Proof. We write

$$H^{r_1}(D)=H(D)\times\underbrace{H(D)\times\cdots\times H(D)}_{r}.$$

The space $H(D)$ is separable, therefore, it follows from [22] that

$$\mathcal{B}(H^{r_1}(D))=\mathcal{B}(H(D))\times\underbrace{\mathcal{B}\bigl(H(D)\bigr)\times\cdots\times\mathcal{B}\bigl(H(D)\bigr)}_{r}.$$

Thus, it suffices to consider the measure $P_{\underline{\zeta}}$ on the sets of the form

$$B=A\times A_1\times\cdots\times A_r,\quad A,A_j\in\mathcal{B}\bigl(H(D)\bigr),\quad j=1,\dots,r.$$

Since the measure $\underline{m}_H$ is the product of the measures $\hat{m}_H, m_{1H},\dots,m_{rH}$, the definition of $P_{\underline{\zeta}}$ gives the equality

$$P_{\underline{\zeta}}(B)=\underline{m}_H(A\times A_1\times\cdots\times A_r)=\hat{m}_H(A)m_{1H}(A_1)\cdots m_{rH}(A_r). \qquad (7)$$

In [16], it is proved that the support of the random element $\zeta(s,\hat{\omega})$ is the set S. The algebraic independence of the numbers $\alpha_1,\dots,\alpha_r$ implies their transcendence. Therefore, by [2] the support the random element $L(\lambda_j,\alpha_j,s,\omega_j)$ is the space $H(D)$, $j=1,\dots,r$. On the other hand, the distribution P_ζ of $\zeta(s,\hat{\omega})$ is

$$P_\zeta(A)=\hat{m}_H\bigl(\hat{\omega}\in\hat{\Omega}\colon\ \zeta(s,\hat{\omega})\in A\bigr),\quad A\in\mathcal{B}\bigl(H(D)\bigr),$$

and the distribution P_{L_j} of $L(\lambda_j,\alpha_j,s,\omega_j)$, $j=1,\dots,r$, is

$$P_{L_j}(A_j)=m_{jH}\bigl(\omega_j\in\Omega_j\colon\ L(\lambda_j,\alpha_j,s,\omega_j)\in A_j\bigr),\quad A_j\in\mathcal{B}\bigl(H(D)\bigr).$$

In view of (7),

$$P_{\underline{\zeta}}(B) = P_\zeta(A)P_{L_1}(A_1)\cdots P_{L_r}(A_r).$$

Hence, obviously, $P_{\underline{\zeta}}(\underline{S}) = 1$. Moreover, if $A \in \mathcal{B}(H(D))$ with $A \nsubseteq S$, or $A_j \in \mathcal{B}(H(D))$ with $A_j \nsubseteq H(D)$, for some j, then, in view of the minimality of S and $H(D)$ for $P_\zeta(A)$ and $P_{L_j}(A_j)$, respectively, we have that $P_\zeta(A) < 1$ or $P_{L_j}(A_j) < 1$. Thus, then $P_{\underline{\zeta}}(B) < 1$. Hence, the minimality of $\underline{S}$ follows. □

4 Universality theorem

In this section, we will prove Theorem 4. Its proof is based on Theorems 5 and 6 as well as on the Mergelyan theorem on the approximation of analytic functions by polynomials. We state this theorem as the next lemma.

Lemma 5. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\epsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K}\left|f(s) - p(s)\right| < \epsilon.$$

Proof. The proof of the lemma can be found in [23], see also [24]. □

Proof of Theorem 4. By Lemma 5, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\left|f(s) - p(s)\right| < \frac{\epsilon}{4}. \tag{8}$$

Since $f(s) \neq 0$ on K, $p(s) \neq 0$ on K as well provided ϵ is small enough. Thus, we can define on K a continuous branch of $\log p(s)$ which will be analytic in the interior of K. Applying Lemma 5 once more, we obtain that there exists a polynomial $q(s)$ such that

$$\sup_{s\in K}\left|p(s) - \mathrm{e}^{q(s)}\right| < \frac{\epsilon}{4}.$$

This together with (8) shows that

$$\sup_{s\in K}\left|f(s) - \mathrm{e}^{q(s)}\right| < \frac{\epsilon}{2}. \tag{9}$$

Again, by Lemma 5, there exist polynomials $p_j(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|f_j(s) - p_j(s)\right| < \frac{\epsilon}{2}. \tag{10}$$

Define

$$G = \biggl\{(g, g_1, \dots, g_r) \in H^{r_1}(D)\colon \sup_{s\in K}\left|g(s) - \mathrm{e}^{q(s)}\right| < \frac{\epsilon}{2},$$
$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|g_j(s) - p_j(s)\right| < \frac{\epsilon}{2}\biggr\}.$$

Then G is an open set, and, in view of Theorem 6, $\mathrm{e}^{q(s)}, p_1(s), \dots, p_r(s))$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, an equivalent of the weak convergence of probability measures in terms of open sets, see Theorem 2.1 of [22], together with Theorem 5 and properties of the support give the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0,T]\colon\ \underline{\zeta}(s+\mathrm{i}\tau, \underline{\alpha}, \underline{\lambda}) \in G\bigr\} \geqslant P_{\underline{\zeta}}(G) > 0.$$

Hence, by the definition of G, we find that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\biggl\{\tau \in [0,T]\colon\ \sup_{s\in K}\bigl|\zeta(s+\mathrm{i}\tau) - \mathrm{e}^{q(s)}\bigr| < \frac{\epsilon}{2},\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\bigl|L(\lambda_j,\alpha_j,s+\mathrm{i}\tau) - p_j(s)\bigr| < \frac{\epsilon}{2}\biggr\} > 0. \tag{11}$$

Inequalities (9) and (10) show that

$$\biggl\{\tau \in [0,T]\colon\ \sup_{s\in K}\bigl|\zeta(s+\mathrm{i}\tau) - \mathrm{e}^{q(s)}\bigr| < \frac{\epsilon}{2},\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\bigl|L(\lambda_j,\alpha_j,s+\mathrm{i}\tau) - p_j(s)\bigr| < \frac{\epsilon}{2}\biggr\}$$
$$\subset \biggl\{\tau \in [0,T]\colon\ \sup_{s\in K}\bigl|\zeta(s+\mathrm{i}\tau) - f(s)\bigr| < \epsilon,\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\bigl|L(\lambda_j,\alpha_j,s+\mathrm{i}\tau) - f_j(s)\bigr| < \epsilon\biggr\}.$$

Combining this with (11) gives the assertion of the theorem. □

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References

1. A. Laurinčikas, The universality of the Lerch zeta-function, *Liet. Matem. Rink.*, **37**(3):365–375, 1997 (in Russian); *Lith. Math. J.*, **37**(3):275–280, 1997.

2. A. Laurinčikas, R. Garunkštis, *The Lerch Zeta-Function*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

3. S.M. Gonek, *Analytic Properties of Zeta and L-Functions*, PhD Thesis, University of Michigan, 1979.

4. A. Laurinčikas, On the joint universality of Lerch zeta-functions, *Mat. Zamietki*, **88**(3):428–437, 2010 (in Russian); *Math. Notes*, **88**(3):386–394, 2010.

5. A. Laurinčikas, K. Matsumoto, The joint universality and the functional independendence for Lerch zeta-functions, *Nagoya Math. J.*, **157**:211–227, 2000.

6. A. Laurinčikas, K. Matsumoto, Joint value distribution theorems on Lerch zeta-functions. III, in: A. Laurinčikas et al. (Eds.), *Analytic and Probabilistic Methods in Number Theory. Proceedings of fourth conference in honour of J. Kubilius, Palanga, Lithuania, 2006*, TEV, Vilnius, 2007, pp. 87–98.
7. T. Nakamura, The existence and the non-existence of joint t-universality for Lerch zeta-functions, *J. Number Theory*, **125**(2):424–441, 2007.
8. A. Laurinčikas, The joint universality of Hurwitz zeta-functions, *Šiauliai Math. Semin.*, **3**(11):169–187, 2008.
9. J. Genys, R. Macaitienė, S. Račkauskienė, D. Šiaučiūnas, A mixed joint universality theorem for zeta-functions, *Math. Model. Anal.*, **15**(4):431–446, 2010.
10. H. Mishou, The joint value-distribution of the Riemann zeta function and Hurwitz zeta functions, *Lith. Math. J.*, **47**(1):32–47, 2007.
11. R. Kačinskaitė, A. Laurinčikas, The joint distribution of periodic zeta-functions, *Stud. Sci. Math. Hung.*, **48**(2):257–279, 2011.
12. A. Laurinčikas, Joint universality of zeta-functions with periodic coefficients, *Izv. Ross. Akad. Nauk., Ser. Mat.*, **74**(3):79–102, 2010 (in Russian); *Izv. Math.*, **74**(3):515–539, 2010.
13. H. Mishou, H. Nagoshi, Functional distribution of $L(s, \chi_d)$ with real characters and denseness of quadratic class numbers, *Trans. Am. Math. Soc.*, **358**(10):4343–4366, 2006.
14. K.M. Bitar, N.N. Khuri, H.C. Ren, Path integrals and Voronin's theorem on the universality of the Riemann zeta function, *Ann. Phys.*, **211**:172–196, 1991.
15. B. Bagchi, *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series*, PhD Thesis, Indian Statistical Institute, Calcutta, 1981.
16. A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1996.
17. A. Laurinčikas, The universality of zeta-functions, *Acta Appl. Math.*, **78**(1–3):251–271, 2003.
18. K. Matsumoto, Probabailistic value-distribution theory of zeta-functions, *Sugaku*, **53**:279–296, 2001 (in Japanese); *Sugaku Expo.*, **17**:51–71, 2004.
19. K. Matsumoto, Some problems on mean values and the universality of zeta and multiple zeta-functions, in: A. Dubickas et al. (Eds.), *Analytic and Probabilistic Methods in Number Theory. Proceedings of third conference in honour of J. Kubilius, Palanga, Lithuania, 2001*, TEV, Vilnius, 2002, pp. 195–199.
20. J. Steuding, *Value-Distribution of L-Functions*, Lect. Notes Math., Vol. 1877, Springer-Verlag, Berlin, Heidelberg, 2007.
21. J.B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1973.
22. P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968; Willey-Interscience, 1999 (2nd edition).
23. S. N. Mergelyan, Uniform approximations to functions of complex variable, *Usp. Mat. Nauk*, **7**:31–122, 1952 (in Russian).
24. J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloq. Publ., Am. Math. Soc., Vol. 20, 1960.