

Limit Theorems for Twists of L -Functions of Elliptic Curves. IV

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Abstract. In this paper, we prove a multidimensional limit theorem for moduli of twists of L -functions of elliptic curves. The limit measure in this theorem is defined by the characteristic transforms.

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1 Introduction

In [3], we have obtained a limit theorem for the modulus of twisted with Dirichlet character L -functions of elliptic curves with an increasing modulus of the character. Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$. For each prime number p , denote by E_p the reduction of the curve E modulo p which is a curve over the finite field \mathbb{F}_p , and define $\lambda(p)$ by the equality

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ is the number of points of E_p . The L -function $L_E(s)$, $s = \sigma + it$, of the elliptic curve E is defined, for $\sigma > \frac{3}{2}$, by the product

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

and continues analytically to the whole complex plane.

Let χ be a Dirichlet character modulo q . The twist $L_E(s, \chi)$ with character χ for the function $L_E(s)$ is defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s, \chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}$$

and is analytically continued to an entire function.

Suppose that the modulus q of the character χ is a prime number, denote, as usual, by χ_0 the principal character modulo q , and, for $Q \geq 2$, define

$$M_Q = \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} 1 \quad \text{and} \quad \mu_Q(\dots) = M_Q^{-1} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} 1,$$

where in place of dots we will write a condition satisfied by a pair $(q, \chi(\text{mod } q))$. Moreover, denote by $\mathcal{B}(X)$ the Borel σ -field of the space X . Then, in [3], for $\sigma > \frac{3}{2}$, we have proved a limit theorem on the weak convergence of the measure

$$\mu_Q(|L_E(s, \chi)| \in A), \quad A \in \mathcal{B}(\mathbb{R})$$

as $Q \rightarrow \infty$. The limit measure P in that theorem is defined by its characteristic transforms

$$w_k(\tau) = \int_{\mathbb{R} \setminus \{0\}} |x|^{i\tau} \text{sgn}^k \, dP = \sum_{m=1}^{\infty} \frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R}, \quad k = 0, 1,$$

where $a_\tau(m)$ and $b_\tau(m)$ are certain explicitly given multiplicative functions.

We note that the first theorems of the above type were obtained by P. Elliott [1] and [2], and E. Stankus [5] for Dirichlet L -functions.

The aim of this note is to prove a joint limit theorem for the moduli of the twists of L -functions of elliptic curves. For $j = 1, \dots, r$, let E_j be an elliptic curve given by the Weierstrass equation

$$y^2 = x^3 + a_j x + b_j, \quad a_j, b_j \in \mathbb{Z}$$

with non-zero discriminant $\Delta_j = -16(4a_j^3 + 27b_j^2)$. Similarly as above, we define the quantities $\lambda_j(p)$, and the twist

$$L_{E_j}(s, \chi) = \prod_{p|\Delta_j} \left(1 - \frac{\lambda_j(p)\chi(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta_j} \left(1 - \frac{\lambda_j(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}, \quad \sigma > \frac{3}{2}.$$

This, for $\sigma > \frac{3}{2}$, can be rewritten in the form

$$L_{E_j}(s, \chi) = \prod_{p|\Delta_j} \left(1 - \frac{\lambda_j(p)\chi(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta_j} \left(1 - \frac{\alpha_j(p)\chi(p)}{p^s}\right)^{-1} \times \left(1 - \frac{\beta_j(p)\chi(p)}{p^s}\right)^{-1},$$

where $\alpha_j(p)$ and $\beta_j(p)$ are conjugate complex numbers such that $\alpha_j(p) + \beta_j(p) = \lambda_j(p)$ and $\alpha_j(p)\beta_j(p) = p$. For brevity, let $\eta = \eta(\tau) = \frac{i\tau}{2}$, $\tau \in \mathbb{R}$, and, for primes p and $k \in \mathbb{N}$, we set

$$d_\tau(p^k) = \frac{\eta(\eta + 1) \cdots (\eta + k - 1)}{k!}.$$

For $j = 1, \dots, r$, $p \nmid \Delta_j$ and $k \in \mathbb{N}$, we define

$$a_{j;\tau}(p^k) = \sum_{l=0}^k d_\tau(p^l) \alpha_j^l(p) d_\tau(p^{k-l}) \beta_j^{k-l}(p),$$

$$b_{j;\tau}(p^k) = \sum_{l=0}^k d_\tau(p^l) \bar{\alpha}_j^l(p) d_\tau(p^{k-l}) \bar{\beta}_j^{k-l}(p),$$

where $\bar{\alpha}_j(p)$ and $\bar{\beta}_j(p)$ denote the conjugates of $\alpha_j(p)$ and $\beta_j(p)$, respectively. For $j = 1, \dots, r$, $p \mid \Delta_j$ and $k \in \mathbb{N}$, we define

$$a_{j;\tau}(p^k) = b_{j;\tau}(p^k) = d_\tau(p^k) \lambda_j^k(p).$$

Now, for $j = 1, \dots, r$ and $m \in \mathbb{N}$, define

$$a_{j;\tau}(m) = \prod_{p^l \parallel m} a_{j;\tau}(p^l), \quad b_{j;\tau}(m) = \prod_{p^l \parallel m} b_{j;\tau}(p^l),$$

where $p^l \parallel m$ means that $p^l \mid m$ but $p^{l+1} \nmid m$. Thus, $a_{j;\tau}(m)$ and $b_{j;\tau}(m)$ are multiplicative functions.

Now, on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, we will define a certain probability measure $P^{(r)}$. For this, we will use the characteristic transforms. Let P be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$. Denote by P_j , $j = 1, \dots, r$, P_{j_1, j_2} , $j_2 > j_1 = 1, \dots, r - 1, \dots, P_{1, \dots, j-1, j+1, \dots, r}$, $j = 1, \dots, r$, the one-dimensional, two-dimensional, ..., $(r - 1)$ -dimensional marginal measures of P , i.e.,

$$P_j(A) = P(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{j-1} \times A \times \mathbb{R} \times \cdots \times \mathbb{R}), \quad A \in \mathcal{B}(\mathbb{R}), \quad j = 1, \dots, r,$$

$$P_{j_1, j_2}(A_1 \times A_2) = P(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{j_1-1} \times A_1 \times \mathbb{R} \times \cdots \times \mathbb{R} \times A_2 \times \mathbb{R} \times \cdots \times \mathbb{R}),$$

$$\underbrace{\hspace{10em}}_{j_2-1}$$

$$A_1, A_2 \in \mathcal{B}(\mathbb{R}), \quad j_2 > j_1 = 1, \dots, r - 1,$$

.....

$$P_{1, \dots, j-1, j+1, \dots, r}(A_1 \times \cdots \times A_{j-1} \times A_{j+1} \times \cdots \times A_r)$$

$$= P(A_1 \times \cdots \times A_{j-1} \times \mathbb{R} \times A_{j+1} \times \cdots \times A_r),$$

$$A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_r \in \mathcal{B}(\mathbb{R}), \quad j = 1, \dots, r.$$

Then the functions

$$w_k(\tau) = \int_{\mathbb{R} \setminus \{0\}} |x|^{i\tau} \operatorname{sgn}^k x \, dP_j, \quad \tau \in \mathbb{R}, \quad k = 0, 1, \quad j = 1, \dots, r,$$

$$w_{k_1, k_2}(\tau_1, \tau_2) = \int_{\mathbb{R} \setminus \{0\}} \int_{\mathbb{R} \setminus \{0\}} (|x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1) (|x_2|^{i\tau_2} \operatorname{sgn}^{k_2} x_2) \, dP_{j_1, j_2},$$

$$\tau_1, \tau_2 \in \mathbb{R}, \quad k_1, k_2 = 0, 1, \quad j_2 > j_1 = 1, \dots, r,$$

.....

$$w_{k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r)$$

$$= \int_{\mathbb{R} \setminus \{0\}} \dots \int_{\mathbb{R} \setminus \{0\}} (|x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1) \dots (|x_{j-1}|^{i\tau_{j-1}} \operatorname{sgn}^{k_{j-1}} x_{j-1})$$

$$(|x_{j+1}|^{i\tau_{j+1}} \operatorname{sgn}^{k_{j+1}} x_{j+1}) \dots (|x_r|^{i\tau_r} \operatorname{sgn}^{k_r} x_r) \, dP_{1, \dots, j-1, j+1, \dots, r},$$

$$\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r \in \mathbb{R}, \quad k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r = 0, 1, \quad j = 1, \dots, r,$$

$$w_{k_1, \dots, k_r}(\tau_1, \dots, \tau_r) = \int_{\mathbb{R} \setminus \{0\}} \dots \int_{\mathbb{R} \setminus \{0\}} (|x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1) \dots (|x_r|^{i\tau_r} \operatorname{sgn}^{k_r} x_r) \, dP,$$

$$\tau_1, \dots, \tau_r \in \mathbb{R}, \quad k_1, \dots, k_r = 0, 1$$

are called the characteristic transforms of the measure P . The characteristic transforms of multidimensional distribution functions were introduced in [4]. It is easily seen that the results of [4] remain valid when distribution functions are replaced by probability measures. Thus, we have that the measure P on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ is uniquely determined by its characteristic transforms $\{w_k(\tau), w_{k_1, k_2}(\tau_1, \tau_2), \dots, w_{k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r), w_{k_1, \dots, k_r}(\tau_1, \dots, \tau_r)\}$.

Now let $P^{(r)}$ be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ given by the following characteristic transforms:

$$w_k(\tau) = \sum_{m=1}^{\infty} \frac{a_{j; \tau}(m) b_{j; \tau}(m)}{m^{2\sigma_j}}, \quad j = 1, \dots, r,$$

$$w_{k_1, k_2}(\tau_1, \tau_2) = \sum_{m=1}^{\infty} \sum_{m_1 m_2 = m} \frac{a_{j_1; \tau_1}(m_1) a_{j_2; \tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n_1 n_2 = m} \frac{b_{j_1; \tau_1}(n_1) b_{j_2; \tau_2}(n_2)}{n_1^{s_1} n_2^{s_2}},$$

$$j_2 > j_1 = 1, \dots, r - 1,$$

.....

$$w_{k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r)$$

$$= \sum_{m=1}^{\infty} \sum_{m_1 \dots m_{j-1} m_{j+1} \dots m_r = m} \frac{a_{1; \tau_1}(m_1) \dots a_{j-1; \tau_{j-1}}(m_{j-1}) a_{j+1; \tau_{j+1}}(m_{j+1}) \dots a_{r; \tau_r}(m_r)}{m_1^{s_1} \dots m_{j-1}^{s_{j-1}} m_{j+1}^{s_{j+1}} \dots m_r^{s_r}}$$

$$\times \sum_{n_1 \dots n_{j-1} n_{j+1} \dots n_r = m} \frac{b_{1; \tau_1}(n_1) \dots b_{j-1; \tau_{j-1}}(n_{j-1}) b_{j+1; \tau_{j+1}}(n_{j+1}) \dots b_{r; \tau_r}(n_r)}{n_1^{s_1} \dots n_{j-1}^{s_{j-1}} n_{j+1}^{s_{j+1}} \dots n_r^{s_r}},$$

$$j = 1, \dots, r,$$

uniformly in $|\tau| \leq c$ and $s_j \in R_j$ with arbitrary $c > 0$, as $Q \rightarrow \infty$, $j = 1, \dots, r$. Therefore, it remains to consider the characteristic transforms $w_{j_1, j_2; Q}(\tau_1, \tau_2), \dots, w_{1, \dots, j-1, j+1, \dots, r; Q}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r), w_Q(\tau_1, \dots, \tau_r)$.

In [3], it was obtained that, for $s_j \in R_j$,

$$|L_{E_j}(s_j, \chi)|^{i\tau} = \sum_{m=1}^{\infty} \frac{\hat{a}_{j;\tau}(m)}{m^{s_j}} \sum_{n=1}^{\infty} \frac{\hat{b}_{j;\tau}(m)}{n^{\bar{s}_j}}, \tag{2.3}$$

where $\hat{a}_{j;\tau}(m)$ and $\hat{b}_{j;\tau}(m)$ are multiplicative functions given, for $p \nmid \Delta_j$ and $k \in \mathbb{N}$, by

$$\hat{a}_{j;\tau}(p^k) = \sum_{l=0}^k d_{\tau}(p^l) \alpha_j^l(p) \chi(p^l) d_{\tau}(p^{k-l}) \beta_j^{k-l}(p) \chi(p^{k-l}) \tag{2.4}$$

$$\hat{b}_{j;\tau}(p^k) = \sum_{l=0}^k d_{\tau}(p^l) \bar{\alpha}_j^l(p) \bar{\chi}(p^l) d_{\tau}(p^{k-l}) \bar{\beta}_j^{k-l}(p) \bar{\chi}(p^{k-l}), \tag{2.5}$$

and, for $p \mid \Delta_j$, $k \in \mathbb{N}$, by

$$\hat{a}_{j;\tau}(p^k) = d_k(p^k) \lambda_j^k(p) \chi^k(p), \quad \hat{b}_{j;\tau}(p^k) = d_{\tau}(p^k) \lambda_j^k(p) \bar{\chi}^k(p), \tag{2.6}$$

$j = 1, \dots, r$. Moreover, using the Hasse estimate

$$|\lambda_j(p)| \leq 2\sqrt{p},$$

it was observed in [3] that, for $|\tau_j| \leq c$,

$$|\hat{a}_{j;\tau_j}(m)| \leq m^{\frac{1}{2}} d^{c_1}(m), \quad |\hat{b}_{j;\tau_j}(m)| \leq m^{\frac{1}{2}} d^{c_1}(m), \tag{2.7}$$

where $d(m)$ denotes the divisor function, and c_1 is a positive constant depending only on c .

For simplicity, we will consider only the case $w_{1,2;Q}(\tau_1, \tau_2) \stackrel{def}{=} w_Q(\tau_1, \tau_2)$ and $w_Q(\tau_1, \dots, \tau_r)$. Other characteristic transforms of the measure P_Q are evaluated similarly. From (2.1) and (2.3), we find that, for $s_j \in R_j$,

$$\begin{aligned} w_Q(\tau_1, \tau_2) &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \sum_{m_1=1}^{\infty} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \\ &\quad \times \sum_{n_1=1}^{\infty} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \sum_{m_2=1}^{\infty} \frac{\hat{a}_{2;\tau}(m_2)}{m_2^{s_2}} \sum_{n_2=1}^{\infty} \frac{\hat{b}_{2;\tau}(n_2)}{n_2^{\bar{s}_2}}, \\ &\dots\dots\dots \\ w_Q(\tau_1, \dots, \tau_r) &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \sum_{m_1=1}^{\infty} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \\ &\quad \sum_{n_1=1}^{\infty} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \dots \sum_{m_r=1}^{\infty} \frac{\hat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r=1}^{\infty} \frac{\hat{b}_{r;\tau_r}(n_r)}{n_r^{\bar{s}_r}}, \end{aligned} \tag{2.8}$$

where $\hat{a}_{j;\tau_j}(m)$ and $\hat{b}_{j;\tau_j}(m)$ are multiplicative functions defined by (2.4)–(2.6), and satisfying estimates (2.7), $j = 1, \dots, r$.

3 Asymptotics of Characteristic Transforms of P_Q

In this section, we consider the characteristic transforms of the measure P_Q as $Q \rightarrow \infty$. Let, for brevity, $N = \log Q$. Then the well-known estimate $d(m) = O_\varepsilon(m^\varepsilon)$ with arbitrary $\varepsilon > 0$ together with estimates (2.7) shows that, for $|\tau_j| \leq c$ and $s_j \in R_j$,

$$\sum_{m > N} \frac{\hat{a}_{j;\tau_j}(m)}{m^{s_j}} \ll \sum_{m > N} \frac{m^{\frac{1}{2}d^{c_1}(m_j)}}{m^{\frac{3}{2}+\delta}} \ll_\varepsilon \sum_{m > N} \frac{1}{m^{1+\delta-\varepsilon}} \ll_\varepsilon N^{-\delta+\varepsilon},$$

$$\sum_{n > N} \frac{\hat{b}_{j;\tau_j}(n)}{n^{\bar{s}_j}} = \ll_\varepsilon N^{-\delta+\varepsilon},$$

$j = 1, \dots, r$. Since, in view of (2.7), for $|\tau_j| \leq c$ and $s_j \in R_j$,

$$\sum_{m \leq N} \frac{\hat{a}_{j;\tau_j}(m)}{m^{s_j}} = \mathcal{O}(1) \quad \text{and} \quad \sum_{n \leq N} \frac{\hat{b}_{j;\tau_j}(n)}{n^{\bar{s}_j}} = \mathcal{O}(1),$$

$j = 1, \dots, r$, from this and (2.8) we find that, in the above regions of τ_j and s_j ,

$$w_Q(\tau_1, \tau_2) = \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \left(\sum_{m_1 \leq N} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \right. \\ \times \left. \sum_{n_1 \leq N} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \sum_{m_2 \leq N} \frac{\hat{a}_{2;\tau_2}(m_2)}{m_2^{s_2}} \sum_{n_2 \leq N} \frac{\hat{b}_{2;\tau_2}(n_2)}{n_2^{\bar{s}_2}} \right) + \mathcal{O}_\varepsilon(N^{-\delta+\varepsilon}),$$

.....

$$w_Q(\tau_1, \dots, \tau_r) = \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \left(\sum_{m_1 \leq N} \frac{\hat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \right. \\ \times \left. \sum_{n_1 \leq N} \frac{\hat{b}_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \dots \sum_{m_r \leq N} \frac{\hat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r \leq N} \frac{\hat{b}_{r;\tau_r}(n_r)}{n_r^{\bar{s}_r}} \right) + \mathcal{O}_\varepsilon(N^{-\delta+\varepsilon}). \quad (3.1)$$

The multiplicativity of the functions $\hat{a}_{j;\tau_j}(m)$ and $\hat{b}_{j;\tau_j}(m)$, and the complete multiplicativity of the character χ together with (2.4)–(2.6) and the definition of the multiplicative functions $a_{j;\tau_j}(m)$ and $b_{j;\tau_j}(m)$ show that

$$\hat{a}_{j;\tau_j}(m) = a_{j;\tau_j}(m)\chi(m), \quad \hat{b}_{j;\tau_j}(m) = b_{j;\tau_j}(m)\bar{\chi}(m), \quad (3.2)$$

$j = 1, \dots, r$. Therefore, the main terms on the right-hand sides of (3.1) are of the form

$$\sum_{m_1 \leq N} \frac{a_{1;\tau_1}(m_1)}{m_1^{s_1}} \sum_{n_1 \leq N} \frac{b_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \sum_{m_2 \leq N} \frac{a_{2;\tau_2}(m_2)}{m_2^{s_2}} \sum_{n_2 \leq N} \frac{b_{2;\tau_2}(n_2)}{n_2^{\bar{s}_2}} \\ \times \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \chi(m_1 m_2) \bar{\chi}(n_1 n_2)$$

$$\begin{aligned}
 &= \sum_{m \leq N^2} \sum_{m_1 m_2 = m} \frac{a_{1;\tau_1}(m_1) a_{2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n \leq N^2} \sum_{n_1 n_2 = n} \frac{b_{1;\tau_1}(n_1) b_{2;\tau_2}(n_2)}{n_1^{\bar{s}_1} n_2^{\bar{s}_2}} \\
 &\quad \times \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \chi(m) \bar{\chi}(n), \\
 &\dots\dots\dots \\
 &\sum_{m_1 \leq N} \frac{a_{1;\tau_1}(m_1)}{m_1^{s_1}} \sum_{n_1 \leq N} \frac{b_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \dots \sum_{m_r \leq N} \frac{a_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r \leq N} \frac{b_{r;\tau_r}(n_r)}{n_r^{\bar{s}_r}} \\
 &\quad \times \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \chi(m_1 \dots m_r) \bar{\chi}(n_1 \dots n_r) \\
 &= \sum_{m \leq N^r} \sum_{m_1 \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_r^{s_r}} \sum_{n \leq N^r} \sum_{n_1 \dots n_r = n} \frac{b_{1;\tau_1}(n_1) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_r^{\bar{s}_r}} \\
 &\quad \times \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \chi(m) \bar{\chi}(n). \tag{3.3}
 \end{aligned}$$

Consider two cases $m = n$ and $m \neq n$. If $m = n$, then

$$\begin{aligned}
 \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} \chi(m) \bar{\chi}(n) &= \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} |\chi(m)|^2 = M_Q - \sum_{\substack{q | m \\ q \leq N^j}} (q - 2) \\
 &= M_Q + \mathcal{O}\left(\sum_{q \leq N^j} q\right) = M_Q + \mathcal{O}(N^{2j}), \tag{3.4}
 \end{aligned}$$

$j = 2, \dots, r$. Moreover, in view of the estimate $\sum_{d_1 \dots d_j = m} 1 = \mathcal{O}_\varepsilon(m^\varepsilon)$, $j = 2, \dots, r$, taking into account (2.7), (3.2), we deduce that, for $|\tau_j| \leq c$ and $s_j \in R_j$,

$$\begin{aligned}
 &\sum_{m_1 \dots m_j = m} \frac{a_{1;\tau_1}(m_1) \dots a_{j;\tau_j}(m_j)}{m_1^{s_1} \dots m_j^{s_j}} \sum_{n_1 \dots n_j = n} \frac{b_{1;\tau_1}(n_1) \dots b_{j;\tau_j}(n_j)}{n_1^{\bar{s}_1} \dots n_j^{\bar{s}_j}} \\
 &= \mathcal{O}\left(\sum_{m_1 \dots m_j = m} \frac{d^{c_1}(m_1) \dots d^{c_1}(m_j)}{m^{1+\delta}} \sum_{n_1 \dots n_j = m} \frac{d^{c_1}(n_1) \dots d^{c_1}(n_j)}{n^{1+\delta}}\right) = \mathcal{O}_\varepsilon\left(\frac{1}{m^{2+2\delta-\varepsilon}}\right).
 \end{aligned}$$

Since [1] $M_Q = \frac{Q^2}{2 \log Q} + \mathcal{O}\left(\frac{Q^2}{\log^2 Q}\right)$, the latter estimate and (3.4) show that the case $m = n$ contributes to (3.1)

$$\begin{aligned}
 &\sum_{m=1}^\infty \sum_{m_1 m_2 = m} \frac{a_{1;\tau_1}(m_1) a_{2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n_1 n_2 = m} \frac{b_{1;\tau_1}(n_1) b_{2;\tau_2}(n_2)}{n_1^{\bar{s}_1} n_2^{\bar{s}_2}} + o(1), \\
 &\dots\dots\dots \\
 &\sum_{m=1}^\infty \sum_{m_1 \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_r^{s_r}} \sum_{n_1 \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_r^{\bar{s}_r}} + o(1), \tag{3.5}
 \end{aligned}$$

uniformly in $|\tau_j| \leq c$ and $s_j \in R_j, j = 1, \dots, r$, as $Q \rightarrow \infty$.

Now suppose that $m \neq n$. Then, for $m, n \leq N^j, j = 2, \dots, r$, similarly, as in [3], we obtain that

$$\sum_{q \leq Q} \sum_{\substack{\chi = \chi(\pmod q) \\ \chi \neq \chi_0}} \chi(m) \bar{\chi}(n) = \mathcal{O}\left(\frac{Q}{\log Q}\right).$$

This, (3.1), (3.3) and (3.5) show that, uniformly in $|\tau_j| \leq c$ and $s_j \in R_j, j = 1, \dots, r$,

$$\begin{aligned} w_Q(\tau_1, \tau_2) &= \sum_{m=1}^{\infty} \sum_{m_1 m_2 = m} \frac{a_{1;\tau_1}(m_1) a_{2;\tau_2}(m_2)}{m_1^{s_1} m_2^{s_2}} \sum_{n_1 n_2 = m} \frac{b_{1;\tau_1}(n_1) b_{2;\tau_2}(n_2)}{n_1^{\bar{s}_1} n_2^{\bar{s}_2}} + o(1), \\ &\dots\dots\dots \\ w_Q(\tau_1, \dots, \tau_r) &= \sum_{m=1}^{\infty} \sum_{m_1 \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_r^{s_r}} \\ &\quad \times \sum_{n_1 \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_r^{\bar{s}_r}} + o(1) \end{aligned} \tag{3.6}$$

as $Q \rightarrow \infty$.

4 Proof of Theorem 1

By (2.2) and (3.6), we have that the characteristic transforms of the measure P_Q converge uniformly in $|\tau_j| \leq c$ and $s_j \in R_j, j = 1, \dots, r$, to the functions defined by formulae (1.1) as $Q \rightarrow \infty$. The uniform convergence ensure the continuity of the functions (1.1). Thus, the application of a continuity theorem for characteristic transforms of the measures on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, Theorem 3 of [4], completes the proof of Theorem 1.

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