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Ultimate Time Survival Probability in Three-Risk Discrete Time Risk Model

Andrius Grigutis † and Jonas Šiaulys *,†

Institute of Mathematics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania; andrius.grigutis@mif.vu.lt

* Correspondence: jonas.siaulys@mif.vu.lt

† Both authors contributed equally to this work.

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Abstract: In this paper, we prove recursive formulas for ultimate time survival probability when three random claims X, Y, Z in the discrete time risk model occur in a special way. Namely, we suppose that claim X occurs at each moment of time $t \in \{1, 2, \ldots\}$, claim Y additionally occurs at even moments of time $t \in \{2, 4, \ldots\}$ and claim Z additionally occurs at every moment of time, which is a multiple of three $t \in \{3, 6, \ldots\}$. Under such assumptions, the model that is obtained is called the three-risk discrete time model. Such a model is a particular case of a nonhomogeneous risk renewal model. The sequence of claims has the form $\{X, X+Y, X+Z, X+Y, X, X+Y+Z, \ldots\}$. Using the recursive formulas, algorithms were developed to calculate the exact values of survival probabilities for the three-risk discrete time model. The running of algorithms is illustrated via numerical examples.

Keywords: multi-risk model; discrete-time risk model; ruin probability; survival probability; ultimate time; net profit condition

MSC: 91B30; 60G40

1. Introduction

According to the Andersen's proposed risk model [1], the insurers surplus process W has a following expression

$$W(t) = u + ct - \sum_{i=1}^{\Theta(t)} R_i,\ t \geqslant 0. \tag{1}$$

Here:

- $u \geqslant 0$ denotes the initial insurer's surplus;
- $c > 0$ denotes the premium rate per unit of time;
- The cost of claims $R_1, R_2, \ldots$ are independent copies of a nonnegative random variable (r.v.) R;
- The inter-occurrence times of claims $\{\theta_1, \theta_2, \ldots\}$ are another sequence of independent copies of a nonnegative r.v. θ, which is not degenerate at zero;
- The sequences $\{R_1, R_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent;
- $\Theta(t) = \#\{n \geqslant 1 : T_n \in [0, t]\}$ is the renewal process generated by r.v. θ, where for each natural n $T_n = \theta_1 + \theta_2 + \ldots + \theta_n$.

Let us denote $j|i$ if number j divides number i, and let us suppose $c = 1$, $\theta \equiv 1$ and

$$R_i = X_i + Y_i \mathbb{1}_{\{2|i\}} + Z_i \mathbb{1}_{\{3|i\}}, i \in \mathbb{N}, \tag{2}$$

where $\{X_i,\, Y_i,\, Z_i\}$ correspondingly are independent copies of integer valued nonnegative independent r.v.s $\{X,\, Y,\, Z\}$. Under such restrictions the main equation of the Andersen's risk model (1) becomes the following equation

$$W(t) = u + t - \sum_{i=1}^{\lfloor t \rfloor} R_i = u + t - \sum_{i=1}^{\lfloor t \rfloor} X_i - \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j - \sum_{k=1}^{\lfloor t/3 \rfloor} Z_k, \tag{3}$$

where $\lfloor . \rfloor$ denotes the floor function. Since r.v.s X, Y, Z are discrete, for the defined model it is enough to consider $u \in \mathbb{N}_0 := \{0, 1, \ldots\}$ and $t \in \mathbb{N}$. The model given in (3), we call the three-risk discrete time model, and our motivation to investigate it is research is done in [2–5], where discrete time risk models were investigated with the following occurrence order of claims $\{X,\, Y,\, X,\, Y, \ldots\}$, $\{X,\, Y,\, Z,\, X,\, Y,\, Z, \ldots\}$ and $\{X, X+Y, X, X+Y, \ldots\}$.

If we suppose that $Y_i \equiv Z_i \equiv 0$ for all i, then the three-risk discrete time model becomes the discrete time risk model generated by a nonnegative r.v. X. Such a model is simple to consider. A lot of various results on the behaviour of the discrete time risk model can be found; for instance, in [6–16] and in references therein.

If in the main equation of the Andersen's model (1), we suppose that $c = 1$, $\theta \equiv 1$ and all r.v.s $Z_1, Z_2, \ldots$ are integer values but not necessarily identically distributed, then we obtain so called nonhomogeneous discrete time risk model. Such a model is more general with respect to the model described by Equation (3). There are a number of results on the calculation of finite time ruin probability, the probability that $W(t) \leqslant 0$ for some $t \in \{1, .., T\}$, for models with non identically distributed claims. See, for instance, [17–26] and reference therein. However, there are no results on the calculation of the ultimate time ruin probability, the probability that $W(t) \leqslant 0$ for some $t \in \mathbb{N}$, and of the ultimate time survival probability, the probability that $W(t) > 0$ for all $t \in \mathbb{N}$, for such a general, nonhomogenous, discrete time risk model. To get suitable algorithms for the calculations of these ultimate time probabilities, we are forced to limit the set of the discrete time risk models significantly.

In this paper, we consider the three-risk model described by Equation (3), which requires much more effort to express its ultimate time survival probability compared to the previous research in [2–4]. It also gives us a grip on the most general case for ultimate time, when n claims generate a *multi-risk discrete time model* instead of two or three claims. If $n = 1$, we get the homogenous discrete time risk model. For such model, a basic knowledge from probability theory is sufficient to find a suitable algorithm for ruin or survival probabilities calculation. With increasing n there decreases a "level of models homogeneity," meaning that models move closer to the real life, but then, as a side effect, various additional elements occur: dependence on distribution, number of initial values needed for recursive formula, non-singularity of certain recursive matrices and so on.

The main critical characteristics for the three-risk discrete time model are the *time of ruin* τ_u:

$$\tau_u = \begin{cases} \inf\{t \in \mathbb{N} : W(t) \leqslant 0\}, \\ \infty, \text{ if } \; W(t) > 0 \; \text{ for all } \; t \in \mathbb{N}; \end{cases}$$

the *finite time ruin probability* $\psi(u, T) := \mathbb{P}(\tau_u \leqslant T)$, $T \in \mathbb{N}$, the *ultimate time ruin probability* $\psi(u) := \mathbb{P}(\tau_u < \infty)$; and the *ultimate time survival probability* $\varphi(u) := \mathbb{P}(\tau_u = \infty) = 1 - \psi(u)$. It is obvious that

$$\psi(u, T) = \mathbb{P}\left(\bigcup_{t=1}^{T} \{\, W(t) \leqslant 0 \,\}\right), \;\; \psi(u) = \mathbb{P}\left(\bigcup_{t=1}^{\infty} \{\, W(t) \leqslant 0 \,\}\right) \;\text{ and }\; \varphi(u) = \mathbb{P}\left(\bigcap_{t=1}^{\infty} \{\, W(t) > 0 \,\}\right).$$

Let us denote the probability distribution function (PDF) and the cumulative distribution function (CDF) for a certain convolution of r.v.s:

$$\mathbb{P}(X_1+\ldots+X_i+Y_1+\ldots Y_j+Z_1+\ldots+Z_k=m)=:a^i*b^j*c^k(m),$$
$$\mathbb{P}(X_1+\ldots+X_i+Y_1+\ldots Y_j+Z_1+\ldots+Z_k\leqslant m)=:A^i*B^j*C^k(m),$$

where $i=\overline{0,6}$, $j=\overline{0,3}$, $k=\overline{0,2}$ and $m\in\mathbb{N}_0$.

We note that

- If any of i,j,k equals zero, then we do not include a corresponding r.v. (r.v.s) into a certain convolution;
- We use notations $a^i*b^j*c^k(m)=a^i*b^j*c^k_m$ and $A^i*B^j*C^k(m)=A^i*B^j*C^k_m$ interchangeably and $\overline{a^i*b^j*c^k}(m)=1-a^i*b^j*c^k(m)$, $\overline{A^i*B^j*C^k}(m)=1-A^i*B^j*C^k(m)$.

For the three-risk discrete time model, the values of the ultimate and finite time ruin probabilities are closely related with the following condition. It is said that the *net profit condition holds* if

$$\mathbb{E}X+\frac{\mathbb{E}Y}{2}+\frac{\mathbb{E}Z}{3}<1. \tag{4}$$

This condition makes a direct impact for the risk model (3). Intuitively, this impact can be explained as follows. Let us take the expectation of (3) according to the probability space of r.v.s R_i, $i\in\mathbb{N}$. For time moments $t=6N$ and $t=6N+1$ with arbitrary $N\in\mathbb{N}_0$, we obtain

$$\mathbb{E}W(6N)=u+6N\left(1-\mathbb{E}X-\frac{\mathbb{E}Y}{2}-\frac{\mathbb{E}Z}{3}\right),$$
$$\mathbb{E}W(6N+1)=u+6N\left(1-\mathbb{E}X-\frac{\mathbb{E}Y}{2}-\frac{\mathbb{E}Z}{3}\right)+1-\mathbb{E}X.$$

The similar expressions hold up to time moment $t=6N+5$. Now we can see how the net profit condition influences the sign of $\mathbb{E}W(t)$ and possibility that $W(t)\leqslant 0$ at least once in time.

Another intuitive derivation of the net profit condition is the following. Suppose that index i is uniformly distributed. Under such assumption we get from the law of total expectation that

$$\mathbb{E}R_i=\mathbb{E}\left(\mathbb{E}(R_i\,|\,i)\right)=\mathbb{E}\left(\mathbb{E}\left(X_i+Y_i\mathbb{1}_{\{2|i\}}+Z_i\mathbb{1}_{\{3|i\}}\,|\,i\right)\right)=\mathbb{E}X+\frac{\mathbb{E}Y}{2}+\frac{\mathbb{E}Z}{3},$$

where $\mathbb{E}(R_i\,|\,i)$ denotes the conditional expectation. Then we obtain the net profit condition by comparing this expression with the premium rate $c=1$ per unit of time.

The precise impact of the net profit condition is described in theorems of Section 2 below. For instance, unsatisfied condition (4) causes almost unavoidable ruin according to Theorem 4 below. The simplest case to make the net profit condition unsatisfied is $\mathbb{P}(X=0)=a(0)=0$. Indeed, if $\mathbb{P}(X=0)=0$ for some integer valued nonnegative r.v. X, then $\mathbb{P}(X\geqslant 1)=1$, which implies $\mathbb{E}X\geqslant 1$ and $\mathbb{E}X+\mathbb{E}Y/2+\mathbb{E}Z/3\geqslant 1$. Of course, $a(0)=0$ is not only the reason causing the net profit condition to be unsatisfied.

Let us denote $I_n^m:=\bigcap_{t=n}^m\{W(t)>0\}$. For the ultimate time survival probability we have that

$$1-\psi(u)=\varphi(u)=\mathbb{P}\left(I_1^\infty\right)=\mathbb{P}\left(I_1^6,I_7^\infty\right)=\mathbb{P}\left(I_7^\infty|I_1^6\right)\mathbb{P}\left(I_1^6\right).$$

In our three-risk model defined by (3) $R_i\overset{d}{=}R_{i+6}$ for every $i=1,2,\ldots$ Consequently,

$$\mathbb{P}\left(I_7^\infty\right)=\mathbb{P}\left(\bigcap_{t=7}^\infty\left\{u+t-\sum_{i=1}^6R_i-\sum_{i=7}^tR_i>0\right\}\right)=\mathbb{P}\left(\bigcap_{t=1}^\infty\left\{u+6+t-\sum_{i=1}^6R_i-\sum_{i=1}^tR_i>0\right\}\right),$$

and

$$\begin{aligned}\varphi(u) &= \mathbb{P}\left(I_7^{\infty}|I_1^6\right)\mathbb{P}\left(I_1^6\right)\\ &= \sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1\\ i_3\leqslant u+2-i_1-i_2\\ i_4\leqslant u+3-i_1-i_2-i_3\\ i_5\leqslant u+4-i_1-i_2-i_3-i_4\\ i_6\leqslant u+5-i_1-i_2-i_3-i_4-i_5}} a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{i_4}\cdot a_{i_5}\cdot a*b*c_{i_6}\,\varphi(u+6-i_1-\ldots-i_6). \end{aligned} \tag{5}$$

Formula (5) shows how the values of probability we seek to calculate are recursively related. For example, to get a value of $\varphi(u+6)$ for $u=0,1,\ldots$ we need to know $\varphi(0),\varphi(1),\ldots,\varphi(u+5)$. Supplementing and subtracting terms of the previous sum, we rewrite the Equation (5) in the following way.

$$\begin{aligned}\varphi(u) =& \sum_{k=0}^{u+5} a^6*b^3*c^2(u+5-k)\varphi(k+1)\\ &-\sum_{k=0}^{4}\varphi(k+1)\sum_{j=1}^{5-k} a_{u+j}\cdot a^5*b^3*c^2(5-k-j)\\ &-\sum_{k=0}^{3}\varphi(k+1)\sum_{j=2}^{5-k}\sum_{i_1\leqslant u} a_{i_1}\cdot a*b_{u+j-i_1}\cdot a^4*b^2*c^2(5-k-j)\\ &-\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1}} a_{i_1}\cdot a*b_{i_2}\cdot a*c_{u+j-i_1-i_2}\cdot a^3*b^2*c(5-k-j)\\ &-\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1\\ i_3\leqslant u+2-i_1-i_2}} a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{u+j-i_1-i_2-i_3}\cdot a^2*b*c(5-k-j)\\ &-\varphi(1)a*b*c(0)\sum_{\substack{i_1\leqslant u\\ i_2\leqslant u+1-i_1\\ i_3\leqslant u+2-i_1-i_2\\ i_4\leqslant u+3-i_1-i_2-i_3}} a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{i_4}\cdot a_{u+5-i_1-i_2-i_3-i_4}\\ =:& \sum_{k=0}^{u+5} a^6*b^3*c^2(u+5-k)\varphi(k+1)-\sum_{k=1}^{5}S_k(u), \end{aligned} \tag{6}$$

where $S_1(u),\ldots,S_5(u)$ denote the previously written sums by the same order of occurrence. On the other hand, collecting the terms at $\varphi(1),\ldots,\varphi(5)$ from $S_1(u),\ldots,S_5(u)$ in (6), we derive that

$$\varphi(u)=\sum_{k=0}^{u+5} a^6*b^3*c^2(u+5-k)\varphi(k+1)-\sum_{k=1}^{5} z_k(u)\varphi(k), \tag{7}$$

where

$$\begin{aligned} z_k(u) =& \sum_{i=1}^{6-k}\sum_{j=i}^{6-k}\sum_{\bigcap_{l=1}^{i} I_{l,i}} a_{k_1}^{\lfloor (4+i)/5\rfloor}(a*b_{k_2})^{\lfloor (3+i)/5\rfloor}(a*c_{k_3})^{\lfloor (2+i)/5\rfloor}(a*b_{k_4})^{\lfloor (1+i)/5\rfloor}a_{k_5}^{\lfloor i/5\rfloor}\times\\ &\times a^{6-i}*b^{3-\lfloor i/2\rfloor}*c^{2-\lfloor i/3\rfloor}(6-j-k), \end{aligned} \tag{8}$$

and $I_{l,i}$ is the following summation region

$$I_{l,i}=\left\{k_l;0;u+l-1-\sum_{m=1}^{l-1}k_m\right\}\mathbb{1}_{\{5|4+i-l\}}\bigcup\left\{k_l;u+j-\sum_{m=1}^{l-1}k_m;u+j-\sum_{m=1}^{l-1}k_m\right\}\mathbb{1}_{\{5|5+l-i\}}.$$

Here, we suppose that $\sum_{m=1}^{0} k_m =: 0$ and the corresponding structure of the summation region is the following $\{\text{summation variable; summation starts; summation ends}\}$.

Our future work, more or less spins around the Equation (7), and the part $\sum_{k=1}^{5} z_k(u)\varphi(k)$, being dependent on $b(0)$ and $c(0)$, dictates the complexity of our statements. This, keeping in mind that $a(0) = 0$ leads to the unsatisfied net profit condition, can be briefly described in Table 1.

Table 1. All possible different cases of three-risk models under the net profit condition.

Statement	$\boldsymbol{a(0)}$	$\boldsymbol{b(0)}$	$\boldsymbol{c(0)}$	***Comments***
Theorem 2	$\neq 0$	$= 0$	$= 0$	The most simple case
Theorem 3	$\neq 0$	$\neq 0$	$\neq 0$	Initial values
Corollary 1 from Theorem 3	$\neq 0$	$\neq 0$	$\neq 0$	The most general case
Corollaries 2 and 3 from Theorem 3	$\neq 0$	$\neq 0$	$= 0$	Middle case
Corollary 4 from Theorem 3	$\neq 0$	$= 0$	$\neq 0$	Middle case

2. Main Results

We start with a statements on $\varphi(u)$, when the net profit condition (4) holds.

Theorem 1. *Let us consider the three-risk discrete time model with generating r.v.s X, Y and Z. If* $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 < 1$, *then*

$$\lim_{u\to\infty} \varphi(u) = 1. \tag{9}$$

Theorem 2. *Let us consider the three-risk discrete time model with generating r.v.s X, Y and Z. If* $\{a(0) \neq 0, b(0) = 0, c(0) = 0\}$ *and the net profit condition (4) holds, then*

$$\begin{aligned}
\varphi(0) &= 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z, \\
\varphi(1) &= \frac{\varphi(0)}{a^6 * b^3 * c^2(5)}, \\
\varphi(u) &= \frac{1}{a^6 * b^3 * c^2(5)} \left(\varphi(u-1) - \sum_{k=1}^{u-1} a^6 * b^3 * c^2(u+5-k)\varphi(k) \right), u \geqslant 2.
\end{aligned}$$

Remark 1. *Restrictions in Theorem 2 ensure that* $5 \leqslant 6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z < 6$. *The upper bound follows straightforward from the definition of the net profit condition, and the lower bound follows from the observation that* $X \geqslant 0$, $Y \geqslant 1$, *and* $Z \geqslant 1$.

Remark 2. *Conditions of Theorem 2 also ensure that* $a^6 * b^3 * c^2(5) > 0$. *This follows from the observation that* $\min Y = 2$ *or* $\min Z = 3$ *leads to the unsatisfied net profit condition. If* $\min Y = 1$ *and* $\min Z = 2$, *then the net profit condition is ruined too. Therefore, probabilities* $\mathbb{P}(Y = 1)$ *and* $\mathbb{P}(Z = 1)$ *are positive, and consequently, we have that* $a^6 * b^3 * c^2(5) = a_0^6 \cdot b_1^3 \cdot c_1^2 > 0$.

The next theorem provides an algorithm of finding initial values of $\varphi(0), \ldots, \varphi(4)$ for Formula (7), when $\{a(0) \neq 0, b(0) \neq 0, c(0) \neq 0\}$. But before we can start formulating the statement, unavoidably we need some tedious notation. Let us define six recurrent sequences β_n^0, β_n^1, β_n^2, β_n^3, β_n^4 and γ_n. For $n \in \{0, \ldots, 5\}$ we present values of these sequences in Table 2,

Table 2. Initial values of sequences β_n^0, β_n^1, β_n^2, β_n^3, β_n^4 and γ_n.

n	β_n^0	β_n^1	β_n^2	β_n^3	β_n^4	γ_n
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	0	1	0	0	0
3	0	0	0	1	0	0
4	0	0	0	0	1	0
5	$-\frac{1}{a^5*b^3*c^2(0)}$	$-\frac{\hat{z}_1}{a^5*b^3*c^2(0)}$	$-\frac{\hat{z}_2}{a^5*b^3*c^2(0)}$	$-\frac{\hat{z}_3}{a^5*b^3*c^2(0)}$	$-\frac{\hat{z}_4}{a^5*b^3*c^2(0)}$	$\frac{1}{a^5*b^3*c^2(0)}$

and for $n = 6, 7, \ldots$ we define that

$$\beta_n^0 = \frac{1}{a^6 * b^3 * c^2(0)} \left(\beta_{n-6}^0 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k) \beta_{n-k}^0 - a(n-5) \right),$$
$$\beta_n^1 = \frac{1}{a^6 * b^3 * c^2(0)} \left(\beta_{n-6}^1 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k) \beta_{n-k}^1 + z_1(n-6) - a(n-5)\hat{z}_1 \right),$$
$$\beta_n^2 = \frac{1}{a^6 * b^3 * c^2(0)} \left(\beta_{n-6}^2 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k) \beta_{n-k}^2 + z_2(n-6) - a(n-5)\hat{z}_2 \right),$$
$$\beta_n^3 = \frac{1}{a^6 * b^3 * c^2(0)} \left(\beta_{n-6}^3 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k) \beta_{n-k}^3 + z_3(n-6) - a(n-5)\hat{z}_3 \right),$$
$$\beta_n^4 = \frac{1}{a^6 * b^3 * c^2(0)} \left(\beta_{n-6}^4 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k) \beta_{n-k}^4 + z_4(n-6) - a(n-5)\hat{z}_4 \right),$$
$$\gamma_n = \frac{1}{a^6 * b^3 * c^2(0)} \left(\gamma_{n-6} - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k) \gamma_{n-k} + a(n-5) \right),$$

where coefficients $z_1(n-6)$, $z_2(n-6)$, $z_3(n-6)$, $z_4(n-6)$ are defined in (8), $a(n-5) = z_5(n-6)/\hat{z}_5$ according to the proof of Theorem 2, while coefficients $\hat{z}_5, \ldots, \hat{z}_1$ have the following expressions:

$$\hat{z}_5 = a^5 * b^3 * c^2(0), \tag{10}$$
$$\hat{z}_4 = A^6 * B^3 * C^2(1) + \overline{A}(0) \cdot a^5 * b^3 * c^2(1) + \overline{A}(1) \cdot a^5 * b^3 * c^2(0) + \overline{A*B}(1) \cdot a^4 * b^2 * c^2(0), \tag{11}$$
$$\begin{aligned} \hat{z}_3 = {} & A^6 * B^3 * C^2(2) + \sum_{i_1=1}^{3} \overline{A}(i_1 - 1) \cdot a^5 * b^3 * c^2(3 - i_1) + \sum_{i_2=2}^{3} \overline{A*B}(i_2 - 1) \cdot a^4 * b^2 * c^2(3 - i_2) \\ & + a^3 * b^2 * c(0) \cdot \left(\overline{A*C}(2) + \overline{A*B}(0) \cdot a * c(2) \right), \end{aligned} \tag{12}$$
$$\begin{aligned} \hat{z}_2 = {} & A^6 * B^3 * C^2(3) + \sum_{i_1=1}^{4} \overline{A}(i_1 - 1) \cdot a^5 * b^3 * c^2(4 - i_1) + \sum_{i_2=2}^{4} \overline{A*B}(i_2 - 1) \cdot a^4 * b^2 * c^2(4 - i_2) \\ & + \sum_{i_3=3}^{4} a^3 * b^2 * c(4 - i_3) \left(\overline{A*C}(i_3 - 1) + \overline{A*B}(0) \cdot a * c(i_3 - 1) \right) \\ & + a^4 * b^2 * c^2(0) \cdot \overline{A*B}(3) + a^3 * b^2 * c(0) \left(a * c(1) \cdot \overline{A*B}(2) + \overline{A*B}(1) \cdot \overline{A*C}(1) \right) \\ & + a^2 * b * c(0) \cdot \overline{A*B}(0) \left(a * c(0) \cdot \overline{A*B}(2) + \overline{A*C}(0) \cdot \overline{A*B}(1) \right), \end{aligned} \tag{13}$$

$$\begin{aligned}
\hat{z}_1 &= A^6 * B^3 * C^2(4) + \sum_{i_1=1}^{5} \overline{A}(i_1-1) \cdot a^5 * b^3 * c^2(5-i_1) + \sum_{i_2=2}^{5} \overline{A*B}(i_2-1) \cdot a^4 * b^2 * c^2(5-i_2) \\
&+ \sum_{i_3=3}^{5} a^3 * b^2 * c(5-i_3)(\overline{A*C}(i_3-1) + \overline{A*B}(0) \cdot a * c(i_3-1)) \\
&+ a^2 * b * c(0) \sum_{i_4=4}^{5} \overline{A*B}(i_4-1) \cdot a^2 * b * c(5-i_4) \\
&+ a * b(0) \cdot a * c(1) \sum_{i_4=4}^{5} \overline{A*B}(i_4-2) \cdot a^2 * b * c(5-i_4) \\
&+ a * b(0) \overline{A*C}(1) \sum_{i_4=4}^{5} \overline{A*B}(i_4-3) \cdot a^2 * b * c(5-i_4) \\
&+ \overline{A*B}(0) * \overline{A*C}(0) \sum_{i_4=4}^{5} \overline{A*B}(i_4-3) \cdot a^2 * b * c(5-i_4) \\
&+ a * c(0) \cdot \overline{A*B}(0) \sum_{i_4=4}^{5} \overline{A*B}(i_4-2) \cdot a^2 * b * c(5-i_4) \\
&+ a * b * c(0) \sum_{k=0}^{2} \overline{A}(4-k) \cdot a^4 * b^2 * c(k) + a * b * c(0) \cdot \overline{A^4 * B^2 * C}(2) \cdot \overline{A}(1) \\
&- a^4 * b^3 * c(0) \cdot a_2 \cdot \overline{A*C}(2) - a^2 * b^2 * c(0) \cdot a_2 \cdot \overline{A^2 * B}(0) \cdot \overline{A*C}(1) \\
&+ a^3 * b^3 * c(0) \cdot a_2 \cdot a * c_2 \cdot \overline{A}(0) - a * b * c(0) \sum_{i_2=0}^{1} \sum_{k=0}^{1-i_2} a^2 * b * c(k) \cdot a_{3-i_2-k} \cdot \overline{A*B}(i_2+1) \\
&- a * b * c(0) \sum_{i_1=0}^{2} \sum_{k=0}^{2-i_1} a^3 * b^2 * c(k) \cdot a_{4-i_1-k} \cdot \overline{A}(i_1).
\end{aligned} \tag{14}$$

In (27) and (28) bellow, we will show that

$$\varphi(0) + \hat{z}_1\varphi(1) + \hat{z}_2\varphi(2) + \hat{z}_3\varphi(3) + \hat{z}_4\varphi(4) + \hat{z}_5\varphi(5) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z, \tag{15}$$

and

$$\varphi(n) = \beta_n^0\varphi(0) + \beta_n^1\varphi(1) + \beta_n^2\varphi(2) + \beta_n^3\varphi(3) + \beta_n^4\varphi(4) + \gamma_n(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z)$$

for $n \in \mathbb{N}_0$. That will be the starting point to gather the initial values for Formula (7) via the following theorem.

Theorem 3. *Let us consider the three-risk discrete time model with generating r.v.s X, Y and Z. If $\{a(0) \neq 0,$ $b(0) \neq 0, c(0) \neq 0\}$ and the net profit condition (4) holds, then the following matrix equation holds for each $n \in \mathbb{N}_0$*

$$\begin{pmatrix}
\beta_{n+1}^0 - \beta_n^0 & \beta_{n+1}^1 - \beta_n^1 & \beta_{n+1}^2 - \beta_n^2 & \beta_{n+1}^3 - \beta_n^3 & \beta_{n+1}^4 - \beta_n^4 \\
\beta_{n+2}^0 - \beta_n^0 & \beta_{n+2}^1 - \beta_n^1 & \beta_{n+2}^2 - \beta_n^2 & \beta_{n+2}^3 - \beta_n^3 & \beta_{n+2}^4 - \beta_n^4 \\
\beta_{n+3}^0 - \beta_n^0 & \beta_{n+3}^1 - \beta_n^1 & \beta_{n+3}^2 - \beta_n^2 & \beta_{n+3}^3 - \beta_n^3 & \beta_{n+3}^4 - \beta_n^4 \\
\beta_{n+4}^0 - \beta_n^0 & \beta_{n+4}^1 - \beta_n^1 & \beta_{n+4}^2 - \beta_n^2 & \beta_{n+4}^3 - \beta_n^3 & \beta_{n+4}^4 - \beta_n^4 \\
\beta_{n+5}^0 - \beta_n^0 & \beta_{n+5}^1 - \beta_n^1 & \beta_{n+5}^2 - \beta_n^2 & \beta_{n+5}^3 - \beta_n^3 & \beta_{n+5}^4 - \beta_n^4
\end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \end{pmatrix}$$

$$+\begin{pmatrix}\gamma_{n+1}-\gamma_n\\ \gamma_{n+2}-\gamma_n\\ \gamma_{n+3}-\gamma_n\\ \gamma_{n+4}-\gamma_n\\ \gamma_{n+5}-\gamma_n\end{pmatrix}\times(6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z)=\begin{pmatrix}\varphi(n+1)-\varphi(n)\\ \varphi(n+2)-\varphi(n)\\ \varphi(n+3)-\varphi(n)\\ \varphi(n+4)-\varphi(n)\\ \varphi(n+5)-\varphi(n)\end{pmatrix}. \quad (16)$$

Remark 3. *From the definition of survival probability, it is clear that the sequence $\varphi(n)$ is not decreasing and $\varphi(n) \leqslant 1$ for all $n \in \mathbb{N}_0$. Therefore, the right hand side of (16) tends to $(0,\ldots,0)^T_{1\times 5}$ as $n \to \infty$. Unfortunately, we can not prove that nor the system matrix in (16), nor its simpler versions in Corollaries 2 and 4, are nonsingular. On the other hand, for certain chosen distributions, we never find such matrixes to be singular. See also comments after the Corollaries 3 and 4, and the conjecture bellow.*

Corollary 1. *Suppose all conditions of Theorem 3 are satisfied. Then the probabilities $\varphi(0),\ldots,\varphi(4)$ satisfy system (16) for all $n \in \mathbb{N}_0$. In addition,*

$$\varphi(5)=\frac{6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z-\varphi(0)-\hat{z}_1\varphi(1)-\hat{z}_2\varphi(2)-\hat{z}_3\varphi(3)-\hat{z}_4\varphi(4)}{a^5*b^3*c^2(0)},$$

$$\varphi(u)=\frac{\varphi(u-6)-\sum_{k=1}^{u-1}a^6*b^3*c^2(u-k)\varphi(k)+\sum_{k=1}^{5}z_k(u-6)\varphi(k)}{a^6*b^3*c^2(0)},$$

where $u \geqslant 6$, and the quantities $z_k(u-6)$, $\hat{z}_k$ for suitable k are defined in Equations (8) and (11)–(14).

It is evident the statement of Corollary 1 follows from Theorem 3 and equations (7) and (15).

The next our statements on $\varphi(u)$ deals with the case $\{a(0) \neq 0, b(0) \neq 0, c(0) = 0\}$. This case, in turn, splits in two cases too: $c(1) \neq 0$ and $c(1) = 0$. For the case $\{a(0) \neq 0, b(0) \neq 0, c(0) = 0, c(1) \neq 0\}$ we have that $\hat{z}_5 = \hat{z}_4 = 0$, $\hat{z}_3 = a^5 * b^3 * c^2(2)$, and expressions of $\hat{z}_2$, $\hat{z}_1$ become relatively simpler than those in (13) and (14). In addition, the recurrence relation of (7) becomes the following, substantially simpler relation:

$$\varphi(u)=\sum_{k=0}^{u+3}a^6*b^3*c^2(u+5-k)\varphi(k+1)-\sum_{k=1}^{3}z_k(u)\varphi(k)$$

with coefficients $z_k(u)$ defined in (8). Now, we introduce four modified recurrent sequences while comparing to the previous β and γ sequences defined before Theorem 3. For $n \in \{0,1,2,3\}$, we present the values of these new sequences in Table 3,

Table 3. Initial values of sequences $\widehat{\beta}^0_n$, $\widehat{\beta}^1_n$, $\widehat{\beta}^2_n$ and $\widehat{\gamma}_n$.

n	$\widehat{\beta}^0_n$	$\widehat{\beta}^1_n$	$\widehat{\beta}^2_n$	$\widehat{\gamma}_n$
0	1	0	0	0
1	0	1	0	0
2	0	0	1	0
3	$-\frac{1}{a^5b^3c^2(2)}$	$-\frac{\hat{z}_1}{a^5b^3c^2(2)}$	$-\frac{\hat{z}_2}{a^5b^3c^2(2)}$	$\frac{1}{a^5b^3c^2(2)}$

and for $n \in \{4,5,\ldots\}$ we suppose that

$$\widehat{\beta}^0_n=\frac{1}{a^6*b^3*c^2(2)}\left(\widehat{\beta}^0_{n-4}-\sum_{k=1}^{n-1}a^6*b^3*c^2(k+2)\widehat{\beta}^0_{n-k}-\frac{z_3(n-4)}{\hat{z}_3}\right),$$

$$\widehat{\beta}_n^1 = \frac{1}{a^6 * b^3 * c^2(2)} \left(\widehat{\beta}_{n-4}^1 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+2)\widehat{\beta}_{n-k}^1 + z_1(n-4) - \frac{z_3(n-4)}{\hat{z}_3}\hat{z}_1 \right),$$
$$\widehat{\beta}_n^2 = \frac{1}{a^6 * b^3 * c^2(2)} \left(\widehat{\beta}_{n-4}^2 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+2)\widehat{\beta}_{n-k}^2 + z_2(n-4) - \frac{z_3(n-4)}{\hat{z}_3}\hat{z}_2 \right),$$
$$\widehat{\gamma}_n = \frac{1}{a^6 * b^3 * c^2(2)} \left(\widehat{\gamma}_{n-4} - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+2)\widehat{\gamma}_{n-k} + \frac{z_3(n-4)}{\hat{z}_3} \right).$$

Corollary 2. *Let us consider the three-risk discrete time model for generating r.v.s X, Y and Z. If $\{a(0) \neq 0, b(0) \neq 0, c(0) = 0, c(1) \neq 0\}$ and the net profit condition (4) is satisfied, then the following system of equations holds for all $n \in \mathbb{N}_0$.*

$$\begin{pmatrix} \widehat{\beta}_{n+1}^0 - \widehat{\beta}_n^0 & \widehat{\beta}_{n+1}^1 - \widehat{\beta}_n^1 & \widehat{\beta}_{n+1}^2 - \widehat{\beta}_n^2 \\ \widehat{\beta}_{n+2}^0 - \widehat{\beta}_n^0 & \widehat{\beta}_{n+2}^1 - \widehat{\beta}_n^1 & \widehat{\beta}_{n+2}^2 - \widehat{\beta}_n^2 \\ \widehat{\beta}_{n+3}^0 - \widehat{\beta}_n^0 & \widehat{\beta}_{n+3}^1 - \widehat{\beta}_n^1 & \widehat{\beta}_{n+3}^2 - \widehat{\beta}_n^2 \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \end{pmatrix}$$
$$+ \begin{pmatrix} \widehat{\gamma}_{n+1} - \widehat{\gamma}_n \\ \widehat{\gamma}_{n+2} - \widehat{\gamma}_n \\ \widehat{\gamma}_{n+3} - \widehat{\gamma}_n \end{pmatrix} \times (6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) = \begin{pmatrix} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \\ \varphi(n+3) - \varphi(n) \end{pmatrix}.$$

In addition,

$$\varphi(3) = \frac{6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z - \varphi(0) - \hat{z}_1\varphi(1) - \hat{z}_2\varphi(2)}{a^5 * b^3 * c^2(2)},$$

and

$$\varphi(u) = \frac{\varphi(u-4) - \sum_{k=1}^{u-1} a^6 * b^3 * c^2(u+2-k)\varphi(k) + \sum_{k=1}^{3} z_k(u-4)\varphi(k)}{a^6 * b^3 * c^2(2)}$$

for $u \geqslant 4$, where quantities $z_k(u-6)$, $\hat{z}_k$, for suitable k are defined in Equations (8) and (13)–(14) respectively.

For the next case $\{a(0) \neq 0, b(0) \neq 0, c(0) = c(1) = 0\}$, it follows from (10)–(14) that $\hat{z}_5 = \hat{z}_4 = \hat{z}_3 = \hat{z}_2 = 0$, $\hat{z}_1 = a^3 * b^2 * c(2)$, and recursive Equation (7) simplifies to the relation

$$\varphi(u) = \sum_{k=0}^{u+1} a^6 * b^3 * c^2(u+5-k)\varphi(k+1) - z_1(u)\varphi(1),$$

where coefficient $z_1(u)$ is defined in (8). Now we define again two modified recurrent sequences comparing to the previous β and γ sequences:

$$\widetilde{\beta}_0^0 = 1,\ \widetilde{\beta}_1^0 = -\frac{1}{a^3 * b^2 * c(2)},\ \widetilde{\gamma}_0 = 0,\ \widetilde{\gamma}_1 = \frac{1}{a^3 * b^2 * c(2)},$$
$$\widetilde{\beta}_n^0 = \frac{1}{a^6 * b^3 * c^2(4)} \left(\widetilde{\beta}_{n-2}^0 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+4)\widetilde{\beta}_{n-k}^0 - \frac{z_1(n-2)}{\hat{z}_1} \right), n \geqslant 2,$$
$$\widetilde{\gamma}_n = \frac{1}{a^6 * b^3 * c^2(4)} \left(\widetilde{\gamma}_{n-2} - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+4)\widetilde{\gamma}_{n-k} + \frac{z_1(n-2)}{\hat{z}_1} \right), n \geqslant 2.$$

Corollary 3. *Let us consider the three-risk discrete time model for generating r.v.s X, Y and Z. If $\{a(0) \neq 0, b(0) \neq 0, c(0) = c(1) = 0\}$ and the net profit condition (4) holds, then*

$$\varphi(0) = \frac{\varphi(n+1) - \varphi(n) - (\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n)(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z)}{\widetilde{\beta}_{n+1}^0 - \widetilde{\beta}_n^0},\ n \geqslant 0,$$

$$\varphi(1) = \frac{6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z - \varphi(0)}{a^3 * b^2 * c(2)},$$
$$\varphi(u) = \frac{\varphi(u-2) - \sum_{k=1}^{u-1} a^6 * b^3 * c^2(u+4-k)\varphi(k) + z_1(u-2)\varphi(1)}{a^6 * b^3 * c^2(4)}, \; u \geqslant 2.$$

Differently than in Theorem 3 and Corollaries 1, 2, 4, where we can not show that in general the system matrixes formed by β coefficients are nonsingular, in the Corollary 3 we can show that

$$|\widetilde{\beta}^0_{n+1} - \widetilde{\beta}^0_n| \geqslant 2 \tag{17}$$

for all $n \in \mathbb{N}$. In fact this estimate follows from inequalities

$$\widetilde{\beta}^0_{2m} \geqslant \widetilde{\beta}^0_{2(m-1)} \geqslant 1, \; \widetilde{\beta}^0_{2m+1} \leqslant \widetilde{\beta}^0_{2(m-1)} \leqslant -\frac{1}{a^3 * b^2 * c(2)} \tag{18}$$

provided for $m \in \mathbb{N}$. Indeed, if inequalities (18) hold and n is even, then $n+1$ is odd, and consequently,

$$|\widetilde{\beta}^0_{n+1} - \widetilde{\beta}^0_n| = \widetilde{\beta}^0_n - \widetilde{\beta}^0_{n+1} \geqslant 1 + \frac{1}{a^3 * b^2 * c(2)} \geqslant 2.$$

If inequalities (18) hold and n is is odd, then $n+1$ is even and we get again that

$$|\widetilde{\beta}^0_{n+1} - \widetilde{\beta}^0_n| = \widetilde{\beta}^0_{n+1} - \widetilde{\beta}^0_n \geqslant 1 + \frac{1}{a^3 * b^2 * c(2)} \geqslant 2.$$

Hence, to establish lower bound (17) we need to prove inequalities (18). For this we use induction. If $m = 1$, then we have

$$\widetilde{\beta}^0_2 = \frac{1}{a^6 * b^3 * c^2(4)} \left(1 + \frac{a^6 * b^3 * c^2(5) - z_1(0)}{a^3 * b^2 * c(2)}\right) \geqslant 1 = \widetilde{\beta}^0_0.$$

Similarly,

$$\widetilde{\beta}^0_3 - \widetilde{\beta}^0_1 = \frac{1}{a^6 * b^3 * c^2(4)} \left(-\frac{1 - a^6 * b^3 * c^2(6) + z_1(1)}{a^3 * b^2 * c(2)} - a^6 * b^3 * c^2(5)\widetilde{\beta}^0_2\right) - \frac{1}{a^3 * b^2 * c(2)} \leqslant 0.$$

By supposing that inequalities (18) hold for $m = N$ we get

$$\begin{aligned}
\widetilde{\beta}^0_{2N+2} &= \frac{1}{a^6 * b^3 * c^2(4)} \left(\widetilde{\beta}^0_{2N} - \sum_{k=1}^{2N+1} a^6 * b^3 * c^2(k+4)\widetilde{\beta}^0_{2N+2-k} - \frac{z_1(2N)}{a^3 * b^2 * c(2)}\right) \\
&\geqslant \frac{1}{a^6 * b^3 * c^2(4)} \\
&\quad \times \Big(\widetilde{\beta}^0_{2N} - a^6 * b^3 * c^2(6)\widetilde{\beta}^0_{2N} - a^6 * b^3 * c^2(8)\widetilde{\beta}^0_{2N} - \ldots - a^6 * b^3 * c^2(2N+4)\widetilde{\beta}^0_{2N} \\
&\quad - a^6 * b^3 * c^2(5)\widetilde{\beta}^0_1 - a^6 * b^3 * c^2(7)\widetilde{\beta}^0_1 - \ldots - a^6 * b^3 * c^2(2N+5)\widetilde{\beta}^0_1 - \frac{z_1(2N)}{a^3 * b^2 * c(2)}\Big) \\
&= \frac{1}{a^6 * b^3 * c^2(4)} \Big(\widetilde{\beta}^0_{2N} \left(1 - a^6 * b^3 * c^2(6) - a^6 * b^3 * c^2(8) - \ldots - a^6 * b^3 * c^2(2N+4)\right) \\
&\quad - \widetilde{\beta}^0_1 \left(a^6 * b^3 * c^2(5) + a^6 * b^3 * c^2(7) + \ldots + a^6 * b^3 * c^2(2N+5)\right) - \frac{z_1(2N)}{a^3 * b^2 * c(2)}\Big) \\
&\geqslant \frac{1}{a^6 * b^3 * c^2(4)} \left(\widetilde{\beta}^0_{2N} a^6 * b^3 * c^2(4) + \frac{a^6 * b^3 * c^2(2N+5) - z_1(2N)}{a^3 * b^2 * c(2)}\right) \geqslant \widetilde{\beta}^0_{2N},
\end{aligned}$$

where the last inequality can be derived from the fact that for the certain distribution with property $\{a(0) \neq 0, b(0) \neq 0, c(0) = c(1) = 0\}$ we have

$$z_1(2N) - a_{2N+1} \cdot a^5 * b^3 * c^2(4) = a^3 * b^2 * c(2) \sum_{i_1=0}^{2N} \sum_{i_2=0}^{2N+1-i_1} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{2N+3-i_1-i_2}$$
$$\leqslant a^3 * b^2 * c(2) \sum_{i_1=0}^{2N+1} \sum_{i_2=0}^{2N+1-i_1} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{2N+3-i_1-i_2} = a^3 * b^2 * c(2) \sum_{k=0}^{2N+1} a^2 * b(k) \cdot a * c_{2N+3-k},$$

and

$$a^6 * b^3 * c^2(2N+5) - a_{2N+1} \cdot a^5 * b^3 * c^2(4) - a^3 * b^2 * c(2) \sum_{k=0}^{2N+1} a^2 * b(k) \cdot a * c_{2N+3-k} \geqslant 0.$$

Proceeding with induction hypothesis for $\widetilde{\beta}_n^0$ with odd indices, we obtain

$$\widetilde{\beta}_{2N+3}^0 = \frac{1}{a^6 * b^3 * c^2(4)} \left(\widetilde{\beta}_{2N+1}^0 - \sum_{k=1}^{2N+2} a^6 * b^3 * c^2(k+4) \widetilde{\beta}_{2N+3-k}^0 - \frac{z_1(2N+1)}{a^3 * b^2 * c(2)} \right)$$
$$\leqslant \frac{\widetilde{\beta}_{2N+1}^0 \left(1 - a^6 * b^3 * c^2(6) - a^6 * b^3 * c^2(8) - \ldots - a^6 * b^3 * c^2(2N+6)\right)}{a^6 * b^3 * c^2(4)} \leqslant \widetilde{\beta}_{2N+1}^0.$$

Consequently, estimates (18) are correct. We note only that the similar technic of proving $|\widetilde{\beta}_{n+1}^0 - \widetilde{\beta}_n^0| \geqslant 2$ but for a seasonal bi-risk model is used in [2].

According to Table 1 for the remaining case of possible distributions in three-risk models, we have $\{a(0) \neq 0, b(0) = 0, c(0) \neq 0\}$. In such a case, we obtain that $\hat{z}_5 = \hat{z}_4 = \hat{z}_3 = 0, \hat{z}_2 = a^4 * b^2 * c^2(2)$, and expression of $\hat{z}_1$ is relatively simpler with respect to that in (14). Substituting these coefficients into (7), we get the following recursive equation for survival probability in the case under consideration

$$\varphi(u) = \sum_{k=0}^{u+2} a^6 * b^3 * c^2(u+5-k)\varphi(k+1) - \sum_{k=1}^{2} z_k(u)\varphi(k).$$

Once again, arguing the same as before the Theorem 3, we define three recurrent sequences:

$$\overline{\beta}_0^0 = 1,\ \overline{\beta}_1^0 = 0,\ \overline{\beta}_2^0 = -\frac{1}{a^4b^2c^2(2)},\ \overline{\beta}_0^1 = 0,\ \overline{\beta}_1^1 = 1,\ \overline{\beta}_2^1 = -\frac{\hat{z}_1}{a^4b^2c^2(2)},$$
$$\overline{\gamma}_0 = 0,\ \overline{\gamma}_1 = 0,\ \overline{\gamma}_2 = \frac{1}{a^4b^2c^2(2)},$$
$$\overline{\beta}_n^0 = \frac{1}{a^6 * b^3 * c^2(3)} \left(\overline{\beta}_{n-3}^0 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+3) \overline{\beta}_{n-k}^0 - \frac{z_2(n-3)}{\hat{z}_2} \right),\ n \geqslant 3,$$
$$\overline{\beta}_n^1 = \frac{1}{a^6 * b^3 * c^2(3)} \left(\overline{\beta}_{n-3}^1 - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+3) \overline{\beta}_{n-k}^1 + z_1(n-3) - \frac{z_2(n-3)}{\hat{z}_2} \hat{z}_1 \right),\ n \geqslant 3,$$
$$\overline{\gamma}_n = \frac{1}{a^6 * b^3 * c^2(3)} \left(\overline{\gamma}_{n-3} - \sum_{k=1}^{n-1} a^6 * b^3 * c^2(k+3) \overline{\gamma}_{n-k} + \frac{z_2(n-3)}{\hat{z}_2} \right),\ n \geqslant 3.$$

Corollary 4. *Let us consider the three-risk discrete time model with generating r.v.s X, Y and Z. If $\{a(0) \neq 0, b(0) = 0, c(0) \neq 0\}$ and the net profit condition (4) is satisfied, then the following equations hold:*

$$\begin{pmatrix} \overline{\beta}^0_{n+1} - \overline{\beta}^0_n & \overline{\beta}^1_{n+1} - \overline{\beta}^1_n \\ \overline{\beta}^0_{n+2} - \overline{\beta}^0_n & \overline{\beta}^1_{n+2} - \overline{\beta}^1_n \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \end{pmatrix}$$
$$+ \begin{pmatrix} \overline{\gamma}_{n+1} - \overline{\gamma}_n \\ \overline{\gamma}_{n+2} - \overline{\gamma}_n \end{pmatrix} \times (6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) = \begin{pmatrix} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \end{pmatrix}, n \in \mathbb{N}_0,$$
$$\varphi(2) = \frac{6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z - \varphi(0) - \hat{z}_1\varphi(1)}{a^4 * b^2 * c^2(2)},$$
$$\varphi(u) = \frac{\varphi(u-3) - \sum_{k=1}^{u-1} a^6 * b^3 * c^2(u+3-k)\varphi(k) + \sum_{k=1}^{2} z_k(u-3)\varphi(k)}{a^6 * b^3 * c^2(3)}, \; u \geqslant 3.$$

The system matrix $\overline{\beta}_{2\times 2}$ in Corollary 4 is the most simple one, and is compared with the system matrixes in Theorem 3 and Corollaries 1, 2, to show its non singularity. One may check that

$$\begin{vmatrix} \overline{\beta}^0_{n+1} - \overline{\beta}^0_n & \overline{\beta}^1_{n+1} - \overline{\beta}^1_n \\ \overline{\beta}^0_{n+2} - \overline{\beta}^0_n & \overline{\beta}^1_{n+2} - \overline{\beta}^1_n \end{vmatrix} = \begin{vmatrix} \overline{\beta}^0_{n+1} & \overline{\beta}^1_{n+1} \\ \overline{\beta}^0_{n+2} & \overline{\beta}^1_{n+2} \end{vmatrix} + \begin{vmatrix} \overline{\beta}^0_{n} & \overline{\beta}^1_{n} \\ \overline{\beta}^0_{n+1} & \overline{\beta}^1_{n+1} \end{vmatrix} - \begin{vmatrix} \overline{\beta}^0_{n} & \overline{\beta}^1_{n} \\ \overline{\beta}^0_{n+2} & \overline{\beta}^1_{n+2} \end{vmatrix}$$
$$=: D_{n+1} + D_n - \widetilde{D}_n.$$

The last equality leads to the following conjecture.

Conjecture. *For all $n \in \mathbb{N}_0$ it holds that $D_{n+1} > D_n > 0$, and $\widetilde{D}_{n+1} < \widetilde{D}_n < 0$.*

Obviously, the presented conjecture implies that the system matrix $\overline{\beta}_{2\times 2}$ in the Corollary 4 is nonsingular. Moreover, numerical calculations with some chosen distributions always approve the conjecture.

At the end of this section, we formulate the statement on the values of survival probability $\varphi(u)$, when the net profit condition (4) is unsatisfied.

Theorem 4. *For each three-risk discrete time model, with generating r.v.s X, Y and Z, the following assertions are true:*

- *If $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 > 1$, then $\varphi(u) = 0$ for $u \in \mathbb{N}_0$.*
- *If $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 1$ and $\mathbb{P}(X + Y/2 + Z/3 = 1) < 1$, then $\varphi(u) = 0$ for $u \in \mathbb{N}_0$.*
- *If $\mathbb{P}(X + Y/2 + Z/3 = 1) = 1$, then $\varphi(0) = 0$ and $\varphi(u) = 1$ for $u \in \mathbb{N}$.*

3. Proofs

In this section we present all proofs of previously formulated statements.

Proof of Theorem 1. Our aim is to prove that

$$\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 < 1 \Rightarrow \lim_{u\to\infty} \varphi(u) = 1.$$

According to definition, the survival probability for the model (3) is

$$\varphi(u) = \mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{\sum_{i=1}^{n}(R_i - 1) < u\right\}\right) = \mathbb{P}\left(\sup_{n\geqslant 1}\eta_n < u\right), \tag{19}$$

where $\eta_n = \sum_{i=1}^{n}(R_i - 1)$ and r.v.s $R_i, i \in \mathbb{N}$ are defined by equality (2).

For every $j = \overline{0,5}$ and $N \in \mathbb{N}$ we have

$$\frac{\eta_{6N+j}}{6N+j} = \frac{1}{6N+j}\sum_{i=1}^{6N+j}(R_i - 1) = \frac{1}{6N+j}\left(\frac{N+1}{N+1}\sum_{i=0}^{N}\sum_{k=1}^{j}(R_{6i+k} - 1) + \frac{N}{N}\sum_{i=0}^{N-1}\sum_{k=j+1}^{6}(R_{6i+k} - 1)\right),$$

with $\sum_{k=1}^{0} := 0$ by default.

Since $R_i \overset{d}{=} R_{i+6}$ for each $i \in \mathbb{N}$, the strong law of large numbers implies that

$$\lim_{n\to\infty} \frac{\eta_n}{n} = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}(R_i - 1) = \frac{6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z - 6}{6} < 0 \tag{20}$$

almost surely, and following the proof of Theorem 2.3 from [2] (see page 936), one can get that

$$\liminf_{u\to\infty} \varphi(u) \geqslant 1 - \varepsilon,$$

with an arbitrarily small, positive ε. □

Proof of Theorem 2. One can see that $a^i * b^j * c^k(m) = 0$ if $j + k > m$ and $b(0) = c(0) = 0$. We recall that $i = 0$ or $j = 0$ or $k = 0$ means that a corresponding r.v. is not included into a convolution. Hence, the condition $\{a(0) \neq 0, b(0) = c(0) = 0\}$ and relation (7) imply that

$$\varphi(u) = \sum_{k=0}^{u} a^6 * b^3 * c^2(u + 5 - k)\varphi(k + 1). \tag{21}$$

Summing up both sides of the last equality by u from 0 up to some nonnegative, integer and sufficiently large v and changing the order of summation, we obtain

$$\begin{aligned}\sum_{u=0}^{v} \varphi(u) &= \sum_{u=0}^{v}\sum_{k=0}^{u} a^6 * b^3 * c^2(u + 5 - k)\varphi(k + 1) = \sum_{k=0}^{v} \varphi(k + 1) \sum_{u=k}^{v} a^6 * b^3 * c^2(u + 5 - k)\\ &= \sum_{k=0}^{v} \varphi(k + 1) \left(A^6 * B^3 * C^2(v + 5 - k) - A^6 * B^3 * C^2(4)\right),\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\sum_{k=0}^{v+6} \varphi(k)\overline{A^6 * B^3 * C^2}(v + 6 - k) - \sum_{k=v+1}^{v+6} \varphi(k)\overline{A^6 * B^3 * C^2}(v + 6 - k)\\ &= -\varphi(0)A^6 * B^3 * C^2(v + 6) + \varphi(v + 1)A^6 * B^3 * C^2(5).\end{aligned} \tag{22}$$

According to Theorem 1, for the first sum in (22), we have

$$\sum_{k=0}^{v+6} \varphi(k)\overline{A^6 * B^3 * C^2}(v + 6 - k) \underset{v\to\infty}{\to} 6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z.$$

Indeed, the upper limit for the sum is

$$\limsup_{v\to\infty} \sum_{k=0}^{v+6} \varphi(k)\overline{A^6 * B^3 * C^2}(v + 6 - k) = \sum_{k=0}^{\infty} \overline{A^6 * B^3 * C^2}(k) = 6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z,$$

while the lower limit for the sum is the same, due to the assertion of Theorem 1 and the following estimate provided for each $M \geqslant 2$:

$$\begin{aligned}&\liminf_{v\to\infty} \sum_{k=0}^{v+6} \varphi(k)\overline{A^6 * B^3 * C^2}(v + 6 - k) = \liminf_{v\to\infty} \left(\sum_{k=0}^{M} + \sum_{k=M+1}^{v+6}\right) \varphi(k)\overline{A^6 * B^3 * C^2}(v + 6 - k)\\ &\geqslant \inf_{k\geqslant M+1} \varphi(k) \liminf_{v\to\infty} \sum_{k=M+1}^{v+6} \overline{A^6 * B^3 * C^2}(v + 6 - k) = \inf_{k\geqslant M+1} \varphi(k) \liminf_{v\to\infty} \sum_{k=0}^{v+5-M} \overline{A^6 * B^3 * C^2}(k)\\ &= \inf_{k\geqslant M+1} \varphi(k)(6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z).\end{aligned}$$

Consequently, as $v \to \infty$, from (22) we get

$$\varphi(0) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z.$$

The rest of the assertion can easily be derived from (21). Theorem 2 is proven. □

Proof of Theorem 3. Recall that

$$\varphi(u) = \sum_{k=0}^{u+5} a^6 * b^3 * c^2(u+5-k)\varphi(k+1) - \sum_{k=1}^{5} S_k(u), \tag{23}$$

where sums $S_k(u)$ are defined in (6). As in the proof of Theorem 2, we sum both sides of Equation (23) by u from 0 up to some nonnegative, sufficiently large integer v. We get

$$\sum_{u=0}^{v} \varphi(u) = \sum_{u=0}^{v} \sum_{k=0}^{u+5} a^6 * b^3 * c^2(u+5-k)\varphi(k+1) - \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u). \tag{24}$$

By changing the order of summation, we obtain

$$\begin{aligned}
\sum_{u=0}^{v} \varphi(u) &= \sum_{k=0}^{4} \sum_{u=0}^{v} a^6 * b^3 * c^2(u+5-k)\varphi(k+1) \\
&\quad + \sum_{k=5}^{v+5} \sum_{u=k-5}^{v} a^6 * b^3 * c^2(u+5-k)\varphi(k+1) - \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u) \\
&= \sum_{k=0}^{4} \varphi(k+1) \left(A^6 * B^3 * C^2(v+5-k) - A^6 * B^3 * C^2(4-k) \right) \\
&\quad + \sum_{k=5}^{v+5} \varphi(k+1) A^6 * B^3 * C^2(v+5-k) - \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u) \\
&= \sum_{k=1}^{5} \varphi(k) \left(A^6 * B^3 * C^2(v+6-k) - A^6 * B^3 * C^2(5-k) \right) \\
&\quad - \sum_{k=0}^{5} \varphi(k) A^6 * B^3 * C^2(v+6-k) + \sum_{k=0}^{v+6} \varphi(k) A^6 * B^3 * C^2(v+6-k) - \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\sum_{k=0}^{v+6} \varphi(k)\overline{A^6 * B^3 * C^2}(v+6-k) &= \sum_{k=v+1}^{v+6} \varphi(k) + \sum_{k=1}^{5} \varphi(k)(A^6 * B^3 * C^2(v+6-k) \\
&- A^6 * B^3 * C^2(5-k)) - \sum_{k=0}^{5} \varphi(k) A^6 * B^3 * C^2(v+6-k) - \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u).
\end{aligned} \tag{25}$$

By the proof of Theorem 2, equality (25) implies that

$$6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z = 6 + \sum_{k=1}^{5} \varphi(k)\overline{A^6 * B^3 * C^2}(5-k) - \sum_{k=0}^{5} \varphi(k) - \lim_{v\to\infty} \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u).$$

Consequently,

$$\varphi(0) + \sum_{k=1}^{5} \varphi(k) A^6 * B^3 * C^2(5-k) = 6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z - \lim_{v\to\infty} \sum_{u=0}^{v} \sum_{k=1}^{5} S_k(u). \tag{26}$$

For the first of five remaining sums in the right hand side of the last equality we have

$$\begin{aligned}\sum_{u=0}^{v} S_1(u) &= \sum_{k=0}^{4} \varphi(k+1) \sum_{j=1}^{5-k} a^5 * b^3 * c^2(5-k-j) \sum_{u=0}^{v} a_{u+j} \\ &= \sum_{k=0}^{4} \varphi(k+1) \sum_{j=1}^{5-k} a^5 * b^3 * c^2(5-k-j)(A(v+j) - A(j-1)).\end{aligned}$$

Therefore,

$$\lim_{v\to\infty} \sum_{u=0}^{v} S_1(u) = \sum_{k=0}^{4} \varphi(k+1) \sum_{j=1}^{5-k} a^5 * b^3 * c^2(5-k-j) \cdot \overline{A}(j-1).$$

For the second of five remaining sums in the right hand of (26) we obtain

$$\begin{aligned}\sum_{u=0}^{v} S_2(u) &= \sum_{k=0}^{3} \varphi(k+1) \sum_{u=0}^{v} \sum_{i_1=0}^{u} \sum_{j=2-i_1}^{5-i_1-k} a_{i_1} \cdot a * b_{u+j} \cdot a^4 * b^2 * c^2(5-i_1-j-k) \\ &= \sum_{k=0}^{3} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=2-i_1}^{5-i_1-k} \sum_{u=i_1}^{v} a * b_{u+j} \cdot a^4 * b^2 * c^2(5-i_1-j-k) \\ &= \sum_{k=0}^{3} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=2-i_1}^{5-i_1-k} (A * B(j+v) - A * B(j-1))a^4 * b^2 * c^2(5-i_1-j-k) \\ &= \sum_{k=0}^{3} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{j=2}^{5-k} (A * B(v+j-i_1) - A * B(j-1))a^4 * b^2 * c^2(5-k-j) \\ &= \sum_{k=0}^{3} \varphi(k+1) \sum_{j=2}^{5-k} a^4 * b^2 * c^2(5-k-j) \sum_{i_1=0}^{v} a_{i_1} (A * B(v+j-i_1) - A * B(j-1)).\end{aligned}$$

Consequently,

$$\lim_{v\to\infty} \sum_{u=0}^{v} S_2(u) = \sum_{k=0}^{3} \varphi(k+1) \sum_{j=2}^{5-k} a^4 * b^2 * c^2(5-k-j) \cdot \overline{A * B}(j-1).$$

Further,

$$\begin{aligned}\sum_{u=0}^{v} S_3(u) &= \sum_{k=0}^{2} \varphi(k+1) \sum_{u=0}^{v} \sum_{i_1=0}^{u} a_{i_1} \sum_{i_2=0}^{u+1-i_1} a * b_{i_2} \sum_{j=3}^{5-k} a * c_{u+j-i_1-i_2} \cdot a^3 * b^2 * c(5-k-j) \\ &= \sum_{k=0}^{2} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{u=i_1}^{v} \sum_{i_2=0}^{u+1-i_1} a * b_{i_2} \sum_{j=3}^{5-k} a * c_{u+j-i_1-i_2} \cdot a^3 * b^2 * c(5-k-j) \\ &= a * b_0 \sum_{k=0}^{2} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{u=i_1}^{v} \sum_{j=3}^{5-k} a * c_{j+u-i_1} \cdot a^3 * b^2 * c(5-k-j) \\ &+ \sum_{k=0}^{2} \varphi(k+1) \sum_{i_1=0}^{v} a_{i_1} \sum_{i_2=1}^{v+1-i_1} a * b_{i_2} \sum_{u=i_2+i_1-1}^{v} \sum_{j=5}^{5-k} a * c_{j+u-i_1-i_2} \cdot a^3 * b^2 * c(5-k-j) \\ &= a * b_0 \sum_{k=0}^{2} \varphi(k+1) \sum_{j=3}^{5-k} \sum_{i_1=0}^{v} a_{i_1} \left(A * C(v+j-i_1) - A * C(j-1)\right) a^3 * b^2 * c(5-k-j)\end{aligned}$$

$$+\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\sum_{i_1=0}^{v}\sum_{i_2=1}^{v+1-i_1}a_{i_1}\cdot a*b_{i_2}\left(A*C(v+j-i_1-i_2)-A*C(j-2)\right)a^3*b^2*c(5-k-j)$$

$$=a*b_0\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\sum_{i_1=0}^{v}a_{i_1}\left(A*C(v+j-i_1)-A*C(j-1)\right)a^3*b^2*c(5-k-j)$$

$$+\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\sum_{i_1=0}^{v}\sum_{i_2=0}^{v+1-i_1}a_{i_1}\cdot a*b_{i_2}\left(A*C(v+j-i_1-i_2)-A*C(j-2)\right)a^3*b^2*c(5-k-j)$$

$$-a*b_0\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\sum_{i_1=0}^{v}a_{i_1}\left(A*C(v+j-i_1)-A*C(j-2)\right)a^3*b^2*c(5-k-j),$$

and

$$\begin{aligned}\lim_{v\to\infty}\sum_{u=0}^{v}S_3(u)&=a*b_0\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\overline{A*C}(j-1)\\&\quad+\overline{A*B}_0\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}\overline{A*C}(j-2)a^3*b^2*c(5-k-j)\\&=\sum_{k=0}^{2}\varphi(k+1)\sum_{j=3}^{5-k}a^3*b^2*c(5-k-j)\left(\overline{A*C}(j-1)+\overline{A*B}_0\cdot a*c_{j-1}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{u=0}^{v}S_4(u)&=\sum_{k=0}^{1}\varphi(k+1)\sum_{u=0}^{v}\sum_{i_1=0}^{u}\sum_{i_2=0}^{u+1-i_1}\sum_{i_3=0}^{u+2-i_1-i_2}\sum_{j=4}^{5-k}a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{u+j-i_1-i_2-i_3}\\&\quad\times a^2*b*c(5-k-j)\\&=\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}\sum_{u=i_1}^{v}\sum_{i_2=0}^{u+1-i_1}\sum_{i_3=0}^{u+2-i_1-i_2}\sum_{j=4}^{5-k}a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{u+j-i_1-i_2-i_3}\\&\quad\times a^2*b*c(5-k-j)\\&=a*b_0\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}\sum_{u=i_1}^{v}\sum_{i_3=0}^{u+2-i_1}\sum_{j=0}^{u+2-i_1-i_2}\sum_{j=4}^{5-k}a_{i_1}\cdot a*c_{i_3}\cdot a*b_{u+j-i_1-i_3}\\&\quad\times a^2*b*c(5-k-j)\\&\quad+\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}\sum_{i_2=1}^{v+1-i_1}\sum_{u=i_1+i_2-1}^{v}\sum_{i_3=0}^{u+2-i_1-i_2}\sum_{j=4}^{5-k}a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{u+j-i_1-i_3}\\&\quad\times a^2*b*c(5-k-j)\\&=a*b_0\cdot a*c_0\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}a_{i_1}\sum_{j=4}^{5-k}\sum_{u=i_1}^{v}a*b_{u+j-i_1}\cdot a^2*b*c(5-k-j)\\&\quad+a*b_0\cdot a*c_1\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}a_{i_1}\sum_{j=4}^{5-k}\sum_{u=i_1}^{v}a*b_{u+j-1-i_1}\cdot a^2*b*c(5-k-j)\\&\quad+a*c_0\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}a_{i_1}\sum_{i_2=1}^{v+1-i_1}\sum_{j=4}^{5-k}\sum_{u=i_1+i_2-1}^{v}a*b_{i_2}\cdot a*b_{u+j-1-i_1-i_2}\\&\quad\times a^2*b*c(5-k-j)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{1}\varphi(k+1)\sum_{i_1=0}^{v}\sum_{i_2=1}^{v+1-i_1}\sum_{i_3=1}^{v+2-i_1-i_2}\sum_{u=i_1+i_2+i_3-1}^{v}\sum_{j=4}^{5-k}a_{i_1}\cdot a*b_{i_2}\cdot a*b_{u+j-1-i_1-i_2}\cdot a*c_{i_3}\\
&\qquad\times a^2*b*c(5-k-j)\\
&\quad= a^2*b*c(0)\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{i_1=0}^{v}a_{i_1}(A*B(v+j-i_1)-A*B(j-1))\\
&\qquad\times a^2*b*c(5-k-j)\\
&+a*b_0\cdot a*c_1\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{i_1=0}^{v}a_{i_1}(A*B(v-1+j-i_1)-A*B(j-2))\\
&\qquad\times a^2*b*c(5-k-j)\\
&+a*b_0\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{i_1=0}^{v}a_{i_1}\sum_{i_3=2}^{v+2-i_1}a*c_{i_3}(A*B(v+j-i_1-i_3)-A*B(j-3))\\
&\qquad\times a^2*b*c(5-k-j)\\
&+a*b_0\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{i_1=0}^{v}a_{i_1}\sum_{i_2=1}^{v+1-i_1}a*b_{i_2}(A*B(v+j-i_1-i_2)-A*B(j-2))\\
&\qquad\times a^2*b*c(5-k-j)\\
&+\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\sum_{i_1=0}^{v}a_{i_1}\sum_{i_2=1}^{v+1-i_1}a*b_{i_2}\sum_{i_3=1}^{v+2-i_1-i_2}a*c_{i_3}\\
&\qquad\times(A*B(v+j-i_1-i_2-i_3)-A*B(j-3))\,a^2*b*c(5-k-j),
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
\lim_{v\to\infty}\sum_{u=0}^{v}S_4(u) &= a^2*b*c(0)\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\overline{A*B}(j-1)\cdot a^2*b*c(5-k-j)\\
&\quad+a*b_0\cdot a*c_1\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\overline{A*B}(j-2)\cdot a^2*b*c(5-k-j)\\
&\quad+a*b_0\cdot\overline{A*C}(1)\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\overline{A*B}(j-3)\cdot a^2*b*c(5-k-j)\\
&\quad+a*c_0\cdot\overline{A*B}(0)\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\overline{A*B}(j-2)\cdot a^2*b*c(5-k-j)
\end{aligned}
$$

$$
+\overline{A*B}_0\cdot\overline{A*C}(0)\sum_{k=0}^{1}\varphi(k+1)\sum_{j=4}^{5-k}\overline{A*B}(j-3)\cdot a^2*b*c(5-k-j).
$$

For the remaining case, we first rewrite the sum $S_5(u)$ in the following way

$$
\begin{aligned}
S_5(u) &= a*b*c(0)\,\varphi(1)\sum_{i_1=0}^{u+3}\sum_{i_2=0}^{u+3-i_1}\sum_{i_3=0}^{u+3-i_1-i_2-i_3}a_{i_1}\cdot a*b_{i_2}\cdot a*c_{i_3}\cdot a*b_{i_4}\cdot a_{u+5-i_1-i_2-i_3-i_4}\\
&\quad-a^2*b^2*c(0)\cdot a_2\,\varphi(1)\sum_{i_1=0}^{u+1}\sum_{i_2=0}^{u+1-i_1}a_{i_1}\cdot a*b_{i_2}\cdot a*c_{u+3-i_1-i_2}\\
&\quad+a^2*b^2*c(0)\cdot a_2\cdot a*b_0\cdot a*c_2\cdot a_{u+1}\,\varphi(1)
\end{aligned}
$$

$$
\begin{aligned}
&- a*b*c(0)\,\varphi(1) \sum_{i_1=0}^{u} \sum_{i_2=u+2-i_1}^{u+3-i_1} \sum_{i_3=0}^{u+3-i_1-i_2} \sum_{i_4=0}^{u+3-i_1-i_2-i_3} a_{i_1} \cdot a*b_{i_2} \cdot a*c_{i_3} \cdot a*b_{i_4} \\
&\quad \times a_{u+5-i_1-i_2-i_3-i_4} \\
&- a*b*c(0)\,\varphi(1) \sum_{i_1=u+1}^{u+3} \sum_{i_2=0}^{u+3-i_1} \sum_{i_3=0}^{u+3-i_1-i_2} \sum_{i_4=0}^{u+3-i_1-i_2-i_3} a_{i_1} \cdot a*b_{i_2} \cdot a*c_{i_3} \cdot a*b_{i_4} \\
&\quad \times a_{u+5-i_1-i_2-i_3-i_4} \\
= {}& a*b*c(0)\,\varphi(1) \sum_{k=0}^{u+3} a^4*b^2*c(k) \cdot a_{u+5-k} - a^2*b^2*c(0) \cdot a_2\,\varphi(1) \sum_{k=0}^{u+1} a^2*b(k) \cdot a*c_{u+3-k} \\
&+ a^3*b^3*c(0) \cdot a_2 \cdot a*c_2 \cdot a_{u+1}\,\varphi(1) \\
&- a*b*c(0)\,\varphi(1) \sum_{i_1=0}^{u} a_{i_1} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1-i_2} \sum_{i_4=0}^{1-i_2-i_3} a*b_{i_2} \cdot a*c_{i_3} \cdot a*b_{i_4} \cdot a_{3-i_2-i_3-i_4} \\
&- a*b*c(0)\,\varphi(1) \sum_{i_1=0}^{2} a_{i_1} \sum_{i_2=0}^{2-i_1} \sum_{i_3=0}^{2-i_1-i_2} \sum_{i_4=0}^{2-i_1-i_2-i_3} a*b_{i_2} \cdot a*c_{i_3} \cdot a*b_{i_4} \cdot a_{4-i_1-i_2-i_3-i_4} \\
= {}& a*b*c(0)\,\varphi(1) \sum_{k=0}^{u+3} a^4*b^2*c(k) \cdot a_{u+5-k} - a^2*b^2*c(0) \cdot a_2\,\varphi(1) \sum_{k=0}^{u+1} a^2*b(k) \cdot a*c_{u+3-k} \\
&+ a^3*b^3*c(0) \cdot a_2 \cdot a*c_2 \cdot a_{u+1}\,\varphi(1) \\
&- a*b*c(0)\,\varphi(1) \sum_{i_1=0}^{u} a_{i_1} \sum_{i_2=0}^{1} a*b_{u+2+i_2-i_1} \sum_{k=0}^{1-i_2} a^2*b*c(k) \cdot a_{3-i_2-k} \\
&- a*b*c(0)\,\varphi(1) \sum_{i_1=0}^{2} a_{u+1+i_1} \sum_{k=0}^{2-i_1} a^3*b^2*c(k) \cdot a_{4-i_1-k}.
\end{aligned}
$$

After such simplification, we obtain

$$
\begin{aligned}
\sum_{u=0}^{v} S_5(u) = {}& a*b*c(0)\,\varphi(1) \sum_{u=0}^{v} \sum_{k=0}^{u+3} a_{u+5-k} \cdot a^4*b^2*c(k) \\
&- a^2*b^2*c(0) \cdot a_2\,\varphi(1) \sum_{u=0}^{v} \sum_{k=0}^{u+1} a*c_{u+3-k} \cdot a^2*b(k) \\
&+ a^3*b^3*c(0) \cdot a_1 \cdot a*c_2 \cdot (A(v+1) - A(0))\,\varphi(1) \\
&- a*b*c(0)\,\varphi(1) \sum_{u=0}^{v} \sum_{i_1=0}^{u} \sum_{i_2=0}^{1} a_{i_1} \cdot a*b_{u+2+i_2-i_1} \sum_{k=0}^{1-i_2} a^2*b*c(k) \cdot a_{3-i_2-k} \\
&- a*b*c(0)\,\varphi(1) \sum_{u=0}^{v} \sum_{i_1=0}^{2} a_{u+1+i_1} \sum_{k=0}^{2-i_1} a^3*b^2*c(k) \cdot a_{4-i_1-k} \\
= {}& a*b*c(0)\,\varphi(1) \sum_{k=0}^{2} \sum_{u=0}^{v} a_{u+5-k} \cdot a^4*b^2*c(k) \\
&+ a*b*c(0)\,\varphi(1) \sum_{k=0}^{v+3} \sum_{u=k-3}^{v} a_{u+5-k} \cdot a^4*b^2*c(k) \\
&- a^2*b^2*c(0) \cdot a_2\,\varphi(1) \sum_{u=0}^{v} a*c_{u+3} \cdot a^2*b(0) \\
&- a^2*b*c(0) \cdot a_2\,\varphi(1) \sum_{k=1}^{v+1} \sum_{u=k-1}^{v} a*c_{u+3-k} \cdot a^2*b(k) \\
&+ a^3*b^3*c(0) \cdot a_2 \cdot a*c_2\,(A(v+1) - A(0))\,\varphi(1)
\end{aligned}
$$

$$
\begin{aligned}
&- a*b*c(0)\,\varphi(1)\sum_{i_1=0}^{v} a_{i_1}\sum_{i_2=0}^{1}\sum_{u=i_1}^{v} a*b_{u+2+i_2-i_1}\sum_{k=0}^{1-i_2} a^2*b*c(k)\cdot a_{3-i_2-k}\\
&- a*b*c(0)\,\varphi(1)\sum_{i_1=0}^{2}\sum_{u=0}^{v} a_{u+1+i_1}\sum_{k=0}^{2-i_1} a^3*b^2*c(k)\cdot a_{4-i_1-k}\\
= {}& a*b*c(0)\,\varphi(1)\sum_{k=0}^{2}(A(v+5-k)-A(4-k))\,a^4*b^2*c(k)\\
&+ a*b*c(0)\,\varphi(1)\sum_{k=0}^{v+3}(A(v+5-k)-A(1))a^4*b^2*c(k)\\
&- a^4*b^3*c(0)\cdot a_2\,\varphi(1)(A*C(v+3)-A*C(2))\\
&- a^2*b^2*c(0)\cdot a_2\,\varphi(1)\sum_{k=1}^{v+1}(A*C(v+3-k)-A*C(1))\,a^2*b(k)\\
&+ a^3*b^3*c(0)\cdot a_2\cdot a*c_2\,(A(v+1)-A(0))\,\varphi(1)\\
&- a*b*c(0)\varphi(1)\sum_{i_2=0}^{1}\sum_{k=0}^{1-i_2} a^2*b*c(k)\cdot a_{3-i_2-k}\sum_{i_1=0}^{v} a_{i_1}\,(A*B(v+2+i_2-i_1)-A*B(i_2+1))\\
&- a*b*c(0)\varphi(1)\sum_{i_1=0}^{2}(A(v+1+i_1)-A(i_1))\sum_{k=0}^{2-i_1} a^3*b^2*c(k)\cdot a_{4-i_1-k},
\end{aligned}
$$

and also

$$
\begin{aligned}
\lim_{v\to\infty}\sum_{u=0}^{v} S_5(u) = \varphi(1)\Bigg(& a*b*c(0)\sum_{k=0}^{2}\overline{A}(4-k)\cdot a^4*b^2*c(k)\\
&+ a*b*c(0)\cdot\overline{A^4*B^2*C}(2)\cdot\overline{A}(1) - a^4*b^3*c(0)\cdot a_2\cdot\overline{A*C}(2)\\
&- a^2*b^2*c(0)\cdot a_2\cdot\overline{A^2*C}(0)\cdot\overline{A*C}(1) + a^3*b^3*c(0)\cdot a_2\cdot a*c(2)\cdot\overline{A}(0)\\
&- a*b*c(0)\sum_{i_2=0}^{1}\sum_{k=0}^{1-i_2} a^2*b*c(k)\cdot a_{3-i_2-k}\cdot\overline{A*B}(i_2+1)\\
&- a*b*c(0)\sum_{i_1=0}^{2}\sum_{k=0}^{2-i_1} a^3*b^2*c(k)\cdot a_{4-i_1-k}\cdot\overline{A}(i_1)\Bigg).
\end{aligned}
$$

By substituting the derived asymptotic relations into Equation (26), we get

$$
\varphi(0)+\hat{z}_1\varphi(1)+\hat{z}_2\varphi(2)+\hat{z}_3\varphi(3)+\hat{z}_4\varphi(4)+\hat{z}_5\varphi(5) = 6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z, \tag{27}
$$

where coefficients $\hat{z}_1,\ldots,\hat{z}_5$ are defined in (10)–(14). It is important to observe that, $\hat{z}_1,\ldots,\hat{z}_5$ are nothing but collections of terms at $\varphi(1),\ldots,\varphi(5)$ in term

$$
\sum_{k=1}^{5}\varphi(k)A^6*B^3*C^2(v+6-k)+\lim_{v\to\infty}\sum_{u=0}^{v}\sum_{k=1}^{5}S_k(u)
$$

from equality (26).

We observe that equality (27) relates quantities $\varphi(0),\ldots,\varphi(5)$. Setting $u=0$ in (23), we get a relation between $\varphi(1),\ldots,\varphi(6)$. If we suppose $u=1$, then we get relation between $\varphi(1),\ldots,\varphi(7)$, and so on. This allows us to derive an expression for $\varphi(n)$, $n\in\mathbb{N}_0$. We will prove by induction that

$$
\varphi(n) = \beta_n^0\varphi(0)+\beta_n^1\varphi(1)+\beta_n^2\varphi(2)+\beta_n^3\varphi(3)+\beta_n^4\varphi(4)+\gamma_n(6-6\mathbb{E}X-3\mathbb{E}Y-2\mathbb{E}Z) \tag{28}
$$

for all $n\in\mathbb{N}_0$.

If $n \in \{0, \ldots, 4\}$, then relation (28) follows from the definition of coefficients β and γ presented before Theorem 3. If $n = 5$, then the desired relation follows from Equation (27). Now, we prove that (28) is true for $n = N + 1$, assuming that it holds for $n \leqslant N$.

Substituting $u = N - 5$ into equation (7), we get

$$\varphi(N-5) = \sum_{k=1}^{N} a^6 * b^3 * c^2(k)\, \varphi(N-k+1) + a^6 * b^3 * c^2(0)\, \varphi(N+1) - \sum_{k=1}^{5} z_k(N-5)\, \varphi(k).$$

Therefore,

$$\varphi(N+1) = \frac{1}{a^6 * b^3 * c^2(0)} \Big(\varphi(N-5) - \sum_{k=1}^{N} a^6 * b^3 * c^2(k)\, \varphi(N-k+1) + \sum_{k=1}^{5} z_k(N-5)\, \varphi(k) \Big),$$

and by the induction hypothesis we obtain

$$\begin{aligned}
\varphi(N+1) = \frac{1}{a^6 * b^3 * c^2(0)} \Big(& \beta^0_{N-5}\varphi(0) + \beta^1_{N-5}\varphi(1) + \beta^2_{N-5}\varphi(2) + \beta^3_{N-5}\varphi(3) + \beta^4_{N-5}\varphi(4) \\
& + \gamma_{N-5}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \Big(\beta^0_{N-k+1}\varphi(0) + \beta^1_{N-k+1}\varphi(1) \\
& + \beta^2_{N-k+1}\varphi(2) + \beta^3_{N-k+1}\varphi(3) + \beta^4_{N-k+1}\varphi(4) \\
& + \gamma_{N-k+1}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) \Big) + \sum_{k=1}^{5} z_k(N-5)\varphi(k) \Big) \\
= \varphi(0) \Big(& \frac{1}{a^6 * b^3 * c^2(0)} \Big(\beta^0_{N-5} - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \beta^0_{N-k+1} \Big) \Big) \\
& + \varphi(1) \Big(\frac{1}{a^6 * b^3 * c^2(0)} \Big(\beta^1_{N-5} - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \beta^1_{N-k+1} \Big) + z_1(N-5) \Big) \\
& + \varphi(2) \Big(\frac{1}{a^6 * b^3 * c^2(0)} \Big(\beta^2_{N-5} - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \beta^2_{N-k+1} \Big) + z_2(N-5) \Big) \\
& + \varphi(3) \Big(\frac{1}{a^6 * b^3 * c^2(0)} \Big(\beta^3_{N-5} - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \beta^3_{N-k+1} \Big) + z_3(N-5) \Big) \\
& + \varphi(4) \Big(\frac{1}{a^6 * b^3 * c^2(0)} \Big(\beta^4_{N-5} - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \beta^4_{N-k+1} \Big) + z_4(N-5) \Big) \\
& + (6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z) \Big(\frac{1}{a^6 * b^3 * c^2(0)} \Big(\gamma_{N-5} - \sum_{k=1}^{N} a^6 * b^3 * c^2(k) \gamma_{N-k+1} \Big) \Big) \\
& + \varphi(5) \Big(\frac{1}{a^6 * b^3 * c^2(0)} z_5(N-5) \Big).
\end{aligned}$$

Since

$$\varphi(5) = \frac{6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z - \varphi(0) - \hat{z}_1\varphi(1) - \hat{z}_2\varphi(2) - \hat{z}_3\varphi(3) - \hat{z}_4\varphi(4)}{a^5 * b^3 * c^2(0)},$$

and $z_5(N-5) = a(N-4)\, a^5 * b^3 * c^2(0)$, we have

$$\begin{aligned}
\varphi(N+1) = {} & \beta^0_{N+1}\varphi(0) + \beta^1_{N+1}\varphi(1) + \beta^2_{N+1}\varphi(2) + \beta^3_{N+1}\varphi(3) + \beta^4_{N+1}\varphi(4) \\
& + \gamma_{N+1}(6 - 6\mathbb{E}X - 3\mathbb{E}Y - 2\mathbb{E}Z).
\end{aligned}$$

Therefore, Equation (28) holds for all $n \in \mathbb{N}_0$. Now, by seting differences $\varphi(n+5) - \varphi(n), \ldots, \varphi(n+1) - \varphi(n)$ we get the desired formula (16) of Theorem 3. □

Proof of Theorem 4. First we prove that $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 > 1$ implies $\varphi(u) = 0$ for $u \in \mathbb{N}_0$ Similarly to in (20), we obtain that

$$\lim_{n\to\infty} \frac{\eta_n}{n} = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}(R_i - 1) = \frac{6\mathbb{E}X + 3\mathbb{E}Y + 2\mathbb{E}Z - 6}{6} > 0$$

almost surely, and, consequently, $1 - \varphi(u) = \mathbb{P}\big(\sup_{n\geqslant 1} \eta_n \geqslant u\big) = 1$ according to expression (19). The first part of Theorem 4 is proven.

Now let us prove that conditions $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 1$ and $\mathbb{P}(X + Y/2 + Z/3 = 1) < 1$ imply $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$.

First, let us assume that $a(0) \neq 1$. We observe that this condition together with conditions of the second part of Theorem imply that $a(0) > 0$, $a(1) < 1$ and $a(\kappa) > 0$ for some $\kappa \geqslant 1$. This allows us to define a discrete non-negative random vector (X, X^*), with $a^*(0) > a(0)$ and $a^*(\kappa) < a(\kappa)$ for some $\kappa \geqslant 1$. More precisely, we define that

$$\mathbb{P}(X^* = k, X = k) = a(k),\ k \in \mathbb{N}_0, k \neq \kappa; \quad \mathbb{P}(X^* = \kappa, X = \kappa) = a(\kappa) - \frac{\varepsilon}{\kappa},$$
$$\mathbb{P}(X^* = 0, X = \kappa) = \frac{\varepsilon}{\kappa}; \quad \mathbb{P}(X^* = k, X = l) = 0,\ k, l \in \mathbb{N}_0, k \neq l, k \neq 0, l \neq \kappa,$$

where positive ε is sufficiently small; i.e., $\varepsilon \in (0, \min\{(1 - a(0))/\kappa, \kappa a(\kappa)\})$. The distribution of vector (X, X^*) can be expressed by Table 4.

Table 4. Distribution of vector (X, X^*).

$X^*\backslash X$	0	1	2	...	$\kappa - 1$	κ	$\kappa + 1$	...	Σ
0	$a(0)$	0	0	...	0	ε/κ	0	...	$a(0) + \varepsilon/\kappa$
1	0	$a(1)$	0	...	0	0	0	...	$a(1)$
2	0	0	$a(2)$	...	0	0	0	...	$a(2)$
$\vdots$				...				...	$\vdots$
$\kappa - 1$	0	0	0	...	$a(\kappa - 1)$	0	0	...	$a(\kappa - 1)$
κ	0	0	0	...	0	$a(\kappa) - \varepsilon/\kappa$	0	...	$a(\kappa) - \varepsilon/\kappa$
$\kappa + 1$	0	0	0	...	0	0	$a(\kappa + 1)$	...	$a(\kappa + 1)$
$\vdots$				...				...	$\vdots$
Σ	$a(0)$	$a(1)$	$a(2)$	...	$a(\kappa - 1)$	$a(\kappa)$	$a(\kappa + 1)$	...	1

From the table above it is easy to see that $\mathbb{E}X^* = \mathbb{E}X - \varepsilon$ and the net profit condition holds when the model is generated by X^*, Y, Z. For each $i \in \mathbb{N}$, let us define r.v.

$$R_i^* := X_i^* + Y_i \mathbb{1}_{\{2|i\}} + Z_i \mathbb{1}_{\{3|i\}},$$

where X_i^*, Y_i, Z_i are independent copies of non-negative discrete r.v.s X^*, Y, Z, and observe that

$$\mathbb{P}(R_1^* \leqslant R_1) = \mathbb{P}(X_1^* \leqslant X_1) = \mathbb{P}(X_1^* = 0, X_1 = \kappa) + \sum_{k=0}^{\infty} \mathbb{P}(X_1^* = k, X_1 = k) = 1.$$

Similarly,

$$\mathbb{P}(R_1^* + R_2^* \leqslant R_1 + R_2) = \mathbb{P}(X_1^* + X_2^* \leqslant X_1 + X_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \mathbb{P}(X_1^* + k \leqslant X_1 + l)\mathbb{P}(X_2^* = k, X_2 = l)$$
$$= \sum_{k=0,\, k\neq\kappa}^{\infty} \mathbb{P}(X_1^* + k \leqslant X_1 + k)a(k) + \mathbb{P}(X_1^* + \kappa \leqslant X_1 + \kappa)\left(a(\kappa) - \frac{\varepsilon}{\kappa}\right) + \mathbb{P}(X_1^* \leqslant X_1 + \kappa)\frac{\varepsilon}{\kappa} = 1$$

due to inequalities $\mathbb{P}(X_1^* \leqslant X_1) = 1$ and $\mathbb{P}(X_1^* \leqslant X_1) \leqslant \mathbb{P}(X_1^* \leqslant X_1 + \kappa)$.

Using the induction method we can show that

$$\mathbb{P}\left(\sum_{k=1}^{n} R_k^* \leqslant \sum_{k=1}^{n} R_k\right) = 1 \tag{29}$$

for all $n \in \mathbb{N}$. Indeed, by supposing that inequality (29) holds for n up to a certain N, we get

$$\mathbb{P}\left(\sum_{k=1}^{N+1} R_k^* \leqslant \sum_{k=1}^{N+1} R_k\right) = \mathbb{P}\left(\sum_{k=1}^{N+1} X_k^* \leqslant \sum_{k=1}^{N+1} X_k\right) = \sum_{k=0,\, k\neq\kappa}^{\infty} \mathbb{P}\left(\sum_{k=1}^{N} X_k^* \leqslant \sum_{k=1}^{N} X_k\right)a(k)$$
$$+\mathbb{P}\left(\sum_{k=1}^{N} X_k^* \leqslant \sum_{k=1}^{N} X_k\right)\left(a(\kappa) - \frac{\varepsilon}{\kappa}\right) + \mathbb{P}\left(\sum_{k=1}^{N} X_k^* \leqslant \sum_{k=1}^{N} X_k + \kappa\right)\frac{\varepsilon}{\kappa} = 1.$$

Defining r.v. R^* we enlarge its probability of small value comparing to r. v. R. Intuitively, this creates a less dangerous claim and leads to enlarged survival probability. More precisely, the equality (29) implies that

$$\mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k^* - 1)\right\} \leqslant \sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k - 1)\right\}\right) = 1,$$

and, therefore, for all $u \in \mathbb{N}_0$,

$$\begin{aligned}
\varphi^*(u) &:= \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k^* - 1)\right\} < u\right) \\
&= \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k - 1)\right\} < u + \sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k - 1)\right\} - \sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k^* - 1)\right\}\right) \\
&\geqslant \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{k=1}^{n}(R_k - 1)\right\} < u\right) = \varphi(u).
\end{aligned} \tag{30}$$

Now, for the three-risk discrete time model, generated by R^*, we can apply Theorems 2 and 3, and Corollaries 1, 2 and 4. For example, applying Theorem 2 when $\{a(0) \neq 0, b(0) = 0, c(0) = 0\}$, we get $\varphi(0) = 0$, because $0 \leqslant \varphi(0) \leqslant \varphi^*(0) = 6\varepsilon$ and $\varepsilon > 0$ can be as small as we want. Remaining formulas from Theorem 2 lead to $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$.

If $\{a(0) \neq 0, b(0) \neq 0, c(0) \neq 0\}$, then from Theorem 3 for sufficiently large $n \in \mathbb{N}$, we obtain

$$\beta_{5\times 5}\begin{pmatrix} \varphi^*(0) \\ \vdots \\ \varphi^*(4) \end{pmatrix} = \gamma_{5\times 1} \cdot \varepsilon_1,$$

where $\beta_{5\times 5}$ and $\gamma_{5\times 1}$ are matrixes from Theorem 3, and $\varepsilon_1 > 0$ is as small as we want. Consequently, $\varphi^*(0), \ldots, \varphi^*(4)$ also can be as close to the zero as we want. From that, inequalities $\varphi(0) \leqslant \varphi^*(0), \ldots, \varphi(4) \leqslant \varphi^*(4)$ and the Corollary 1 imply that $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$. A similar consideration, we can apply for the remaining cases: $\{a(0) \neq 0, b(0) \neq 0, c(0) = 0, c(1) \neq 0\}$ — using Corollary 2;

$\{a(0) \neq 0, b(0) \neq 0, c(0) = c(1) = 0\}$ — using Corollary 3; $\{a(0) \neq 0, b(0) = 0, c(0) \neq 0\}$ — using Corollary 4. If $a(0) = 1$, then the conditions of the statement ensure that $b(0) \neq 1$ or $c(0) \neq 1$. In such a case, by the similar way we can define r.v.s Y^* or Z^*, making the net profit condition of the "marked" model satisfied, and, in a similar way to that with X^*, we can derive that $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$. The second part of the Theorem 4 is proven.

Now let us prove the third part of the theorem. We prove that condition $\mathbb{P}(X + Y/2 + Z/3 = 1) = 1$ implies $\varphi(0) = 0$ and $\varphi(u) = 1$ for all $u \in \mathbb{N}$. Since r.v.s X, Y, Z are non-negative and discrete, the above condition means that only one of the following three cases is possible:

$$\{X \equiv 1,\, Y \equiv 0,\, Z \equiv 0\},\ \ \{X \equiv 0,\, Y \equiv 2,\, Z \equiv 0\},\ \ \{X \equiv 0,\, Y \equiv 0,\, Z \equiv 3\}.$$

It follows from this that there are only three possible cases for r.v.s R_i:

$$\{R_i \equiv 1, i \in \mathbb{N}\},\ \ \{R_{2i} \equiv 2, R_{2i-1} \equiv 0, i \in \mathbb{N}\},\ \ \{R_{3i} \equiv 3, R_{3i-1} \equiv R_{3i-2} \equiv 0, i \in \mathbb{N}\}.$$

Now the statement follows immediately from the main equation of the model (3). Theorem 4 is proved. □

4. Numerical Examples

In this section we present several numerical examples of computing $\varphi(u, T) := 1 - \psi(u, T)$ and $\varphi(u)$ for three-risk discrete time risk models with various distributions of claims. All calculations were carried out using software MATHEMATICA with sufficient precision, and most of the results were rounded up to three decimal places. For $\varphi(u, T)$ calculation, we used Theorem 1 from [4], which gives that

$$\varphi(u,1) = \sum_{i_1 \leqslant u} a_{i_1},\quad \varphi(u,2) = \sum_{\substack{i_1 \leqslant u \\ i_2 \leqslant u+1-i_1}} a_{i_1} \cdot a * b_{i_2},\quad \varphi(u,3) = \sum_{\substack{i_1 \leqslant u \\ i_2 \leqslant u+1-i_1 \\ i_3 \leqslant u+2-i_1-i_2}} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{i_3},$$

$$\varphi(u,4) = \sum_{\substack{i_1 \leqslant u \\ i_2 \leqslant u+1-i_1 \\ i_3 \leqslant u+2-i_1-i_2 \\ i_4 \leqslant u+3-i_1-i_2-i_3}} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{i_3} \cdot a * b_{i_4},\quad \varphi(u,5) = \sum_{\substack{i_1 \leqslant u \\ i_2 \leqslant u+1-i_1 \\ i_3 \leqslant u+2-i_1-i_2 \\ i_4 \leqslant u+3-i_1-i_2-i_3 \\ i_5 \leqslant u+4-i_1-i_2-i_3-i_4}} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{i_3} \cdot a * b_{i_4} \cdot a_{i_5},$$

$$\varphi(u,6) = \sum_{\substack{i_1 \leqslant u \\ i_2 \leqslant u+1-i_1 \\ i_3 \leqslant u+2-i_1-i_2 \\ i_4 \leqslant u+3-i_1-i_2-i_3 \\ i_5 \leqslant u+4-i_1-i_2-i_3-i_4 \\ i_6 \leqslant u+5-i_1-i_2-i_3-i_4-i_5}} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{i_3} \cdot a * b_{i_4} \cdot a_{i_5} \cdot a * b * c_{i_6},$$

$$\varphi(u,T) = \sum_{\substack{i_1 \leqslant u \\ i_2 \leqslant u+1-i_1 \\ i_3 \leqslant u+2-i_1-i_2 \\ i_4 \leqslant u+3-i_1-i_2-i_3 \\ i_5 \leqslant u+4-i_1-i_2-i_3-i_4 \\ i_6 \leqslant u+5-i_1-i_2-i_3-i_4-i_5}} a_{i_1} \cdot a * b_{i_2} \cdot a * c_{i_3} \cdot a * b_{i_4} \cdot a_{i_5} \cdot a * b * c_{i_6}\, \varphi(u + 6 - i_1 - \ldots - i_6, T - 6),\ T \geqslant 7.$$

Example 1. *Let us consider the three-risk model generated by r.v.s X, Y and Z with the following distributions:*

X	0	1
$\mathbb{P}$	0.99	0.01

;

Y	1	2
$\mathbb{P}$	0.99	0.01

;

Z	1	2
$\mathbb{P}$	0.99	0.01

.

It is not difficult to verify that all conditions of Theorem 2 are satisfied together with the net profit condition $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 0.85$. Using the above formulas for $\varphi(u, T)$ and formulas from Theorem 2 for $\varphi(u) = \varphi(u, \infty)$, we filled Table 5 below.

Table 5. Values of survival probabilities for the model of Example 1.

$T \setminus u$	0	1	2
1	0.990	1	1
2	0.970	1	1
3	0.951	0.999	1
4	0.932	0.998	1
5	0.932	0.998	1
6	0.895	0.995	1
7	0.895	0.995	1
8	0.895	0.995	1
9	0.894	0.995	1
10	0.894	0.995	1
20	0.890	0.994	1
∞	0.890	0.994	1

Example 2. *Suppose that r.v.s X, Y and Z generating the three-risk model have the following distributions:*

$$\begin{array}{c|c|c|c} X & 0 & 1 & 2 \\ \hline \mathbb{P} & 0.92 & 0.07 & 0.01 \end{array}; \quad \begin{array}{c|c|c} Y & 1 & 2 \\ \hline \mathbb{P} & 0.95 & 0.05 \end{array}; \quad \begin{array}{c|c|c} Z & 1 & 2 \\ \hline \mathbb{P} & 0.85 & 0.15 \end{array}.$$

Using the same formulas as in Example 1 and observing that $\mathbb{E}X + \mathbb{E}Y/2 + \mathbb{E}Z/3 = 0.998$ for the given case, we found values of survival probabilities and filled Table 6. An impact of the net profit is well seen when comparing Tables 5 and 6.

Table 6. Values of survival probabilities for the model of Example 2.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	15	20
1	0.920	0.990	1	1	1	1	1	1	1	1	1	1	1
2	0.804	0.969	0.997	1	1	1	1	1	1	1	1	1	1
3	0.629	0.916	0.987	0.998	1	1	1	1	1	1	1	1	1
4	0.550	0.872	0.973	0.996	0.999	1	1	1	1	1	1	1	1
5	0.544	0.868	0.972	0.996	0.999	1	1	1	1	1	1	1	1
6	0.376	0.739	0.919	0.981	0.996	0.999	1	1	1	1	1	1	1
7	0.372	0.735	0.918	0.980	0.996	0.999	1	1	1	1	1	1	1
8	0.364	0.727	0.913	0.978	0.996	0.999	1	1	1	1	1	1	1
9	0.344	0.704	0.899	0.972	0.994	0.999	1	1	1	1	1	1	1
10	0.327	0.682	0.885	0.966	0.992	0.998	1	1	1	1	1	1	1
20	0.228	0.532	0.764	0.896	0.960	0.986	0.995	0.999	1	1	1	1	1
30	0.182	0.444	0.669	0.823	0.913	0.961	0.983	0.994	0.998	0.999	1	1	1
40	0.162	0.402	0.619	0.778	0.880	0.939	0.971	0.987	0.995	0.998	0.999	1	1
50	0.145	0.365	0.572	0.732	0.842	0.913	0.954	0.977	0.989	0.995	0.998	1	1
∞	0.010	0.027	0.045	0.064	0.082	0.100	0.117	0.135	0.151	0.168	0.184	0.261	0.330

Example 3. *Let us consider the three-risk renewal model generated by r.v.s X, Y and Z with distributions:*

$$\begin{array}{c|c|c} X & 0 & 1 \\ \hline \mathbb{P} & 0.5 & 0.5 \end{array}; \quad \begin{array}{c|c|c} Y & 0 & 1 \\ \hline \mathbb{P} & 0.6 & 0.4 \end{array}; \quad \begin{array}{c|c|c} Z & 0 & 1 \\ \hline \mathbb{P} & 0.7 & 0.3 \end{array}.$$

According to Theorem 1 we can suppose that $\varphi(u) = 1$ for a sufficiently large u. Since all conditions of Corollary 1 hold, we can use formulas presented in this corollary by supposing that $\varphi(n) = 1$ for all $n \geqslant 100$. By applying the algorithm described in Corollary 1, we calculate values of survival probability presented in Table 7.

Table 7. Values of $\varphi(u, T)$ and $\varphi(u)$ for the model of Example 3.

$T \setminus u$	0	1	2	3	4	5	6	7	20
1	0.500	1	1	1	1	1	1	1	1
2	0.400	0.900	1	1	1	1	1	1	1
3	0.363	0.848	0.985	1	1	1	1	1	1
4	0.333	0.801	0.965	0.997	1	1	1	1	1
5	0.333	0.801	0.965	0.997	1	1	1	1	1
6	0.307	0.754	0.936	0.988	0.999	1	1	1	1
7	0.307	0.754	0.936	0.988	0.999	1	1	1	1
8	0.301	0.741	0.927	0.984	0.998	1	1	1	1
9	0.296	0.732	0.920	0.981	0.997	1	1	1	1
10	0.290	0.721	0.911	0.977	0.995	0.999	1	1	1
20	0.275	0.687	0.881	0.958	0.987	0.996	0.999	1	1
30	0.269	0.676	0.870	0.950	0.982	0.993	0.998	0.999	1
40	0.268	0.672	0.866	0.947	0.980	0.992	0.997	0.999	1
50	0.267	0.670	0.864	0.946	0.979	0.992	0.997	0.999	1
∞	0.266	0.668	0.862	0.944	0.977	0.991	0.996	0.999	1

Example 4. *We say that r.v. ξ has the Poisson distribution with parameter $\lambda > 0$ ($\xi \sim \mathcal{P}(\lambda)$) if $\mathbb{P}(\xi = k) = e^{-\lambda}\lambda^k/k!$ for each $k \in \mathbb{N}_0$. Suppose that r.v.s generating three-risk discrete time models X, Y and Z have the following Poisson distributions: $X \sim \mathcal{P}(1/3)$, $Y \sim \mathcal{P}(1/4)$ and $Z \sim \mathcal{P}(1/5)$.*

By applying the same algorithm which was used for calculations in Example 3, one can find values of survival probability and fill Table 8.

Example 5. *Let us consider the three-risk discrete time model generated by r.v.s X, Y and Z with the following distributions:*

$$\begin{array}{c|c|c} X & 0 & 1 \\ \hline \mathbb{P} & 0.6 & 0.4 \end{array}; \quad \begin{array}{c|c|c} Y & 0 & 1 \\ \hline \mathbb{P} & 0.7 & 0.3 \end{array}; \quad \begin{array}{c|c|c} Z & 1 & 2 \\ \hline \mathbb{P} & 0.9 & 0.1 \end{array}.$$

According to Theorem 1 we can suppose that $\varphi(u) = 1$ for sufficiently large u, and the model described in this example satisfies conditions of Corollary 2. Hence, we can use the algorithm presented in this corollary by supposing that $\varphi(n) = 1$ for all $n \geqslant 100$, and we can calculate values of survival probability which are presented in Table 9.

Table 8. Values of survival probabilities for the model with Poissonian random variables (r.v.s) of Example 4.

$T \setminus u$	0	1	2	3	4	5	6
1	0.717	0.955	0.995	1	1	1	1
2	0.633	0.912	0.983	0.997	1	1	1
3	0.603	0.890	0.975	0.995	0.999	1	1
4	0.584	0.875	0.967	0.992	0.998	1	1
5	0.580	0.871	0.965	0.991	0.998	1	1
6	0.570	0.861	0.960	0.989	0.997	0.999	1
7	0.568	0.860	0.959	0.988	0.997	0.999	1
8	0.566	0.857	0.957	0.988	0.997	0.999	1
9	0.564	0.855	0.956	0.987	0.996	0.999	1
10	0.562	0.854	0.955	0.986	0.996	0.999	1
20	0.559	0.850	0.952	0.985	0.995	0.999	1
∞	0.559	0.850	0.952	0.985	0.995	0.999	1

Table 9. Values of survival probabilities for the model described in Example 5.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20
1	0.6	1	1	1	1	1	1	1	1	1	1	1
2	0.528	0.952	1	1	1	1	1	1	1	1	1	1
3	0.391	0.816	0.968	0.998	1	1	1	1	1	1	1	1
4	0.360	0.776	0.950	0.994	1	1	1	1	1	1	1	1
5	0.360	0.776	0.950	0.994	1	1	1	1	1	1	1	1
6	0.301	0.684	0.891	0.973	0.995	0.999	1	1	1	1	1	1
7	0.301	0.684	0.891	0.973	0.995	0.999	1	1	1	1	1	1
8	0.296	0.675	0.884	0.970	0.994	0.999	1	1	1	1	1	1
9	0.277	0.642	0.857	0.955	0.989	0.998	1	1	1	1	1	1
10	0.271	0.630	0.846	0.949	0.986	0.997	0.999	1	1	1	1	1
20	0.230	0.548	0.764	0.890	0.954	0.982	0.994	0.998	0.999	1	1	1
30	0.209	0.504	0.714	0.847	0.923	0.963	0.983	0.993	0.997	0.999	1	1
40	0.201	0.486	0.692	0.826	0.906	0.951	0.976	0.988	0.955	0.998	0.999	1
50	0.196	0.473	0.676	0.810	0.892	0.941	0.969	0.984	0.992	0.996	0.998	1
∞	0.178	0.433	0.623	0.754	0.840	0.896	0.932	0.956	0.971	0.981	0.988	1

Example 6. *Suppose that three-risk discrete time model is generated by r.v.s X, Y and Z with distributions:*

$$\begin{array}{c|c|c} X & 0 & 1 \\ \hline \mathbb{P} & 0.8 & 0.2 \end{array}; \quad \begin{array}{c|c|c} Y & 0 & 1 \\ \hline \mathbb{P} & 0.8 & 0.2 \end{array}; \quad \begin{array}{c|c|c} Z & 2 & 3 \\ \hline \mathbb{P} & 0.99 & 0.01 \end{array}.$$

To find values of survival probabilities of the described model, we use formulas from Corollary 3 and formulas presented at the beginning of Section 4. To obtain the initial value of ultimate time survival probability $\varphi(0)$, we suppose in Corollary 3 that $\varphi(n) = 1$ for all $n \geqslant 100$. The values of survival probabilities, we present in Table 10 below.

Table 10. Values of survival probabilities for the model described in Example 6.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20
1	0.8	1	1	1	1	1	1	1	1	1	1	1
2	0.768	0.992	1	1	1	1	1	1	1	1	1	1
3	0.406	0.815	0.971	0.998	1	1	1	1	1	1	1	1
4	0.389	0.799	0.965	0.997	1	1	1	1	1	1	1	1
5	0.389	0.799	0.965	0.997	1	1	1	1	1	1	1	1
6	0.297	0.681	0.902	0.978	0.997	1	1	1	1	1	1	1
7	0.297	0.681	0.902	0.978	0.997	1	1	1	1	1	1	1
8	0.296	0.679	0.901	0.978	0.996	1	1	1	1	1	1	1
9	0.263	0.626	0.861	0.960	0.991	0.998	1	1	1	1	1	1
10	0.259	0.620	0.856	0.958	0.990	0.998	1	1	1	1	1	1
20	0.201	0.507	0.748	0.888	0.956	0.985	0.995	0.999	1	1	1	1
30	0.170	0.440	0.669	0.821	0.910	0.959	0.982	0.993	0.997	0.999	1	1
40	0.157	0.410	0.631	0.785	0.882	0.939	0.971	0.987	0.994	0.998	0.999	1
50	0.148	0.387	0.601	0.754	0.856	0.920	0.958	0.979	0.990	0.995	0.998	1
∞	0.097	0.257	0.410	0.534	0.631	0.709	0.770	0.818	0.856	0.886	0.910	0.991

Example 7. *Let us consider the three-risk discrete time model generated by r.v.s X, Y and Z with the following distributions:*

$$\begin{array}{c|c|c} X & 0 & 1 \\ \hline \mathbb{P} & 0.7 & 0.3 \end{array}; \quad \begin{array}{c|c|c} Y & 1 & 2 \\ \hline \mathbb{P} & 0.8 & 0.2 \end{array}; \quad \begin{array}{c|c|c} Z & 0 & 1 \\ \hline \mathbb{P} & 0.9 & 0.1 \end{array}.$$

We observe that in this last our example, conditions of Corollary 4 are satisfied. Using the algorithm presented in this corollary and the above formulas for finite time survival probabilities we fill Table 11. We suppose that $\varphi(n) = 1$ for $n \geqslant 100$ in system of Corollary 4 to calculate the initial values of survival probability $\varphi(0)$ and $\varphi(1)$.

Table 11. Values of survival probabilities for the model described in Example 7.

$T \setminus u$	0	1	2	3	4	5	6	7	8	9	10	20
1	0.7	1	1	1	1	1	1	1	1	1	1	1
2	0.392	0.826	0.982	1	1	1	1	1	1	1	1	1
3	0.380	0.813	0.977	0.999	1	1	1	1	1	1	1	1
4	0.307	0.718	0.933	0.990	0.999	1	1	1	1	1	1	1
5	0.307	0.718	0.933	0.990	0.999	1	1	1	1	1	1	1
6	0.266	0.651	0.887	0.973	0.995	0.999	1	1	1	1	1	1
7	0.266	0.651	0.887	0.973	0.995	0.999	1	1	1	1	1	1
8	0.246	0.616	0.859	0.959	0.991	0.998	1	1	1	1	1	1
9	0.244	0.613	0.856	0.957	0.990	0.998	1	1	1	1	1	1
10	0.299	0.583	0.829	0.942	0.984	0.996	0.999	1	1	1	1	1
20	0.191	0.500	0.740	0.878	0.948	0.980	0.993	0.998	0.999	1	1	1
30	0.174	0.460	0.692	0.835	0.917	0.961	0.983	0.993	0.997	0.999	1	1
40	0.165	0.439	0.664	0.809	0.896	0.946	0.973	0.987	0.994	0.998	0.999	1
50	0.159	0.424	0.645	0.789	0.879	0.993	0.964	0.982	0.991	0.996	0.998	1
∞	0.140	0.374	0.574	0.713	0.806	0.869	0.911	0.940	0.960	0.973	0.982	0.9996

5. Discussion

The proven statements show that for the three-risk model, the values of ruin probabilities for any value of the initial surplus can be found. However, if the value of this surplus is large and the claims of the model have infinite supports, the necessary calculations take time. In such a case, we can use also the upper estimate of ruin probability, which usually decreases with increasing initial capital. The useful estimates for the nonhomogeneous models we can find in [27–31] among others. For instance, results of [28,29] imply that

$$\psi(u) \leqslant c_1 \exp\{-c_2 u\},\ u \geqslant 0,$$

for all above examples with a positive constants c_1, and c_2 depending on the numerical characteristics of the random claims X, Y and Z, generating a three-risk discrete time model. On the other hand, the proven statements ignite the hope that similar algorithms can be found to calculate values of the ultimate time survival probability for the general multi-risk discrete time model. As already mentioned in introduction, if for the Andersen's model (1), we set $c = 1$, $\theta \equiv 1$ and all r.v.s $Z_1, Z_2, \ldots$ are integer valued but not necessarily identically distributed, then we have the nonhomogeneous discrete time risk models. The nonhomogeneous risk models are much closer to the real life, since it is hardly possible that insurers always face the same identically distributed claims. Also, such type models may be applied in various other industries, where increase, decrease and randomness occur: some population development; risk management, where all individuals have savings, get incomes and face unexpected expenses; modelling of rare events or catastrophes.

6. Conclusions

In this paper, we derived the formulas for the calculation of exact values for the ultimate time survival probability, the probability that the random process

$$W(t) = u + t - \sum_{i=1}^{\lfloor t \rfloor} X_i - \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j - \sum_{k=1}^{\lfloor t/3 \rfloor} Z_k$$

is positive for all $t \in \{1, 2, \ldots\}$. Comparing this research to the close previous studies in [4,6,7], where the models

$$W(t) = u + t - \sum_{i=1}^{\lfloor t \rfloor} X_i - \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j,\ W(t) = u + t - \sum_{i=1}^{\lfloor t \rfloor} X_i$$

were investigated, it is easy to see how quickly things get complicated. There occurs a significant dependency on distributions of random claims; the expressions of $\varphi(u)$ are a lot longer and more complicated; the proofs of the non-singularity of certain types of recurrent matrices are still open questions. It is hardly possible to give an exact expressions on $\varphi(u)$ for n-risk model if $n \geqslant 4$. For that, a new ideas and methods are needed.

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