

On the mean square of the periodic zeta-function. II

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Abstract. In the paper, the error term in the Atkinson type formula for the periodic zeta-function in the critical strip is considered, and an asymptotic formula for its mean square is obtained. This formula generalizes that proved for the Riemann zeta-function.

Keywords: periodic zeta-function, Riemann zeta-function, the Atkinson formula.

1 Introduction

Let $s = \sigma + it$ be a complex variable, and $\lambda \in \mathbb{R}$ be a fixed parameter. The periodic zeta-function $\zeta_\lambda(s)$ is defined for $\sigma > 1$ by the series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s}.$$

For $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function $\zeta(s)$, thus, can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. For $\lambda \notin \mathbb{Z}$, $\zeta_\lambda(s)$ is analytically continued to an entire function.

In view of the periodicity of the coefficients $e^{2\pi i \lambda m}$, we may assume that $0 < \lambda \leq 1$.

The moment problem for zeta-functions plays an important role in analytic number theory. It consists of finding the asymptotics or estimates for the mean values of powers of the modulus of zeta-functions.

Since $\zeta_\lambda(s) = e^{2\pi i \lambda} L(\lambda, 1, s)$, where for $0 < \lambda \leq 1$, $L(\lambda, \alpha, s)$ denotes the Lerch zeta-function defined for $\sigma > 1$ by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and by analytic continuation elsewhere, we have that

$$\int_0^T |\zeta_\lambda(\sigma + it)|^{2k} dt = \int_0^T |L(\lambda, 1, \sigma + it)|^{2k} dt.$$

Therefore, the moment problem for $\zeta_\lambda(s)$ reduces to that for the Lerch zeta-function. Let $\zeta(s, \alpha)$ denote the Hurwitz zeta-function,

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1.$$

Then, for example, if σ is fixed, $1/2 < \sigma < 1$, by Theorem 4.2.1 of [11], we have that

$$\begin{aligned} \int_1^T |\zeta_\lambda(\sigma + it)|^2 dt &= \zeta(2\sigma)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \zeta(2-2\sigma, \lambda) T^{2-2\sigma} \\ &\quad + O(T^{1-\sigma} \log T) + O(T^{\sigma/2}). \end{aligned}$$

Suppose that λ is a rational number, $\lambda = a/q$ with $a, q \in \mathbb{N}$, $1 \leq a \leq q$. In [8, 9] and [3], the mean square

$$I_\sigma(q, T) \stackrel{\text{def}}{=} \sum_{a=1}^q \int_0^T |\zeta_{a/q}(\sigma + it)|^2 dt$$

with $1/2 \leq \sigma < 1$ has been studied, and the Atkinson-type formula for the error term of the mean square formula has been obtained. Let

$$E(q, T) = I_{1/2}(q, T) - qT \left(\log \frac{qT}{2\pi} - 2\gamma_0 - 1 \right),$$

where γ_0 is the Euler constant. Then in [2], the formula for

$$\int_2^T E^2(q, T) dt$$

has been obtained. Let $1/2 < \sigma < 3/4$ be fixed. The aim of this note is to obtain a formula for the mean square of

$$E_\sigma(q, T) = I_\sigma(q, T) - q\zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma} (qT)^{2-2\sigma},$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function. Let

$$\sigma_z(m) = \sum_{d|m} d^z, \quad z \in \mathbb{C}.$$

Theorem 1. *Let σ , $1/2 < \sigma < 3/4$, be fixed. Then, for $T \rightarrow \infty$ and $q \leq T^{1-4\sigma/3}$,*

$$\begin{aligned} \int_2^T E_\sigma^2(q, t) dt &= 2(5-4\sigma)^{-1} (2\pi)^{2\sigma-3/2} q^{3/2-2\sigma} T^{5/2-2\sigma} \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} \\ &\quad + O(q^{11/4-2\sigma} T^{7/4-\sigma} \log T). \end{aligned}$$

For $q = 1$, we have the Matsumoto result [12], Theorem 3.

The case $3/4 \leq \sigma < 1$ will be presented in a subsequent paper.

2 The Atkinson formula

In [1], Atkinson proved a formula for the function

$$E(T) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt - T \log \frac{T}{2\pi} - (2\gamma_0 - 1)T.$$

This formula is a sum of two finite sums involving some elementary functions and of the error term $O(\log^2 T)$. The proof of the Atkinson formula is also given in [5]. Matsumoto and Meurman [12, 13] gave the Atkinson-type formula for the error term in the mean square formula of the Riemann-zeta function in the critical strip. Modified versions of the Atkinson formula were obtained in [6, 7, 15, 16].

The Atkinson formula is a powerful tool for the investigation of the error term in the mean square formula, and for various allied applications. For example, Heath-Brown [4] applied the Atkinson formula for estimation of the twelfth moment

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{12} dt.$$

The Atkinson-type formula is also known for Dirichlet L -functions (see [16]) and for the function $\zeta_{a/q}(s)$ (see [3, 8, 9]). In the latter case, the dependence on q is important. For the proof of the main theorem of this paper, we will use a formula obtained in [3]. For its statement, we need some notation. Let $c_1 T < N < c_2 T$ with some positive constants $c_1 < c_2$,

$$N_1 = N_1(q, T, N) = q \left(\frac{T}{2\pi} + \frac{qN}{2} - \left(\left(\frac{qN}{2} \right)^2 + \frac{qNT}{2\pi} \right)^{1/2} \right)$$

and

$$\operatorname{arsinh}(x) = \log(x + \sqrt{1 + x^2}).$$

Define the sums

$$\begin{aligned} \sum_1(q, T) &= 2^{\sigma-1} q^{1-\sigma} \left(\frac{T}{\pi} \right)^{1/2-\sigma} \sum_{m \leq N} \frac{(-1)^{qm} \sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \\ &\quad \times \left(\operatorname{arsinh} \left(\sqrt{\frac{\pi q m}{2T}} \right) \right)^{-1} \left(\frac{T}{2\pi q m} + \frac{1}{4} \right)^{-1/4} \\ &\quad \times \cos \left(2T \operatorname{arsinh} \left(\sqrt{\frac{\pi q m}{2T}} \right) + \sqrt{2\pi q m T + T^2 q^2 m^2} - \frac{\pi}{4} \right) \end{aligned}$$

and

$$\begin{aligned} \sum_2(q, T) = & -2q^{1-\sigma} \left(\frac{T}{2\pi} \right)^{1/2-\sigma} \sum_{m \leq N_1} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\log \left(\frac{qT}{2\pi m} \right) \right)^{-1} \\ & \times \cos \left(T \log \left(\frac{qT}{2\pi m} \right) - T + \frac{\pi}{4} \right). \end{aligned}$$

Then we have the following result [3].

Lemma 1. *Suppose that $1/2 < \sigma < 3/4$. Then, for $q \leq T$,*

$$E_\sigma(q, T) = \sum_1(q, T) + \sum_2(q, T) + R(q, T),$$

where $R(q, T) = O(q^{7/4-\sigma} \log T)$ with the O – constant depending only on σ .

3 Some estimates

In this section, we present some known estimates that will be applied in the proof of Theorem 1.

Lemma 2. *Let $g_j(t)$, $j = 1, \dots, k$, and $f(t)$ be real-valued continuous monotonic functions on $[a, b]$, and let $f(t)$ have a continuous monotonic derivative on $[a, b]$. If $|g_j(t)| \leq M_j$, $j = 1, \dots, k$, and $|f'(t)| \geq M_0^{-1}$ on $[a, b]$, then*

$$\left| \int_a^b \prod_{j=1}^k g_j(t) e^{2\pi i f(t)} dt \right| \leq 2^{k+3} \prod_{j=0}^k M_j.$$

The lemma is Lemma 15.3 of [5].

Lemma 3. *The estimates*

$$\sum_{m \leq x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1} \quad \text{and} \quad \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1}$$

are true.

Proof. Since $1 - 2\sigma < 0$, the series over primes

$$\sum_p \frac{|\sigma_{1-2\sigma}^2(p) - 1| \log p}{p}$$

is convergent. Therefore, from general mean values theorems for multiplicative functions (see, for example, [10]) we obtain that

$$\sum_{m \leq x} \sigma_{1-2\sigma}^2(m) \ll x.$$

This and summing by parts give the estimate of the lemma.

The second estimate of the lemma is obtained similarly. □

Lemma 4. For every $\varepsilon > 0$,

$$\sum_{m \leq x} \sum_{\substack{n \leq x \\ m \neq n}} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} |\sqrt{m} - \sqrt{n}|^{-1} \ll x^{2\sigma-1+\varepsilon}.$$

Proof. We have that

$$\begin{aligned} & \sum_{m \leq x} \sum_{\substack{n \leq x \\ m \neq n}} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} |\sqrt{m} - \sqrt{n}|^{-1} \\ & \ll \sum_{n < m \leq x} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} (\sqrt{m} - \sqrt{n})^{-1} \ll \sum_{n \leq m/2} + \sum_{n > m/2}. \end{aligned}$$

We observe that, for $n \leq m/2$, we have that $m - n \geq m/2$, thus, $(m - n)^{-1} \ll m^{-1}$ for $n \leq m/2$. Therefore, using Lemma 3 and partial summation, we find

$$\begin{aligned} \sum_{n \leq m/2} & \ll \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{5/4-\sigma}} \sum_{n \leq m/2} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} (\sqrt{m} + \sqrt{n})(m - n)^{-1} \\ & \ll \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{3/4-\sigma}} \sum_{n \leq m/2} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} (m - n)^{-1} \\ & \ll \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} \sum_{n \leq m/2} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} \\ & \ll \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} m^{\sigma-1/4} = \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1}, \end{aligned}$$

and, for every $\varepsilon > 0$,

$$\sum_{n > m/2} \ll \sum_{m \leq x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \sum_{m/2 < n < m} \frac{\sigma_{1-2\sigma}(n)}{m - n} \ll x^{2\sigma-1+\varepsilon}. \quad \square$$

Lemma 5. We have

$$\sum_{m \leq x} \sum_{\substack{n \leq x \\ m \neq n}} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \left| \log \frac{m}{n} \right|^{-1} \ll x^{2\sigma} \log x.$$

Proof. In view of Lemma 3 and the estimate [5],

$$\sum_{m \leq x} \left| \log \frac{m}{n} \right|^{-1} \ll x + n \log x,$$

we find that the considered sum is estimated as

$$\begin{aligned}
&\ll \sum_{\substack{m \leq x \\ m \neq n}} \sum_{n \leq x} \left(\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} + \frac{\sigma_{1-2\sigma}^2(n)}{n^{2-2\sigma}} \right) \left| \log \frac{m}{n} \right|^{-1} \\
&\ll \sum_{\substack{m \leq x \\ m \neq n}} \sum_{n \leq x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \left| \log \frac{m}{n} \right|^{-1} \ll \sum_{m \leq x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \sum_{n \leq x} \left| \log \frac{m}{n} \right|^{-1} \\
&\ll x^{2\sigma} + \log x \sum_{m \leq x} \frac{\sigma_{1-2\sigma}^2(m)}{m^{1-2\sigma}} \ll x^{2\sigma} \log x. \quad \square
\end{aligned}$$

4 Proof of Theorem 1

By Lemma 1, we have that

$$\begin{aligned}
\int_T^{2T} E_\sigma^2(q, t) dt &= \int_T^{2T} \sum_1^2(q, T) dt + 2 \int_T^{2T} \sum_1(q, T) \left(\sum_2(q, T) + R(q, t) \right) dt \\
&\quad + \int_T^{2T} \left(\sum_2(q, T) + R(q, t) \right)^2 dt. \quad (1)
\end{aligned}$$

For brevity, we write

$$g_1(t, qm) = \left(\operatorname{arsinh} \left(\sqrt{\frac{\pi qm}{2t}} \right) \right)^{-1}, \quad g_2(t, qm) = \left(\frac{t}{2\pi qm} + \frac{1}{4} \right)^{-1/4}$$

and

$$f(t, qm) = 2t \operatorname{arsinh} \left(\sqrt{\frac{\pi qm}{2t}} \right) + \sqrt{2\pi qmt + t^2 q^2 m^2} - \frac{\pi}{4}.$$

Then, taking $N = T$ in Lemma 1, we have that

$$\begin{aligned}
\sum_1(q, t) &= 2^{\sigma-1} q^{1-\sigma} \left(\frac{t}{\pi} \right)^{1/2-\sigma} \sum_{m \leq N} \frac{(-1)^{qm} \sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \\
&\quad \times g_1(t, qm) g_2(t, qm) \cos(f(t, qm)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_1^2(q, t) &= 2^{2\sigma-2} q^{2-2\sigma} \left(\frac{t}{\pi} \right)^{1-2\sigma} \sum_{m \leq T} \sum_{n \leq T} \frac{(-1)^{qm+qn} \sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\
&\quad \times g_1(t, qm) g_1(t, qn) g_2(t, qm) g_2(t, qn) \cos(f(t, qm)) \cos(f(t, qn))
\end{aligned}$$

$$\begin{aligned}
&= 2^{2\sigma-3} q^{2-2\sigma} \left(\frac{t}{\pi}\right)^{1-2\sigma} \sum_{m \leq T} \sum_{n \leq T} \frac{(-1)^{qm+qn} \sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\
&\quad \times g_1(t, qm) g_1(t, qn) g_2(t, qm) g_2(t, qn) \\
&\quad \times (\cos(f(t, qm) + f(t, qn)) + \cos(f(t, qm) - f(t, qn))). \quad (2)
\end{aligned}$$

Let $S_1(q, t)$ be the part of $\sum_1^2(q, t)$ in (2) with $m = n$. Then

$$\begin{aligned}
&\int_T^{2T} S_1(q, t) dt \\
&= 2^{2\sigma-3} q^{2-2\sigma} \pi^{2\sigma-1} \operatorname{Re} \sum_{m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \int_T^{2T} t^{1-2\sigma} g_1^2(t, qm) g_2^2(t, qm) e^{2if(t, qm)} dt \\
&\quad + 2^{2\sigma-3} q^{2-2\sigma} \pi^{2\sigma-1} \sum_{m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \int_T^{2T} t^{1-2\sigma} g_1^2(t, qm) g_2^2(t, qm) dt. \quad (3)
\end{aligned}$$

It is not difficult to see that, for $t \in [T, 2T]$,

$$g_1(t, qm) \ll \begin{cases} \sqrt{T/(mq)} & \text{if } m \leq T/(4q), \\ 1 & \text{if } m > T/(4q) \end{cases} \quad (4)$$

and

$$g_2(t, qm) \ll \begin{cases} (mq/T)^{1/4} & \text{if } m \leq T/(4q), \\ 1 & \text{if } m > T/(4q). \end{cases} \quad (5)$$

Moreover, for $t \in [T, 2T]$,

$$|f'(t, qm)| \gg \begin{cases} \sqrt{mq/T} & \text{if } m \leq T/(4q), \\ 1 & \text{if } m > T/(4q). \end{cases} \quad (6)$$

In view of estimates (4)–(6), using Lemmas 2 and 3, we find that

$$\begin{aligned}
&2^{2\sigma-3} q^{2-2\sigma} \pi^{2\sigma-1} \operatorname{Re} \sum_{m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \int_T^{2T} t^{1-2\sigma} g_1^2(t, qm) g_2^2(t, qm) e^{2if(t, qm)} dt \\
&\ll q^{1-2\sigma} T^{2-2\sigma} \sum_{m \leq T/4q} \frac{\sigma_{1-2\sigma}^2(m)}{m^{3-2\sigma}} + q^{2-2\sigma} T^{1-2\sigma} \sum_{T/4q < m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \\
&\leq q^{1-2\sigma} T^{2-2\sigma} + q^{2-2\sigma}. \quad (7)
\end{aligned}$$

For the second term in the right-hand side of (3), we use more precise estimates for the functions $g_1(t, qm)$ and $g_2(t, qm)$, namely, for $t \in [T, 2T]$ (see [2]),

$$g_1^2(t, qm) = \begin{cases} 2t/(\pi mq) + O(1) & \text{if } m \leq T/(4q), \\ O(1) & \text{if } m > T/(4q), \end{cases}$$

and

$$g_2^2(t, qm) = \begin{cases} (2\pi qm/t)^{1/2} + O((qm/t)^{3/2}) & \text{if } m \leq T/(4q), \\ O(1) & \text{if } m > T/(4q). \end{cases}$$

Hence, for $t \in [T, 2T]$,

$$\begin{aligned} & t^{1-2\sigma} g_1(t, qm) g_2(t, qm) \\ &= \begin{cases} 2^{2/3} t^{3/2-2\sigma} (\pi mq)^{-1/2} + O(t^{1/2-2\sigma} (qm)^{1/2}) & \text{if } m \leq T/(4q), \\ O(t^{1-2\sigma}) & \text{if } m > T/(4q). \end{cases} \end{aligned}$$

Therefore, by application of Lemma 3,

$$\begin{aligned} & 2^{2\sigma-3} q^{2-2\sigma} \pi^{2\sigma-1} \sum_{m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \int_T^{2T} t^{1-2\sigma} g_1^2(t, qm) g_2^2(t, qm) dt \\ &= 2^{2\sigma-3/2} \pi^{2\sigma-3/2} q^{3/2-2\sigma} \sum_{m \leq T/(4q)} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} \int_T^{2T} t^{3/2-2\sigma} dt \\ &+ O\left(q^{5/2-2\sigma} \sum_{m \leq T/(4q)} \frac{\sigma_{1-2\sigma}^2(m)}{m^{3/2-2\sigma}} \int_T^{2T} t^{1/2-2\sigma} dt\right) \\ &+ O\left(q^{2-2\sigma} \sum_{T/(4q) < m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \int_T^{2T} t^{1-2\sigma} dt\right) \\ &= \frac{2^{2\sigma-3/2} \pi^{2\sigma-3/2} q^{3/2-2\sigma}}{5/2-2\sigma} + \sum_{m \leq T/(4q)} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} ((2T)^{5/2-2\sigma} - T^{5/2-2\sigma}) \\ &+ O\left(q^{5/2-2\sigma} T^{3/2-2\sigma} \sum_{m \leq T/(4q)} \frac{\sigma_{1-2\sigma}^2(m)}{m^{3/2-2\sigma}}\right) \\ &+ O\left(q^{2-2\sigma} T^{2-2\sigma} \sum_{T/(4q) < m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}}\right) \\ &= \frac{2^{2\sigma-3/2} \pi^{2\sigma-3/2} q^{3/2-2\sigma}}{5/2-2\sigma} + \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} ((2T)^{5/2-2\sigma} - T^{5/2-2\sigma}) \\ &+ O(q^{2-2\sigma} T). \end{aligned} \tag{8}$$

Denote by $S_2(q, t)$ the part of $\sum_1^2(q, t)$ in (2) with $m \neq n$. In this case, for $t \in [T, 2T]$, we use the estimate

$$f'(t, qm) \pm f'(t, qn) \gg \begin{cases} \sqrt{q/T} |\sqrt{m} \pm \sqrt{n}| & \text{if } m \leq T/(4q), n \leq T/(4q), \\ |\log(m/n)| & \text{if } m > T/(4q), n > T/(4q), \\ 1 & \text{if } m \leq T/(4q), n > T/(4q) \text{ or } m > T/(4q), n \leq T/(4q). \end{cases}$$

This, Lemmas 2–5 and estimates (4), (5) give the estimate

$$\begin{aligned} \int_T^{2T} S_2(q, t) dt &\ll q^{1-2\sigma} T^{2-2\sigma} \sum_{\substack{m \leq T/(4q) \\ m \neq n}} \sum_{\substack{n \leq T/(4q)}} \frac{\sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} |\sqrt{m} - \sqrt{n}|^{-1} \\ &\quad + q^{2-2\sigma} T^{1-2\sigma} \sum_{\substack{T/(4q) < m \leq T \\ m \neq n}} \sum_{\substack{T/(4q) < n \leq T}} \frac{\sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma} |\log(m/n)|} \\ &\quad + q^{7/4-2\sigma} T^{5/4-2\sigma} \sum_{\substack{m \leq T \\ m \neq n}} \sum_{\substack{n \leq T}} \frac{\sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\ &\ll q^{2-4\sigma} T^{1+\varepsilon} + q^{2-2\sigma} T \log T + q^{7/4-2\sigma} T^{1/4} \ll q^{2-2\sigma} T^{1+\varepsilon}. \end{aligned}$$

This, (3), (7) and (8) show that

$$\begin{aligned} \int_T^{2T} \sum_1^2(q, t) dt &= 2^{2\sigma-3/2} \left(\frac{5}{2} - 2\sigma\right)^{-1} \pi^{2\sigma-3/2} q^{3/2-2\sigma} \\ &\quad \times \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} ((2T)^{5/2-2\sigma} - T^{5/2-2\sigma}) + O(q^{2-2\sigma} T^{1+\varepsilon}). \quad (9) \end{aligned}$$

Now we will estimate

$$\int_T^{2T} \sum_2^2(q, t) dt.$$

By the definition of $\sum_2(q, t)$, we have that

$$\begin{aligned} \sum_2^2(q, t) &= 4q^{2-2\sigma} \left(\frac{t}{2\pi}\right)^{1-2\sigma} \sum_{m \leq N_1} \sum_{n \leq N_1} \frac{\sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\ &\quad \times \left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1} \left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1} \cos(g(t, qm)) \cos(g(t, qn)) \end{aligned}$$

$$\begin{aligned}
&= 2q^{2-2\sigma} \left(\frac{t}{2\pi} \right)^{1-2\sigma} \sum_{m \leq N_1} \sum_{n \leq N_1} \frac{\sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\
&\quad \times \left(\log \left(\frac{qt}{2\pi m} \right) \right)^{-1} \left(\log \left(\frac{qt}{2\pi n} \right) \right)^{-1} \\
&\quad \times (\cos(g(t, qm) + g(t, qn)) + \cos(g(t, qm) - g(t, qn))), \tag{10}
\end{aligned}$$

where

$$g(t, qm) = t \log \left(\frac{qt}{2\pi m} \right) - t + \frac{\pi}{4},$$

and

$$N_1 = N_1(q, t, T) = q \left(\frac{t}{2\pi} + \frac{qT}{2} - \left(\left(\frac{qT}{2} \right)^2 + \left(\frac{qTt}{2\pi} \right)^{1/2} \right) \right).$$

It is not difficult to see, that

$$N_1 \leq \frac{t^2}{4\pi^2 T}.$$

Therefore, for $m \leq N_1$ and $t \in [T, 2T]$,

$$\frac{qt}{2\pi m} \geq \pi q > 3.$$

This implies the estimate

$$\left(\log \frac{qt}{2\pi m} \right)^{-1} \ll 1. \tag{11}$$

Moreover, by the definition of $g(t, qm)$, we find that

$$(g(t, qm) \pm g(t, qn))' \gg \left| \log \frac{m}{n} \right|. \tag{12}$$

Denote by $Z_1(q, t)$ the part of $\sum_1^2(q, t)$ in (10) with $m \neq n$, and let $T_1 \geq T$ be such that $N_1(q, t, T) \geq \max(m, n)$ for $t \geq T_1$. Then we deduce from Lemmas 2 and 5 and from estimates (10)–(12) that

$$\begin{aligned}
\int_T^{2T} Z_1(q, t) dt &= 2q^{2-2\sigma} (2\pi)^{2\sigma-1} \sum_{m \leq N_1(q, t, T)} \sum_{n \leq N_1(q, t, T)} \frac{\sigma_{1-2\sigma}(m) \sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}} \\
&\quad \times t^{1-2\sigma} \left(\log \left(\frac{qt}{2\pi m} \right) \right)^{-1} \left(\log \left(\frac{qt}{2\pi n} \right) \right)^{-1} \\
&\quad \times (\cos(g(t, qm) + g(t, qn)) + \cos(g(t, qm) - g(t, qn))) dt
\end{aligned}$$

$$\begin{aligned}
&= 2q^{2-2\sigma}(2\pi)^{2\sigma-1} \sum_{m \leq N_1(q, 2T, T)} \sum_{n \leq N_1(q, 2T, T)} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}} \\
&\quad \times \int_{T_1}^{2T} t^{1-2\sigma} \left(\log \left(\frac{qt}{2\pi m} \right) \right)^{-1} \left(\log \left(\frac{qt}{2\pi n} \right) \right)^{-1} \\
&\quad \times (\cos(g(t, qm) + g(t, qn)) + \cos(g(t, qm) - g(t, qn))) dt \\
&\ll q^{2-2\sigma} T^{1-2\sigma} \sum_{m \leq T} \sum_{n \leq T} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}} \left| \log \frac{m}{n} \right|^{-1} \\
&\ll q^{2-2\sigma} T \log T. \tag{13}
\end{aligned}$$

Now let $Z_2(q, t)$ be the part of $\sum_2^2(q, t)$ in (10) with $m = n$. Then by Lemma 3, we find that

$$\int_T^{2T} Z_2(q, t) dt \ll q^{2-2\sigma} T^{1-2\sigma} \sum_{m \leq T} \frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \ll q^{2-2\sigma} T.$$

This, (10) and (3) show that

$$\int_T^{2T} \sum_2^2(q, t) dt \ll q^{2-2\sigma} T \log T. \tag{14}$$

Clearly, in virtue of the estimate $R(q, t) = O(q^{7/4-\sigma} \log T)$,

$$\int_T^{2T} R^2(q, t) dt \ll T q^{7/2-2\sigma} \log^2 T.$$

This, together with (14), gives

$$\begin{aligned}
\int_T^{2T} \left(\sum_2(q, t) + R(q, t) \right)^2 dt &\ll \int_T^{2T} \sum_2^2(q, t) dt + \int_T^{2T} R^2(q, t) dt \\
&\ll q^{7/2-2\sigma} T \log^2 T. \tag{15}
\end{aligned}$$

Moreover, the Cauchy–Schwarz inequality and (9), (15) imply the estimate

$$\begin{aligned}
&\int_T^{2T} \sum_1(q, t) \left(\sum_2(q, t) + R(q, t) \right) dt \\
&\ll \left(\int_T^{2T} \sum_1^2(q, t) dt \right)^{1/2} \left(\int_T^{2T} \left(\sum_2(q, t) + R(q, t) \right)^2 dt \right)^{1/2} \\
&\ll q^{11/4-2\sigma} T^{7/4-\sigma} \log T.
\end{aligned}$$

This estimate together with (1), (9) and (15) gives

$$\begin{aligned} \int_T^{2T} E_\sigma^2(q, t) dt &= 2^{2\sigma-3/2} \left(\frac{5}{2} - 2\sigma \right)^{-1} \pi^{2\sigma-3/2} q^{3/2-2\sigma} \\ &\times \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} ((2T)^{5/2-2\sigma} - T^{5/2-2\sigma}) \\ &+ O(q^{7/2-2\sigma} T \log^2 T) + O(q^{11/4-2\sigma} T^{7/4-\sigma} \log T). \end{aligned}$$

Now, taking $T/2, T/2^2, \dots$ in place of T and summing, we obtain the theorem.

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