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Non-Newtonian flows in domains with non-compact boundaries

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ABSTRACT

Non-Newtonian flow of a viscous fluid in unbounded domains with cylindrical outlets is considered. The viscosity is assumed to be dependent on the shear rate. Applying the Banach fixed point theorem and the Hilbert spaces with exponential weight we prove the existence and uniqueness of solutions with an exponential stabilization to the quasi-Poiseuille flows at the outlets if the right hand side decays exponentially. These results may be used for the matching technique for flows in thin tube structures.

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1. Introduction

The asymptotic behavior of solutions of elliptic equations in unbounded domains with some outlets at infinity was considered in [10], for the elasticity equations in [12], for the Stokes and Navier–Stokes equations in [4,5,11,16–18] and for the viscoelastic flows [19].

To our knowledge these results have not been generalized for the case of non-Newtonian flows with the viscosity dependent on the shear strain rate, dependence which is typical for hemorheology (see [5], pp. 84–89, 196–200). On the other hand these theorems are important tools for an asymptotic analysis of non-Newtonian flows in thin structures (modeling blood circulation in blood-vessel network or oil transport in pipelines). Namely they describe the boundary layers appearing at bifurcation zones (see [13,14] for Newtonian flows). The paper [3] justifies the first approximation for the Bingham flow, however the boundary layers there are not yet constructed.

In the present paper the non-Newtonian flow with viscosity depending on the shear strain rate is considered in domains with outlets and their asymptotic behavior at infinity is studied. Such special form of the rheology

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requires important modifications in the existing technique of an asymptotic analysis of the behavior of solutions of the Navier–Stokes equations at infinity. We prove the existence and uniqueness of solutions as well as their exponential stabilization at infinity in the outlets to some analogue of the Poiseuille flow.

The paper has the following structure. Section 2 is devoted to the mathematical analysis of the quasi-Poiseuille flow for the rheology involving the viscosity depending on the shear strain rate. A Poiseuille flow is an exact solution to the equations of the fluid motion (Stokes, Navier–Stokes) in an infinite cylinder with the no slip condition at the boundary, with a linear pressure with respect to the longitudinal variable, and with the velocity vector having only longitudinal component (called normal velocity) different from zero; this normal velocity depends only on transversal variables. A quasi-Poiseuille (or Hagen–Poiseuille) flow is an exact solution having the same structure and corresponding to some non-Newtonian rheology. Such flows for various rheologies were studied in [5–7,20,21]. We introduce the parameter λ, the coefficient of the non-linear part of the viscosity as function of the shear rate. We prove the existence of a positive constant λ_0 such that for all $\lambda < \lambda_0$ there exists a quasi-Poiseuille flow for any given pressure slope (Theorem 2.1). We prove some auxiliary properties of the quasi-Poiseuille flow used further for the analysis of the problem set in a domain with outlets. In Section 3 we prove the Poincaré–Friedrichs and Korn's inequalities in weighted Sobolev spaces Lemmata 3.1 and 3.2, give the weak formulation for the problem of non-Newtonian flow with viscosity depending on the shear rate in a domain with outlets and for any $\lambda < \lambda_0$ we prove the existence and uniqueness of a solution, first in the classical Sobolev space H_0^1 (Theorem 3.4) and then in weighted Sobolev space (Theorem 3.5); this implies an exponential stabilization of the velocity to the quasi-Poiseuille flows in the outlets. Section 4 is devoted to the Dirichlet's boundary value problem in a domain with one outlet at infinity. First we prove that the boundary value function can be extended to the whole domain (with a divergence free extension, Lemma 4.1), then set the problem, give its weak formulation and prove the existence and uniqueness of the solution in the classical and weighted Sobolev spaces (Theorems 4.2 and 4.3).

2. Non-Newtonian Poiseuille flow

Let $n = 2,3$, $\nu_0, \lambda > 0$ be positive constants. Let σ be a bounded domain with Lipschitz boundary in $\mathbb{R}^{n-1}$. Let ν be a bounded C^1-smooth function $\mathbb{R}^{2n-3} \to \mathbb{R}$ such that for all $y \in \mathbb{R}^{2n-3}$, $|\nu(y)| \le C$, $|\nabla \nu(y)| \le C$ and $|\nabla(\nu(y)y)| \le C$ where C is a positive constant.

Consider the Dirichlet boundary value problem for the equation:

$$\begin{cases} -\mathrm{div}((\nu_0 + \lambda \nu(\dot{\gamma}(\mathbf{u})))D(\mathbf{u})) + \nabla p = 0, \ x \in \mathbb{R} \times \sigma \ , \\ \mathrm{div}\mathbf{u} = 0, \ x \in \mathbb{R} \times \sigma \ , \\ \mathbf{u} = 0, \ x \in \partial(\mathbb{R} \times \sigma) \ , \end{cases} \tag{1}$$

where σ is a cross-section of the cylinder $\mathbb{R} \times \sigma$, $D(\mathbf{u})$ is the strain rate matrix with the elements $d_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $\dot{\gamma}(\mathbf{u}) = (d_{12}, d_{13}, d_{23})$ if $n = 3$ and $\dot{\gamma}(\mathbf{u}) = d_{12}$ if $n = 2$.

Define a quasi-Poiseuille flow as a solution to the following problem: find the couple $(\mathbf{V}_{P\alpha}, \mathcal{P}_{P\alpha})$ such that $\mathbf{V}_{P\alpha}(x) = (v_{P\alpha}(x'), 0, \dots, 0)^T$, and $\mathcal{P}_{P\alpha}(x) = -\alpha x_1 + \beta$, $\alpha, \beta \in \mathbb{R}$, $x' = (x_2, \dots, x_n)$, where $v_{P\alpha}$ satisfies

$$\begin{cases} -\dfrac{1}{2}\mathrm{div}_{x'}((\nu_0 + \lambda \nu(\dot{\gamma}_P(v_{P\alpha})))\nabla_{x'} v_{P\alpha}) = \alpha, \quad x' \in \sigma \ , \\ v_{P\alpha} = 0, \ x' \in \partial\sigma \ , \end{cases} \tag{2}$$

where $\dot{\gamma}_P(v_{P\alpha}) = \frac{1}{2}\nabla_{x'} v_{P\alpha}$ if $n = 2$, $\dot{\gamma}_P(v_{P\alpha}) = (\frac{1}{2}\nabla_{x'} v_{P\alpha}, 0)$ if $n = 3$.

Let us remind that the classical steady form of Poiseuille flow is the couple $(\mathbf{V}_{P\alpha}, \mathcal{P}_{P\alpha})$ in the case $\lambda = 0$. In this case function $v_{P\alpha}$ is the solution to the Dirichlet boundary value problem

$$-\frac{\nu_0}{2}\Delta v_{P\alpha} = \alpha, x' \in \sigma, \quad v_{P\alpha} = 0, x' \in \partial\sigma.$$

In particular, if σ is a disc of the radius R then $v_{P\alpha} = \frac{\alpha}{2\nu_0}(R^2 - r^2)$ in polar coordinates.

Let $C_{PF\sigma}$ be the Poincaré–Friedrichs inequality constant for σ, so that for any function $u \in H_0^1(\sigma)$, $\|u\|_{H^1(\sigma)} \le C_{PF\sigma}\|\nabla_{x'}u\|_{L^2(\sigma)}$.

Theorem 2.1. *There exists λ_0 such that $\forall\alpha \in \mathbb{R}, \forall\lambda \in [0, \lambda_0)$ there exists a unique solution to problem (2) from the Sobolev space $H_0^1(\sigma)$. It satisfies the estimate*

$$\|v_{P\alpha}\|_{H_0^1(\sigma)} \le C_{PF\sigma}\frac{|\alpha|}{\nu_0}C_\sigma^0\frac{\nu_0}{\nu_0 - 2\lambda C} \tag{3}$$

where the constant C_σ^0 depends on σ only.

Proof. Let us use a Banach fixed point theorem argument.

Let L be the operator $H_0^1(\sigma) \to H_0^1(\sigma)$, such that for any $U \in H_0^1(\sigma)$, LU is a solution to the problem

$$\begin{cases} -\dfrac{1}{2}\mathrm{div}_{x'}(\nu_0\nabla_{x'}(LU)) = \dfrac{1}{2}\mathrm{div}((\lambda\nu(\dot\gamma_P(U)))\nabla_{x'}U) + \alpha, \quad x' \in \sigma \ , \\ LU = 0, \ x' \in \partial\sigma \ , \end{cases} \tag{4}$$

If $U_1, U_2 \in H_0^1(\sigma)$ then

$$\|\nabla_{x'}((LU_1) - (LU_2))\|_{L^2(\sigma)} \le \frac{2\lambda}{\nu_0}\|\nu(\dot\gamma_P(U_1))\dot\gamma_P(U_1) - \nu(\dot\gamma_P(U_2))\dot\gamma_P(U_2)\|_{L^2(\sigma)}$$
$$\le \frac{2\lambda}{\nu_0}C\|\dot\gamma_P(U_1) - \dot\gamma_P(U_2)\|_{L^2(\sigma)} = \frac{2\lambda}{\nu_0}C\|\nabla_{x'}(U_1 - U_2)\|_{L^2(\sigma)}.$$

Let λ_0 be equal to $\frac{\nu_0}{2C}$, then $\forall\lambda < \lambda_0$, L is a contraction with the contraction factor $q = \frac{2\lambda C}{\nu_0}$ and there exists a unique solution.

Taking an initial approximation as 0 we get, as a consequence of the Banach fixed point theorem, the estimate:

$$\|\nabla_{x'}v_{P\alpha}\|_{L^2(\sigma)} \le \frac{1}{1-q}\|\nabla_{x'}L0\|_{L^2(\sigma)}$$

Consider the Dirichlet problem for $u_0 = L0$:

$$-\nu_0\Delta u_0 = 2\alpha, \ x' \in \sigma; u_0 = 0, \quad x' \in \partial\sigma \tag{5}$$

There exists a constant C_σ^0 depending only on σ, such that $\|u_0\|_{H^1(\sigma)} \le \frac{|\alpha|}{\nu_0}C_\sigma^0$. (Moreover, it follows from the ADN theory [1] that if $\partial\sigma \in C^N$ then there exists a constant C_σ^N, depending on σ and N only, such that $\|u_0\|_{H^N(\sigma)} \le \frac{|\alpha|}{\nu_0}C_\sigma^N$). So, $\|\nabla_{x'}L0\|_{L^2(\sigma)} \le \frac{|\alpha|}{\nu_0}C_\sigma^0$. Applying the Poincaré–Friedrichs inequality, we get:

$$\|v_{P\alpha}\|_{H_0^1(\sigma)} \le C_{PF\sigma}\frac{|\alpha|}{\nu_0}C_\sigma^0\frac{1}{1-q}.$$

The theorem is thus proved.

Corollary 2.2. *Let $\partial\sigma \in C^{2,\beta}$ and let the function ν be of the class $C^1(\mathbb{R}^{2n-3})$. There exists λ_2 such that $\forall\alpha \in \mathbb{R}, \forall\lambda \in [0, \lambda_2)$ there exists a solution to problem (2) from the space $C^{2,\beta}(\sigma)$.*

Proof. The proof follows from [9] Chapter 4, Theorem 8.3.

Define $F_\alpha = \int_\sigma v_{P\alpha}(x')dx'$ the flux corresponding to the pressure slope α. Note that the case of a Newtonian flow (the steady form of Navier–Stokes or Stokes equations) F_α is proportional to α. Here this is not the case.

Lemma 2.3. *Let $\lambda < \lambda_0$. Then $v_{P\alpha}$ is continuous with respect to α in the norm $\|\nabla_{x'} \cdot \|_{L^2(\sigma)}$.*

Proof. Let $v_{P\alpha_1}$ and $v_{P\alpha_2}$ be solutions corresponding to $\alpha = \alpha_1$ and $\alpha = \alpha_2$ respectively. Then as in the proof of Theorem 2.1 we get

$$\|\nabla_{x'}(v_{P\alpha_1} - v_{P\alpha_2})\|_{L^2(\sigma)} \leq \frac{2\lambda}{\nu_0}\|\nu(\dot\gamma_P(v_{P\alpha_1}))\dot\gamma_P(v_{P\alpha_1}) - \nu(\dot\gamma_P(v_{P\alpha_2}))\dot\gamma_P(v_{P\alpha_2})\|_{L^2(\sigma)} + \frac{\lambda}{\nu_0}|\alpha_1 - \alpha_2|\sqrt{\operatorname{mes}\sigma},$$

and for $\lambda < \lambda_0$,

$$\|\nabla_{x'}(v_{P\alpha_1} - v_{P\alpha_2})\|_{L^2(\sigma)} \leq \left(1 - \frac{2\lambda}{\nu_0}C\right)^{-1}\frac{\lambda}{\nu_0}|\alpha_1 - \alpha_2|\sqrt{\operatorname{mes}\sigma},$$

and this estimate completes the proof.

Corollary 2.4. *F_α is continuous with respect to α.*

Corollary 2.5. $\operatorname{sgn}(F_\alpha) = \operatorname{sgn}(\alpha)$.

Proof. Indeed,

$$\begin{aligned}\int_\sigma \alpha v_{P\alpha}(x')dx' &= -\int_\sigma \frac{1}{2}\operatorname{div}_{x'}((\nu_0 + \lambda\nu(\dot\gamma_P(v_{P\alpha})))\nabla_{x'}v_{P\alpha})v_{P\alpha}(x')dx' \\ &= \int_\sigma \frac{1}{2}(\nu_0 + \lambda\nu(\dot\gamma_P(v_{P\alpha})))\nabla_{x'}v_{P\alpha}\cdot\nabla_{x'}v_{P\alpha}(x')dx' \geq 0.\end{aligned}$$

Lemma 2.6. *Let $\lambda < \min\left(\lambda_0, \frac{\nu_0}{2(\nu_0+C)}\right)$. For any $F \in \mathbb{R}$ there exists a unique pair $(v_{P\alpha}, \alpha)$ satisfying (2) such that $F_\alpha = F$. There exists a constant C_{00} such that for any $F \in \mathbb{R}$,*

$$\|v_{P\alpha}\|_{H^1(\sigma)} \leq C_{00}|F|, \quad |\alpha| \leq C_{00}|F|.$$

Proof. Let us define $\tilde H_0^1(\sigma) = \{\varphi \in H_0^1(\sigma) | \int_\sigma \varphi dx' = 0\}$ and let us use an equivalent formulation to problem (2) that is (see [5], section 2.2.2, Proposition 21(ii)):

$$\begin{cases} \displaystyle\int_\sigma \frac{1}{2}((\nu_0 + \lambda\nu(\dot\gamma_P(v_{P\alpha})))\nabla_{x'}v_{P\alpha}\cdot\nabla_{x'}\varphi)dx' = 0, \quad \forall\varphi \in \tilde H_0^1(\sigma)\,, \\ v_{P\alpha} \in H_0^1(\sigma), \quad \displaystyle\int_\sigma v_{P\alpha}dx' = F\,, \\ \alpha = \displaystyle\int_\sigma \frac{1}{2}((\nu_0 + \lambda\nu(\dot\gamma_P(v_{P\alpha})))\nabla_{x'}v_{P\alpha}\cdot\nabla_{x'}\chi)dx', \end{cases} \tag{6}$$

where $\chi \in H^1(\sigma)$, such that, $\int_\sigma \chi(x')dx' = 1$.

Let $A \in C_0^\infty(\sigma)$, $a = FA$, $\int_\sigma A(x')dx' = 1$.

Consider now the problem: find $u \in \tilde H_0^1(\sigma)$ such that

$$\int_\sigma((\nu_0 + \lambda\nu(\dot\gamma_P(u+a)))\nabla_{x'}(u+a)\cdot\nabla_{x'}\varphi)dx' = 0, \quad \forall\varphi \in \tilde H_0^1(\sigma),$$

i.e.

$$\int_\sigma \nu_0 \nabla_{x'} u \cdot \nabla_{x'} \varphi dx' = -\int_\sigma \int_\sigma \nu_0 \nabla_{x'} a \cdot \nabla_{x'} \varphi dx'$$
$$- \int_\sigma \lambda \nu(\dot{\gamma}_P(u+a)) \nabla_{x'}(u+a) \cdot \nabla_{x'} \varphi dx', \quad \forall \varphi \in \tilde{H}^1_0(\sigma). \tag{7}$$

Applying the Banach fixed point theorem and the Riesz representation theorem we get the existence and uniqueness of a solution $u \in \tilde{H}^1_0(\sigma)$. Then $v_{P\alpha} = u + a$.

Let us prove the estimates. We get:

$$\|v_{P\alpha}\|_{H^1(\sigma)} \leq \|u\|_{H^1(\sigma)} + \|a\|_{H^1(\sigma)}.$$

Next the norm $\|u\|_{H^1(\sigma)}$ is evaluated using identity (7) by taking $\varphi = u$, we obtain

$$\|\nabla u\|^2_{L^2(\sigma)} \leq \|\nabla u\|_{L^2(\sigma)} \|\nabla a\|_{L^2(\sigma)}$$
$$+ \frac{\lambda(\nu_0 + C)}{\nu_0} (\|\nabla u\|^2_{L^2(\sigma)} + \|\nabla u\|_{L^2(\sigma)} \|\nabla a\|_{L^2(\sigma)}). \tag{8}$$

Applying the Young inequality to the products of norms in the right-hand side and taking λ less than $\frac{\nu_0}{2(\nu_0+C)}$, we obtain

$$(1/4)\|\nabla u\|^2_{L^2(\sigma)} \leq 64 \|\nabla a\|^2_{L^2(\sigma)}. \tag{9}$$

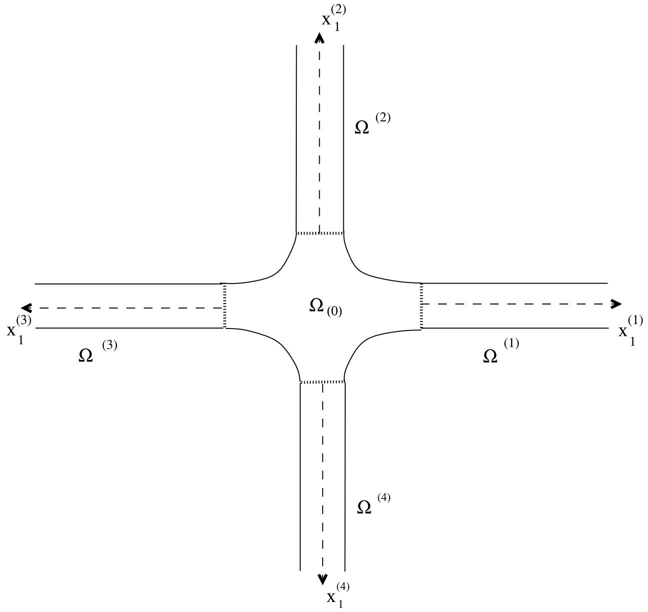
This estimate implies the estimate for the H^1-norm of v_α. Then this estimate with $(6)_3$ is used for evaluation of α. The lemma is proved.

3. Existence, uniqueness and stabilization of a solution to the non-Newtonian flow equations in an unbounded domain with cylindrical outlets to infinity

Consider the domain $\Omega \subset \mathbb{R}^n$ with J cylindrical outlets to infinity, i.e., $\Omega = \Omega_0 \cup \left(\cup_{k=1}^J \Omega^{(k)}\right)$, where Ω_0 is a bounded domain, $\Omega_0 \cap \Omega^{(l)} = \emptyset$ for $l \in \{1, \ldots, J\}$, $\Omega^{(k)} \cap \Omega^{(l)} = \emptyset$ for $k \neq l$, $k, l \in \{1, \ldots, J\}$, and the outlets to infinity $\Omega^{(k)}$ in some coordinate systems $x^{(k)} = (x_1^{(k)}, x^{(k)\prime})$ having the origins within the boundary of domain Ω_0 are given by the relations

$$\Omega^k = \{x^{(k)} \in \mathbb{R}^n,\ x^{(k)\prime} \in \sigma_k\ , x_1^{(k)} \geq 0\},$$

where σ_k are some bounded domains in $\mathbb{R}^{n-1}$, cross-sections of the cylinders. Assume that for any $k \in \{1, \ldots, J\}$ there exists a $\delta_k > 0$ such that the cylinder $\{x^{(k)} \in \mathbb{R}^n,\ x^{(k)\prime} \in \sigma_k\ , -\delta_k < x_1^{(k)} < 0\} \subset \Omega_0$. Denote d_σ the maximal diameter of the cross-sections σ_k. We assume that the boundary $\partial\Omega$ is Lipschitz and that the common part of the boundaries $\partial\Omega \cap \partial\Omega_0 \neq \emptyset$ and has a positive measure. In particular, Ω can be just a semi-infinite cylinder: $\Omega = \{x \in \mathbb{R}^n,\ x' \in \sigma \subset \mathbb{R}^{n-1}, x_1 > 0\}$. Evidently there exists a positive real number $R > d_\sigma$ such that the ball $B_R = \{x \in \mathbb{R}^n,\ |x| < R\}$ contains Ω_0.



Let us define in Ω as usual, weighted function spaces. Denote $\beta = (\beta_1, \dots, \beta_J)$ and set

$$E_\beta(x) = \begin{cases} 1, & x \in \Omega_0, \\ \exp(2\beta_k x_1^{(k)}), & x \in \Omega^{(k)}, \quad k = 1, \dots, J. \end{cases} \tag{10}$$

Denote by $\mathcal{W}^{l,2}_\beta(\Omega)$, $l \geq 0$, the space of functions obtained as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathcal{W}^{l,2}_\beta(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega E_\beta(x) |D^\alpha u(x)|^2 \, dx \Big)^{1/2}$$

and set $\mathcal{W}^{0,2}_\beta(\Omega) = \mathcal{L}^2_\beta(\Omega)$. Notice that for $\beta_k > 0$ elements of the space $\mathcal{W}^{l,2}_\beta(\Omega)$ exponentially vanish as $x_1^{(k)} \to \infty$.

Lemma 3.1 (*Poincaré–Friedrichs Inequality in* $\mathcal{W}^{1,2}_\beta(\Omega)$)**.** *There exists a constant $C_{PF\Omega}$ independent of β, such that for any function $u \in \mathcal{W}^{1,2}_\beta(\Omega)$ the inequality holds:*

$$\|u\|^2_{\mathcal{L}^2_\beta(\Omega)} \leq C_{PF\Omega} \|\nabla u\|^2_{\mathcal{L}^2_{\beta(\Omega)}}.$$

Proof. As the subdomain Ω_0 has a common part of the boundary with Ω, we apply the standard Poincaré–Friedrichs inequality. Then for any k we consider the cylinders $\Omega^{(k)}_{K,K+1} = \Omega^{(k)} \cap \{x_1^{(k)} \in [K, K+1)\}$ and integrate from $x_1^{(k)} = K$ to $x_1^{(k)} = K+1$ with the weight E_β the standard inequality

$$\|u\|^2_{L^2(\sigma_k)} \leq (C_{PF\sigma_k} \|\nabla_{x^{(k)\prime}} u\|_{L^2(\sigma_k)})^2.$$

Finally we add all these inequalities for all k and K and for $\Omega_{(0)}$ and get the assertion of the theorem.

Lemma 3.2 (*Korn's Inequality in the Weighted Space* $\mathcal{W}^{1,2}_\beta(\Omega)$)**.** *Let $\bar\beta$ be $\max_{1 \leq k \leq J} \beta_k$. Then there exists a constant C_Ω independent of β, such that for any vector-valued function $\mathbf{N} \in \mathcal{W}^{1,2}_\beta(\Omega)$ the inequality holds:*

$$\int_\Omega E_\beta(x) |\nabla \mathbf{N}|^2 dx \leq C_\Omega e^{2\bar\beta} \int_\Omega E_\beta(x) D(\mathbf{N}) : D(\mathbf{N}) dx,$$

where : is the Hadamard product, i.e. for two $n \times n$ matrices A and B, $A : B$ is the $n \times n$ matrix having entries $(A : B)_{ij} = A_{ij} B_{ij}$, where A_{ij} and B_{ij} are the entries of matrices A and B respectively.

Proof. First for the subdomain Ω_0 we apply the standard Korn's inequality, then for any k we apply the standard Korn's inequality for a cylinder $(0,1) \times \sigma_k$ with constant C_{σ_k}, to the cylinders $\Omega^{(k)}_{K,K+1} = \Omega^{(k)} \cap \{x_1^{(k)} \in [K, K+1)\}$ as follows:

$$\begin{aligned}\int_{\Omega^{(k)}_{K,K+1}} E_\beta(x) |\nabla \mathbf{N}|^2 dx &\leq \int_{\Omega^{(k)}_{K,K+1}} exp\{2\beta_k (K+1)\} |\nabla \mathbf{N}|^2 dx \\ &\leq exp\{2\beta_k (K+1)\} C_{\sigma_k} \int_{\Omega^{(k)}_{K,K+1}} D(\mathbf{N}) : D(\mathbf{N}(x)) dx \\ &\leq exp\{2\beta_k\} C_{\sigma_k} \int_{\Omega^{(k)}_{K,K+1}} E_\beta(x) D(\mathbf{N}) : D(\mathbf{N}(x)) dx.\end{aligned}$$

Finally we derive the assertion adding all inequalities for cylinders $\Omega^{(k)}_{K,K+1}$ and for $\Omega_{(0)}$ and taking C_Ω as maximum of all constants C_{σ_k} and the Korn's constant for Ω_0. The lemma is proved.

Let us define the cut-off functions χ_k associated to each outlet $\Omega^{(k)}$ as a C^2-smooth function vanishing everywhere in Ω except for the branch $\Omega^{(j)}$, where it depends on the local longitudinal variable $x_1^{(k)}$ only, vanishes if $x_1^{(k)} < 1$ and is equal to one if $x_1^{(k)} > 2$.

Remark 3.3. Define

$$E_\beta^{(K)}(x) = \begin{cases} 1, & x \in \Omega_0, \\ \exp(2\beta_k x_1^{(k)}), & x \in \Omega^{(k)}, x_1^{(k)} < K, \quad k = 1, \dots, J, \\ \exp(2\beta_k K), & x \in \Omega^{(k)}, x_1^{(k)} \geq K, \quad k = 1, \dots, J, \end{cases} \tag{11}$$

Then in the same way we can prove that

$$\int_\Omega E_\beta^{(K)}(x) |\nabla \mathbf{N}|^2 dx \leq C_\Omega e^{2\bar{\beta}} \int_\Omega E_\beta^{(K)}(x) D(\mathbf{N}) : D(\mathbf{N}) dx,$$

and C_Ω is independent of K. The Poincaré–Friedrichs inequality as well holds still true:

$$\int_\Omega E_\beta^{(K)}(x) |\mathbf{N}|^2 dx \leq C_{PF\Omega} \int_\Omega E_\beta^{(K)}(x) |\nabla \mathbf{N}|^2 dx$$

Consider a weak formulation of the following main problem:

$$\begin{cases} -\mathrm{div}((\nu_0 + \lambda \nu(\dot{\gamma}(\mathbf{N} + \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j}))) D(\mathbf{N} + \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j})) \\ +\nabla(P - \sum_{j=1}^J \chi_j(x) \alpha_j x_1^{(e_j)}) = \mathbf{f_0} - \sum_{i=1}^n \frac{\partial \mathbf{f_i}}{\partial x_i}, \ x \in \Omega \ , \\ \mathrm{div}(\mathbf{N} + \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j}) = 0, \ x \in \Omega \ , \\ \mathbf{N} = 0, \ x \in \partial\Omega \ , \end{cases} \tag{12}$$

which is: find $\mathbf{N} \in H_0^1(\Omega)$, such that for any vector-valued divergence free test function $\varphi \in C_0^\infty(\Omega)$ (satisfying $\mathrm{div}\varphi = 0$),

$$\begin{cases} \int_\Omega (\nu_0 + \lambda\nu(\dot\gamma(\mathbf{N} + \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j})))D(\mathbf{N} + \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j}) : D(\varphi) - \\ \nabla(\sum_{j=1}^J \chi_j \alpha_j x_1^{(j)}) \cdot \varphi dx = \int_\Omega \Big(\mathbf{f_0} \cdot \varphi + \sum_{i=1}^n \mathbf{f_i} \cdot \frac{\partial\varphi}{\partial x_i}\Big) dx \;, \\ \mathrm{div}(\mathbf{N} + \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j}) = 0, \;\; x \in \Omega \;. \end{cases} \tag{13}$$

Denote the linear combination of quasi-Poiseuille flows $\mathbf{V}_\chi = \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j}$, $P_\chi = \sum_{j=1}^J \chi_j \alpha_j x_1^{(j)}$.

Theorem 3.4. *Let F_j, $j = 1, \ldots, J$, be given fluxes, such that $\sum_{j=1}^J F_j = 0$. Let $\mathbf{V}_{\mathbf{P}\alpha_\mathbf{1}}, \ldots, \mathbf{V}_{P\alpha_j}$ be the quasi-Poiseuille flows in the branches $\Omega^{(j)}$ of the domain Ω corresponding to the given fluxes. Let λ be positive number satisfying the condition*

$$\lambda < \frac{\nu_0}{CC_\Omega}. \tag{14}$$

Then problem (13) admits a unique weak solution $\mathbf{N} \in H_0^1(\Omega)$.

Proof. Rewrite the variational formulation in the form

$$\begin{aligned} &\int_\Omega \nu_0 D(\mathbf{N}) : D(\varphi)dx + \int_\Omega \lambda\nu(\dot\gamma(\mathbf{N} + \mathbf{V}_\chi))D(\mathbf{N}) : D(\varphi)dx \\ &+ \int_\Omega \lambda\{\nu(\dot\gamma(\mathbf{N} + \mathbf{V}_\chi)) - \nu(\dot\gamma(\mathbf{V}_\chi))\}D(\mathbf{V}_\chi) : D(\varphi)dx \\ &+ \int_\Omega \nu_0 D(\mathbf{V}_\chi) : D(\varphi) + \lambda\nu(\dot\gamma(\mathbf{V}_\chi))D(\mathbf{V}_\chi) : D(\varphi) - \nabla P_\chi \cdot \varphi dx \\ &= \int_\Omega \Big(\mathbf{f_0} \cdot \varphi + \sum_{i=1}^n \mathbf{f_i} \cdot \frac{\partial\varphi}{\partial x_i}\Big) dx \;, \end{aligned} \tag{15}$$

and

$$\mathrm{div}\mathbf{N} = -\mathrm{div}(\mathbf{V}_\chi), x \in \Omega.$$

Note that

$$\begin{aligned} &\int_\Omega (\nu_0 D(\mathbf{V}_\chi) : D(\varphi) + \lambda\nu(\dot\gamma(\mathbf{V}_\chi))D(\mathbf{V}_\chi) : D(\varphi))\, dx - \int_\Omega \nabla P_\chi \cdot \varphi dx \\ &\quad = -\int_\Omega \Big(\mathrm{div}\Big(\nu_0 D(\mathbf{V}_\chi)\Big) + \mathrm{div}\Big(\lambda\nu(\dot\gamma(\mathbf{V}_\chi))D(\mathbf{V}_\chi)\Big) + \nabla P_\chi\Big) \cdot \varphi dx, \end{aligned}$$

where the integrand function vanishes outside the ball B_{R+2} and so has a finite support.

Applying the theorem on the divergence equation (see [2,4,8,15]) we can construct a vector valued function $\Phi \in H_0^1(\Omega)$ with a support within $\Omega \cap B_{R+2}$ such that

$$\mathrm{div}\mathbf{\Phi} = -\mathrm{div}(\mathbf{V}_\chi), x \in \Omega.$$

Note that the right-hand side here has a finite support because $\text{div}\mathbf{V}_{P\alpha_j} = 0$ where $\chi_j = 1$.

Let us introduce a new unknown function $\tilde{\mathbf{N}} = \mathbf{N} + \Phi$. Then clearly, $\text{div}\tilde{\mathbf{N}} = 0$, so that problem (12) is reduced to the corresponding problem for the divergence free unknown $\tilde{\mathbf{N}}$. The existence and uniqueness of a solution is proved by a Banach fixed point theorem argument with λ satisfying condition (14).

Define space $H^1_{div0}(\Omega) = \{\mathbf{u} \in H^1_0(\Omega), \text{div}\mathbf{u} = 0\}$ supplied with the norm $\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ and operator $\mathcal{L} : H^1_{div0}(\Omega) \to H^1_{div0}(\Omega)$ such that for any given $\tilde{\mathbf{N}} \in H^1_{div0}(\Omega)$, $\mathcal{L}\tilde{\mathbf{N}}$ is a solution to the problem: for any vector-valued test function $\varphi \in C_0^\infty(\Omega)$, such that $\text{div}\varphi = 0$,

$$\begin{aligned}
&\int_\Omega \nu_0 D(\mathcal{L}\tilde{\mathbf{N}}) : D(\varphi)dx + \int_\Omega \lambda\nu(\dot{\gamma}(\mathbf{N}+\mathbf{V}_\chi))D(\mathbf{N}+\mathbf{V}_\chi) : D(\varphi)dx \\
&+ \int_\Omega \nu_0 D(\mathbf{V}_\chi) : D(\varphi)dx - \int_\Omega \nabla P_\chi \cdot \varphi dx \\
&= \int_\Omega \Big(\mathbf{f_0}\cdot\varphi + \sum_{i=1}^n \mathbf{f_i}\cdot\frac{\partial\varphi}{\partial x_i}dx + \int_\Omega \nu_0 D(\mathbf{\Phi}) : D(\varphi)\Big)dx\ ,
\end{aligned} \tag{16}$$

where $\mathbf{N} = \tilde{\mathbf{N}} - \mathbf{\Phi}$.

Let us check that $\mathcal{L}$ is contraction for small values of λ. Let $\tilde{\mathbf{N}}_1, \tilde{\mathbf{N}}_2 \in H^1_{div0}(\Omega)$, denote $\tilde{\mathbf{M}}$ their difference, then the difference $\tilde{\mathbf{Q}} = \mathcal{L}\tilde{\mathbf{N}}_1 - \mathcal{L}\tilde{\mathbf{N}}_2$ satisfies the problem:

$$\begin{aligned}
&\int_\Omega \nu_0 D(\tilde{\mathbf{Q}}) : D(\varphi)dx + \int_\Omega \lambda\Big(\nu(\dot{\gamma}(\mathbf{N}_1+\mathbf{V}_\chi))D(\mathbf{N}_1+\mathbf{V}_\chi) \\
&-\nu(\dot{\gamma}(\mathbf{N}_2+\mathbf{V}_\chi))D(\mathbf{N}_2+\mathbf{V}_\chi)\Big) : D(\varphi)dx = 0
\end{aligned} \tag{17}$$

for any vector-valued test function $\varphi \in C_0^\infty(\Omega)$, such that $\text{div}\varphi = 0$, where $\mathbf{N}_i = \tilde{\mathbf{N}}_i - \Phi$.

So,

$$|\int_\Omega \nu_0 D(\tilde{\mathbf{Q}}) : D(\varphi)dx| \le \lambda C \int_\Omega |D(\tilde{\mathbf{M}}) : D(\varphi)|dx.$$

Since $C_0^\infty(\Omega)$ is dense in $H^1_0(\Omega)$, taking $\varphi = \tilde{\mathbf{Q}}$ and applying Korn's inequality we get:

$$\nu_0\|\nabla\tilde{\mathbf{Q}}\|^2_{L^2(\Omega)} \le C_\Omega \int_\Omega \nu_0 D(\tilde{\mathbf{Q}}) : D(\tilde{\mathbf{Q}})dx \le \lambda C C_\Omega \|\nabla\tilde{\mathbf{M}}\|_{L^2(\Omega)}\|\nabla\tilde{\mathbf{Q}}\|_{L^2(\Omega)}.$$

So, we proved that operator $\mathcal{L}$ is a contraction for λ satisfying (14). Then the theorem is proved.

Let us define L as the $\sup_{x\in\Omega\cap B_{2R+2}}|D(\mathbf{V}_\chi)|$.

Theorem 3.5. *Let F_j, $j = 1,\dots,J$, be given fluxes, such that $\sum_{j=1}^J F_j = 0$. Let $\mathbf{V}_{\mathbf{P}\alpha_1},\dots,\mathbf{V}_{P\alpha_j}$ be the quasi-Poiseuille flows in the branches $\Omega^{(j)}$ of the domain Ω corresponding to the given fluxes. Let $\mathbf{f_0},\dots,\mathbf{f_n} \in \mathcal{L}^2_\beta(\Omega)$. Let λ be positive number satisfying the condition*

$$\lambda < \frac{\nu_0}{CC_\Omega} min\{1, \frac{1}{4L}\}. \tag{18}$$

Then there exists a positive β^0 independent of $\mathbf{f_i}, i = 0,\dots,n, F_j, j = 1,\dots,J$, such that if for all $i = 1,\dots,J$, $0 < \beta'_i < \beta^0$, then $\mathbf{N} \in \mathcal{W}^{1,2}_{\beta'}(\Omega)$ and satisfies the following estimate:

$$\|\mathbf{N}\|_{\mathcal{W}^{1,2}_{\beta'}(\Omega)} \le C_{T4}\Big(\|\mathbf{f_0}\|_{\mathcal{L}^2_\beta(\Omega)} + \sum_{i=1}^n \|\mathbf{f_i}\|_{\mathcal{L}^2_\beta(\Omega)} + \sum_{j=1}^{J-1}|F_j|\Big). \tag{19}$$

where C_{T4} is a positive constant independent of $\mathbf{f_i}, i = 0,\dots,n$, and $F_j, j = 1,\dots,J$.

Proof. The proof is similar to the one of Theorem 3.2 of the paper [18]. Namely, let us take in integral identity (13)$\varphi(x) = E_\beta^{(K)}(x)\tilde{\mathbf{N}} + \mathbf{W}^{(K)}(x)$, where $\mathbf{W}^{(K)}$ is a vector field constructed in Lemma I.1.13 of [18] that we cite here for the reader's convenience:

Lemma I.1.13. *Let* $\mathbf{u} \in H_0^1(\Omega)$, $\operatorname{div}\mathbf{u} = 0$ *and* $\int_{\sigma_k} \mathbf{u}\cdot\mathbf{n}ds = 0, k = 1,\dots,J$. *Then there exists a vector-valued function* $\mathbf{W}^{(K)} \in H_0^1(\Omega)$, *such that*

$$supp\mathbf{W}^{(K)} \subset \cup_{k=1}^{J}\{x \in \Omega^{(k)}, x_1^{(k)} < K\}$$

and

$$\operatorname{div}\mathbf{W}^{(K)}(x) = -\operatorname{div}(E_\beta^{(K)}(x)\mathbf{u}(x)), \quad x \in \Omega.$$

Moreover, there holds the estimate

$$\int_\Omega E_{-\beta}^{(K)}(x)|\nabla\mathbf{W}^{(K)}(x)|^2 dx \le c\gamma_*^2 \int_\Omega E_\beta^{(K)}(x)|\mathbf{u}(x)|^2 dx$$
$$\le c\gamma_*^2 \int_\Omega E_\beta^{(K)}(x)|\nabla\mathbf{u}(x)|^2 dx$$

with constant c *independent of* K *and* $\mathbf{u}$, *and* $\gamma_* = 2\max\{\beta_1,\dots,\beta_J\}$.

We can then define $\varphi(x) = E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) + \mathbf{W}^{(K)}(x)$; we get: $\operatorname{div}\varphi = 0$, $\varphi|_{\partial\Omega} = 0$ and

$$\begin{aligned}
&\int_\Omega \nu_0 D(\tilde{\mathbf{N}}) : D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) + \mathbf{W}^{(K)}(x))dx \\
&+\int_\Omega \lambda\nu(\dot\gamma(\tilde{\mathbf{N}} - \mathbf{\Phi} + \mathbf{V}_\chi))D(\tilde{\mathbf{N}}) : D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) + \mathbf{W}^{(K)}(x))dx \\
&- \int_\Omega\Big(\operatorname{div}\Big(\nu_0 D(\mathbf{V}_\chi)\Big)+\operatorname{div}\Big(\lambda\nu\dot\gamma(\mathbf{V}_\chi)D(\mathbf{V}_\chi)\Big)+\nabla P_\chi\Big)\cdot(E_\beta^{(K)}\tilde{\mathbf{N}}+\mathbf{W}^{(K)})dx \\
&+ \int_\Omega \lambda\Big(\nu(\dot\gamma(\tilde{\mathbf{N}} - \mathbf{\Phi} + \mathbf{V}_\chi)) - \nu(\dot\gamma(\mathbf{V}_\chi))\Big)D(\mathbf{V}_\chi) : D(E_\beta^{(K)}\tilde{\mathbf{N}} + \mathbf{W}^{(K)})dx \\
&=\int_\Omega \Big(\mathbf{f_0}\cdot(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) + \mathbf{W}^{(K)}(x)) + \sum_{i=1}^n \mathbf{f_i}\cdot\frac{\partial(E_\beta^{(K)}\tilde{\mathbf{N}} + \mathbf{W}^{(K)})}{\partial x_i}\Big)dx \\
&+ \int_\Omega \nu_0 D(\mathbf{\Phi}) : D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) + \mathbf{W}^{(K)}(x))dx \\
&+ \int_\Omega \lambda\nu(\dot\gamma(\tilde{\mathbf{N}} - \mathbf{\Phi} + \mathbf{V}_\chi))D(\mathbf{\Phi}) : D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) + \mathbf{W}^{(K)}(x))dx\,.
\end{aligned} \tag{20}$$

Let us bound from below the term

$$\begin{aligned}
&\int_\Omega \nu_0 E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)) : D(\tilde{\mathbf{N}}(x))dx \\
&+ \int_\Omega \lambda E_\beta^{(K)}(x)\nu(\dot\gamma(\tilde{\mathbf{N}} - \mathbf{\Phi} + \mathbf{V}_\chi))D(\tilde{\mathbf{N}}(x)) : D(\tilde{\mathbf{N}}(x))dx \\
&\ge\frac{\nu_0}{2}\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)) : D(\tilde{\mathbf{N}}(x))dx
\end{aligned}$$

and bound from above all other terms. Let us start with the right-hand side of (20). Using Hölder inequality and Young ε-inequality, the above cited Lemma I.1.13 and weighted Poincaré and Korn inequalities for

$H_0^1(\Omega)$ (without loss of generality we can consider $\beta_k < 1$) we get

$$
\begin{aligned}
&|\int_\Omega \mathbf{f_0}\cdot(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x)+\mathbf{W}^{(K)}(x))+\sum_{i=1}^n \mathbf{f_i}\cdot\frac{\partial(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x)+\mathbf{W}^{(K)}(x))}{\partial x_i}dx| \\
&\le \frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\Big(|\mathbf{f_0}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2\Big)dx+\varepsilon\Big(\int_\Omega E_\beta^{(K)}(x)|\tilde{\mathbf{N}}|^2dx \\
&+\ c_1\big(\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx+\gamma_*\int_\Omega E_\beta^{(K)}(x)|\tilde{\mathbf{N}}(x)|^2dx \\
&+\ \int_\Omega E_{-\beta}^{(K)}(x)|\mathbf{W}^{(K)}|^2dx+\int_\Omega E_{-\beta}^{(K)}(x)|\nabla\mathbf{W}^{(K)}|^2dx\big)\Big) \\
&\le \frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\Big(|\mathbf{f_0}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2\Big)dx \\
&+\ c_2\varepsilon\Big(\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx+\int_\Omega E_{-\beta}^{(K)}(x)|\nabla\mathbf{W}^{(K)}|^2dx\Big) \\
&\le \frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\Big(|\mathbf{f_0}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2\Big)dx+c_3\varepsilon\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx.
\end{aligned}
$$

In the same way we evaluate the term

$$
\begin{aligned}
&|-\int_\Omega\Bigg(\mathrm{div}\Big(\nu_0 D(\mathbf{V}_\chi)\Big)+\mathrm{div}\Big(\lambda\nu(\dot\gamma(\mathbf{V}_\chi))D(\mathbf{V}_\chi)\Big)+\nabla P_\chi\Bigg)\cdot(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x) \\
&+\ \mathbf{W}^{(K)}(x))dx+\int_\Omega \nu_0 D(\mathbf{\Phi}):D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x)+\mathbf{W}^{(K)}(x))dx \\
&+\ \int_\Omega \lambda\nu(\dot\gamma(\tilde{\mathbf{N}}-\mathbf{\Phi}+\mathbf{V}_\chi))D(\mathbf{\Phi}):D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x)+\mathbf{W}^{(K)}(x))dx| \\
&\le \frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\Big(|\tilde{\mathbf{f}}_\mathbf{0}(x)|^2+\sum_{i=1}^n|\tilde{\mathbf{f}}_\mathbf{i}|^2\Big)dx+c_3\varepsilon\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx,
\end{aligned}
$$

where

$$
\tilde{\mathbf{f}}_\mathbf{0}(x)=\mathrm{div}\Big(\nu_0 D(\mathbf{V}_\chi)\Big)+\mathrm{div}\Big(\lambda\nu(\dot\gamma(\mathbf{V}_\chi))D(\mathbf{V}_\chi)\Big)+\nabla P_\chi,\quad \tilde{\mathbf{f}}_\mathbf{i}(x)=\frac{\partial\mathbf{\Phi}}{\partial x_i}.
$$

Now we evaluate the terms:

$$
\begin{aligned}
&|\int_\Omega \nu_0 D(\tilde{\mathbf{N}}(x)):\frac12\Big(\tilde{\mathbf{N}}(x)(\nabla E_\beta^{(K)}(x))^T+\nabla E_\beta^{(K)}(x)(\tilde{\mathbf{N}}(x))^T\Big)dx \\
&+\ \int_\Omega \lambda\nu(\dot\gamma(\tilde{\mathbf{N}}-\mathbf{\Phi}+\mathbf{V}_\chi))D(\tilde{\mathbf{N}}(x)):\frac12\Big(\tilde{\mathbf{N}}(x)(\nabla E_\beta^{(K)}(x))^T+\nabla E_\beta^{(K)}(x)(\tilde{\mathbf{N}}(x))^T\Big)dx \\
&+\ \int_\Omega \nu_0 D(\tilde{\mathbf{N}}(x)):D(\mathbf{W}^{(K)}(x))dx+\int_\Omega \lambda\nu(\dot\gamma(\tilde{\mathbf{N}}-\mathbf{\Phi}+\mathbf{V}_\chi))D(\tilde{\mathbf{N}}(x)):D(\mathbf{W}^{(K)}(x))dx| \\
&\le(\nu_0+\lambda C)\Big(\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx\Big)^{\frac12} \\
&\times\ \Big(c_4\gamma_*\Big(\int_\Omega E_\beta^{(K)}(x)|\tilde{\mathbf{N}}|^2dx\Big)^{\frac12}+c_5\Big(\int_\Omega E_{-\beta}^{(K)}(x)|\nabla\mathbf{W}^{(K)}|^2dx\Big)^{\frac12}\Big) \\
&\le c_6\gamma_*\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx,
\end{aligned}
$$

and finally

$$
\begin{aligned}
&|\int_\Omega \lambda\{\nu(\dot\gamma(\tilde{\mathbf{N}}(x)-\mathbf{\Phi}+\mathbf{V}_\chi))-\nu(\dot\gamma(\mathbf{V}_\chi))\}D(\mathbf{V}_\chi):D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x)+\mathbf{W}^{(K)}(x))dx| \\
&\le\lambda\int_\Omega CL(|D(\tilde{\mathbf{N}}(x))|+|D(\mathbf{\Phi}(x))|)|D(E_\beta^{(K)}(x)\tilde{\mathbf{N}}(x)+\mathbf{W}^{(K)}(x))|dx \\
&\le\lambda CL\int_\Omega\Big(\gamma_*|\nabla\tilde{\mathbf{N}}(x)||\tilde{\mathbf{N}}(x)|+|\nabla\tilde{\mathbf{N}}(x)|^2\Big)E_\beta^{(K)}(x)+|\nabla\tilde{\mathbf{N}}(x)||\nabla\mathbf{W}^{(K)}|dx+
\end{aligned}
$$

$$\frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\sum_{i=1}^n |\tilde{\mathbf{f}}_{\mathbf{i}}|^2(x)dx + c_3\varepsilon\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx$$
$$\leq \lambda CL\left(\int_\Omega E_\beta^{(K)}(x)\nabla\tilde{\mathbf{N}}(x):\nabla\tilde{\mathbf{N}}(x)dx\right)^{\frac{1}{2}}$$
$$\times\ \left(c_7\gamma_*\Big(\int_\Omega E_\beta^{(K)}(x)|\tilde{\mathbf{N}}|^2dx\Big)^{\frac{1}{2}} + c_8\Big(\int_\Omega E_{-\beta}^{(K)}(x)|\nabla\mathbf{W}^{(K)}|^2dx\Big)^{\frac{1}{2}}\right)$$
$$+\ \lambda CL\int_\Omega E_\beta^{(K)}(x)\nabla\tilde{\mathbf{N}}(x):\nabla\tilde{\mathbf{N}}(x)dx$$
$$\leq (c_9\gamma_* + \lambda CLC_\Omega + c_3\varepsilon)\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx$$
$$+\ \frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\sum_{i=1}^n|\tilde{\mathbf{f}}_{\mathbf{i}}(x)|^2(x)dx,$$

where constants c_i are independent of β_k, C_Ω is the Korn inequality constant.

These upper bounds for four terms yield the inequality

$$\frac{\nu_0}{2}\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx$$
$$\leq\frac{1}{4\varepsilon}\int_\Omega E_\beta^{(K)}(x)\Big(|\mathbf{f_0}|^2+|\tilde{\mathbf{f}}_{\mathbf{0}}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2+2\sum_{i=1}^n|\tilde{\mathbf{f}}_{\mathbf{i}}|^2\Big)dx$$
$$+(3c_3\varepsilon+(c_6+c_9)\gamma_*+\lambda CLC_\Omega)\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx.$$

Let $\gamma_*\leq\frac{\nu_0}{16(c_6+c_9)}$ and $\lambda CLC_\Omega\leq\frac{\nu_0}{4}$. Choose $\varepsilon=\frac{\nu_0}{48c_3}$, then from the last inequality we get

$$\frac{\nu_0}{8}\int_{\Omega:x_1^{(k)}<K}E_\beta(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx$$
$$\leq\frac{\nu_0}{8}\int_\Omega E_\beta^{(K)}(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx$$
$$\leq c\int_\Omega E_\beta^{(K)}(x)\Big(|\mathbf{f_0}|^2+|\tilde{\mathbf{f}}_{\mathbf{0}}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2+2\sum_{i=1}^n|\tilde{\mathbf{f}}_{\mathbf{i}}|^2\Big)dx$$
$$\leq c\int_\Omega E_\beta(x)\Big(|\mathbf{f_0}|^2+|\tilde{\mathbf{f}}_{\mathbf{0}}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2+2\sum_{i=1}^n|\tilde{\mathbf{f}}_{\mathbf{i}}|^2\Big)dx.$$

Since the right hand side is independent of K, we pass $K\to\infty$ in this inequality and get

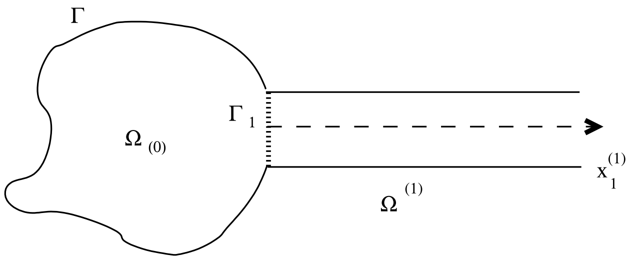
$$\frac{\nu_0}{8}\int_\Omega E_\beta(x)D(\tilde{\mathbf{N}}(x)):D(\tilde{\mathbf{N}}(x))dx$$
$$\leq c\int_\Omega E_\beta(x)\Big(|\mathbf{f_0}|^2+|\tilde{\mathbf{f}}_{\mathbf{0}}|^2+\sum_{i=1}^n|\mathbf{f_i}|^2+2\sum_{i=1}^n|\tilde{\mathbf{f}}_{\mathbf{i}}|^2\Big)dx.$$

Applying the weighted Poincaré and Korn inequalities we prove the assertion of the theorem. The estimate (19) follows from this estimate and from the estimate for $\mathbf{\Phi}$ via the Poiseuille flows $\mathbf{V}_{P\alpha_j}$ and so finally, via the fluxes F_j (see Lemma 2.6).

4. Existence, uniqueness and stabilization of a solution to the non-Newtonian flow equations in an unbounded domain with one outlet to infinity and a non-homogeneous Dirichlet's condition at some part of the boundary

Consider the domain $\Omega = \Omega_0\cup\Omega^{(1)}$, a particular case of the domain in Section 3. Denote $\Gamma = \partial\Omega_0\backslash\partial\Omega^{(1)}$, $\Gamma_1=\partial\Omega_0\cap\partial\Omega^{(1)}$, assume that Γ_1 is a cross-section of the cylinder $\Omega^{(1)}$ and let $\mathbf{g}$ be a vector-valued function belonging to the space $H_0^{1/2}(\Gamma)$ which is the space of traces of functions of $H^1(\Omega_0)$ vanishing at Γ_1, so that the support of function $\mathbf{g}$ belongs to Γ. Define

$$\|u\|_{H^{1/2}(\Gamma)} = \inf_{\tilde{u}\in H^1(\Omega_0),\tilde{u}|_\Gamma=u,\tilde{u}|_{\Gamma_1}=0} \|\tilde{u}\|_{H^1(\Omega_0)}.$$



Lemma 4.1. *There exists a divergence free extension* $\tilde{\mathbf{g}}$ *of* $\mathbf{g}$ *such that they coincide at* Γ, $\tilde{\mathbf{g}} = \mathbf{V}_{P\alpha}$ *at* Γ_1, *where* α *is such that* $-\int_\Gamma \mathbf{g}\cdot\mathbf{n} = \int_\sigma v_{P\alpha}dx'$, $\mathbf{n}$ *is an outer normal vector, and such, that* $\|\tilde{\mathbf{g}}\|_{H^1(\Omega_{(0)})} \le C_4\|\mathbf{g}\|_{H^{1/2}(\Gamma)}$, $C_4 > 0$.

Proof. Let $\bar{\mathbf{g}} \in H^1(\Omega_0)$ be an extension of $\mathbf{g} \in H^{1/2}(\Gamma)$ vanishing at Γ_1 and such that $\|\bar{\mathbf{g}}\|_{H^1(\Omega_0)} \le 2\|\mathbf{g}\|_{H^{1/2}(\Gamma)}$ (this extension exists due to the definition of the $H^{1/2}(\Gamma)$-norm). Consider now a Poiseuille flow function $v_{P\alpha}$ such that the flux $-\int_\Gamma \mathbf{g}\cdot\mathbf{n} = \int_\sigma v_{P\alpha}dx'$, and extend it in $H^1(\Omega_0)$ multiplying $v_{P\alpha}$ by a function depending on the longitudinal variable only, so that the extension $\tilde{v}_{P\alpha}$ vanishes at the boundary except for its part Γ_1. Then using Theorem 2.1 and the relation (3) we prove that there exists a constant $C_5 > 0$ depending on $\Omega_0, \nu_0, C, \lambda$ such that $\|\tilde{v}_{P\alpha}\|_{H^1(\Omega_0)} \le C_5\|\mathbf{g}\|_{H^{1/2}(\Gamma)}$. Consider the vector-valued function $\tilde{\mathbf{V}}_{P\alpha}$ having the first component $\tilde{v}_{P\alpha}$ and other components equal to zero. Let $\mathbf{h}$ be a solution of the divergence equation belonging to $H_0^1(\Omega_0)$:

$$\text{div } \mathbf{h} = -\text{div } (\bar{\mathbf{g}} + \tilde{\mathbf{V}}_{P\alpha}),$$

satisfying the estimate $\|\mathbf{h}\|_{H^1(\Omega_0)} \le C_6\|-\text{div } (\bar{\mathbf{g}}+\tilde{\mathbf{V}}_{P\alpha})\|_{L^2(\Omega_0)}$, $C_6 > 0$, and so $\|\mathbf{h}\|_{H^1(\Omega_0)} \le C_7\|\mathbf{g}\|_{H^{1/2}(\Gamma)}$, $C_7 > 0$. So, we can take $\tilde{\mathbf{g}} = \bar{\mathbf{g}} + \tilde{\mathbf{V}}_{P\alpha} + \mathbf{h}$. So the lemma is proved.

Below we extend function $\tilde{\mathbf{g}}$ to the whole domain Ω taking it equal to its trace at Γ_1 for all points of $\Omega^{(1)}$.

Consider a weak formulation of the following problem set in Ω:

$$\begin{cases} -\text{div}((\nu_0 + \lambda\nu(\dot{\gamma}(\mathbf{N}+\tilde{\mathbf{g}})))D(\mathbf{N}+\tilde{\mathbf{g}})) + \nabla(P - \alpha x_1) \\ = \mathbf{f_0} - \displaystyle\sum_{i=1}^n \frac{\partial \mathbf{f_i}}{\partial x_i},\ x \in \Omega\ , \\ \text{div}(\mathbf{N}+\tilde{\mathbf{g}}) = 0,\ x \in \Omega\ , \\ \mathbf{N} = 0,\ x \in \partial\Omega\ , \end{cases} \tag{21}$$

corresponding to a non-homogeneous boundary condition with a given velocity $\mathbf{g}$ at Γ. Define a weak solution as $\mathbf{N} \in H_0^1(\Omega)$, such that for any vector-valued divergence free test function $\varphi \in C_0^\infty(\Omega)$ (satisfying $\text{div}\varphi = 0$),

$$\begin{cases} \int_\Omega (\nu_0 + \lambda\nu(\dot\gamma(\mathbf{N}+\tilde{\mathbf{g}})))D(\mathbf{N}+\tilde{\mathbf{g}}) : D(\varphi) - \alpha\varphi_1 dx \\ = \int_\Omega \Big(\mathbf{f_0}\cdot\varphi + \sum_{i=1}^n \mathbf{f_i}\cdot\frac{\partial\varphi}{\partial x_i}\Big)dx \,, \\ \mathrm{div}(\mathbf{N}+\tilde{\mathbf{g}}) = 0,\ x\in\Omega\,. \end{cases} \tag{22}$$

Note that condition $\mathrm{div}(\mathbf{N}+\tilde{\mathbf{g}}) = 0$ is equivalent to $\mathrm{div}\mathbf{N} = 0$.

Theorem 4.2. *Let λ be positive number satisfying the condition*

$$\lambda < \frac{\nu_0}{CC_\Omega}. \tag{23}$$

Then problem (22) admits a unique weak solution $\mathbf{N} \in H_0^1(\Omega)$.

Proof. Rewrite the variational formulation in the form

$$\begin{aligned} &\int_\Omega \nu_0 D(\mathbf{N}) : D(\varphi)dx + \int_\Omega \lambda\nu(\dot\gamma(\mathbf{N}+\tilde{\mathbf{g}}))D(\mathbf{N}) : D(\varphi)dx + \\ &\int_\Omega \lambda\left(\nu(\dot\gamma(\mathbf{N}+\tilde{\mathbf{g}})) - \nu(\dot\gamma(\tilde{\mathbf{g}}))\right)D(\tilde{\mathbf{g}}) : D(\varphi)dx \\ &+ \int_\Omega \nu_0 D\tilde{\mathbf{g}} : D(\varphi) + \lambda\nu(\dot\gamma(\tilde{\mathbf{g}}))D\tilde{\mathbf{g}} : D(\varphi) - \alpha\varphi_1 dx \\ &= \int_\Omega \Big(\mathbf{f_0}\cdot\varphi + \sum_{i=1}^n \mathbf{f_i}\cdot\frac{\partial\varphi}{\partial x_i}\Big)dx \,, \end{aligned} \tag{24}$$

and

$$\mathrm{div}\mathbf{N} = 0, x\in\Omega.$$

The existence and uniqueness of a solution is proved by a Banach fixed point theorem argument with λ satisfying condition (14).

Define operator $\mathcal{L} : H^1_{div0}(\Omega) \to H^1_{div0}(\Omega)$ such that for any given $\mathbf{N}\in H^1_{div0}(\Omega)$, $\mathcal{L}\mathbf{N}$ is a solution to the problem: for any vector-valued test function $\varphi\in C_0^\infty(\Omega)$, such that $\mathrm{div}\varphi = 0$,

$$\begin{aligned} &\int_\Omega \nu_0 D(\mathcal{L}\mathbf{N}) : D(\varphi)dx + \int_\Omega \lambda\nu(\dot\gamma(\mathbf{N}+\tilde{\mathbf{g}}))D(\mathbf{N}+\tilde{\mathbf{g}}) : D(\varphi)dx \\ &+ \int_\Omega \nu_0 D\tilde{\mathbf{g}} : D(\varphi) - \int_\Omega \alpha\varphi_1 dx = \int_\Omega \Big(\mathbf{f_0}\cdot\varphi + \sum_{i=1}^n \mathbf{f_i}\cdot\frac{\partial\varphi}{\partial x_i}\Big)dx \,. \end{aligned} \tag{25}$$

Let us check that $\mathcal{L}$ is contraction for small values of λ. Let $\mathbf{N}_1, \mathbf{N}_2 \in H^1_{div0}(\Omega)$, denote $\mathbf{M}$ their difference, then the difference $\mathbf{Q} = \mathcal{L}\mathbf{N}_1 - \mathcal{L}\mathbf{N}_2$ satisfies the problem:

for any vector-valued test function $\varphi\in C_0^\infty(\Omega)$, such that $\mathrm{div}\varphi = 0$,

$$\begin{aligned} &\int_\Omega \nu_0 D(\mathbf{Q}) : D(\varphi)dx + \int_\Omega \lambda\Big(\nu(\dot\gamma(\mathbf{N}_1+\tilde{\mathbf{g}}))D(\mathbf{N}_1+\tilde{\mathbf{g}}) \\ &-\nu(\dot\gamma(\mathbf{N}_2+\tilde{\mathbf{g}}))D(\mathbf{N}_2+\tilde{\mathbf{g}})\Big) : D(\varphi)dx = 0 \end{aligned} \tag{26}$$

The end of the proof repeats the arguments of the proof of Theorem 3.4.

Theorem 4.3. *Let $\mathbf{f_0}, \ldots, \mathbf{f_n} \in \mathcal{L}^2_\beta(\Omega)$. Let λ be a positive number satisfying the condition*

$$\lambda < \frac{\nu_0}{CC_\Omega} min\{1, \frac{1}{4L}\}. \tag{27}$$

Then $\mathbf{N} \in \mathcal{W}^{1,2}_{\beta'}(\Omega)$ with some positive components of β' independent of $\mathbf{f_i}, i = 0, \ldots, n, \mathbf{g}$, and satisfies the following estimate:

$$\|\mathbf{N}\|_{\mathcal{W}^{1,2}_{\beta'}(\Omega)} \le C_{T6}\Big(\|\mathbf{f_0}\|_{\mathcal{L}^2_\beta(\Omega)} + \sum_{i=1}^{n} \|\mathbf{f_i}\|_{\mathcal{L}^2_\beta(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\Gamma)}\Big), \tag{28}$$

where C_{T6} is a positive constant independent of $\mathbf{f_i}, i = 0, \ldots, n, \mathbf{g}$.

Proof. We repeat the proof of Theorem 3.5 with the following modification. Function $\tilde{\mathbf{g}}$ should be presented in the form of the sum $\tilde{\mathbf{g}} = \tilde{\mathbf{g}}(1-\chi_1) + \chi_1\tilde{\mathbf{g}}$, and then the term $\chi_1\tilde{\mathbf{g}}$ is treated as terms $\chi_j \mathbf{V}_{P\alpha_j}$ in the proof of Theorem 3.5 and the term $\tilde{\mathbf{g}}(1-\chi_1)$ respectively as the term $-\mathbf{\Phi}$. We use Lemma 4.1 to evaluate the term $\tilde{\mathbf{g}}$ via $\|\mathbf{g}\|_{H^{1/2}(\Gamma)}$.

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