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A Limit Theorem for Twists of L-Functions of Elliptic Curves

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Abstract. In the paper, a limit theorem for weakly convergent probability measures on the complex plane for twisted with Dirichlet character L-functions of elliptic curves with an increasing modulus of the character is proved.

Keywords: characteristic transform, Dirichlet character, elliptic curve, L-function of elliptic curve, probability measure, weak convergence.

AMS Subject Classification: 11M41; 44K15.

1 Introduction

The present paper is a continuation of [5] and [7]. Therefore, we briefly remind the limit theorems for the modulus and argument for twists of L-functions of elliptic curves obtained in [5] and [4], respectively.

Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with non-zero discriminant $\Delta = -16(4a^3+27b^2)$. For a prime number p, denote by E_p the reduction of the curve E modulo p which is a curve over the finite field $\mathbb{F}_p$, and define $\lambda(p)$ by

$$\left|E(\mathbb{F}_p)\right| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ denotes the number of points of E_p. Let $s = \sigma + it$ be a complex variable. Then the L-function of the elliptic curve E is defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

In view of the Taniyama-Shimura conjecture which was proved in [1], the function $L_E(s)$ continues analytically to an entire function.

Now let χ be a Dirichlet character modulo q. Then the twist $L_E(s,\chi)$ of $L_E(s)$ with the character χ is defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s,\chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1},$$

and continues analytically to an entire function. Moreover, the function $L_E(s)$, for $\sigma > \frac{3}{2}$, can be rewritten in the form

$$L_E(s,\chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-1}, \tag{1.1}$$

where $\alpha(p)$ and $\beta(p)$ are complex conjugate numbers satisfying $\alpha(p)\beta(p) = p$ and $\alpha(p) + \beta(p) = \lambda(p)$.

Define two multiplicative functions $a_z(m)$ and $b_z(m)$, $m \in \mathbb{N}$, where, for $p \nmid \Delta$ and $l \in \mathbb{N}$,

$$a_z(p^l) = \sum_{j=0}^{l} d_z(p^j)\alpha^j(p) d_z(p^{l-j})\beta^{l-j}(p),$$

$$b_z(p^l) = \sum_{j=0}^{l} d_z(p^j)\overline{\alpha}^j(p) d_z(p^{l-j})\overline{\beta}^{l-j}(p),$$

while, for $p \mid \Delta$ and $l \in \mathbb{N}$,

$$a_z(p^l) = b_z(p^l) = d_z(p^l)\lambda^l(p),$$

where

$$d_z(p^l) = \frac{\theta(\theta+1)\cdots(\theta+l-1)}{l!} \quad \text{with} \quad \theta = \tfrac{z}{2}.$$

Suppose that the modulus q of the character χ is a prime number and is not fixed, and put, for $Q \geq 2$,

$$\mu_Q(\ldots) = M_Q^{-1} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0 \\ \ldots}} 1, \quad \text{where} \quad M_Q = \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} 1,$$

χ_0, as usual, denotes the principal character, and in place of dots a condition satisfied by a pair $(q, \chi(\mathrm{mod}\, q))$ is to be written.

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and, on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, define the probability measure $P_{\mathbb{R}}$ by the characteristic transforms

$$w_k(\tau) = \int\limits_{\mathbb{R}\setminus\{0\}} |x|^{i\tau} \operatorname{sgn}^k \mathrm{d}P_{\mathbb{R}} = \sum_{m=1}^{\infty} \frac{a_{i\tau}(m) b_{i\tau}(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R},\ k = 0, 1.$$

Then a limit theorem for $|L_E(s,\chi)|$ proved in [5] is of the form.

Theorem 1. *Suppose that* $\sigma > \frac{3}{2}$. *Then* $\mu_Q\big(\big|L_E(s,\chi)\big| \in A\big)$, $A \in \mathcal{B}(\mathbb{R})$, *converges weakly to the measure* $P_{\mathbb{R}}$ *as* $Q \to \infty$.

In [3], a limit theorem for the argument of the function $L_E(s,\chi)$ has been proved. Denote by γ the unit circle on the complex plane, and let P_γ be a probability measure on $(\gamma, \mathcal{B}(\gamma))$ defined by the Fourier transform

$$g(k) = \int\limits_{\gamma} x^k \mathrm{d}P_\gamma = \sum_{m=1}^{\infty} \frac{a_k(m) b_{-k}(m)}{m^{2\sigma}}, \quad k \in \mathbb{Z}.$$

Then the main result of [4] is the following statement.

Theorem 2. *Suppose that* $\sigma > \frac{3}{2}$. *Then*

$$\mu_Q\big(\exp\{i\arg L_E(s,\chi)\} \in A\big), \quad A \in \mathcal{B}(\gamma),$$

converges weakly to the measure P_γ *as* $Q \to \infty$.

The aim of this note is to consider the weak convergence of the frequency

$$P_{Q,\mathbb{C}}(A) \stackrel{def}{=} \mu_Q\big(L_E(s,\chi) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

as $Q \to \infty$. In other words, we will obtain a limit theorem for the twists $L_E(s,\chi)$ as $Q \to \infty$ on the complex plane $\mathbb{C}$. This theorem joins Theorems 1 and 2.

Let, for brevity,

$$\theta_\pm = \theta(\tau, \pm k) = \frac{i\tau \pm k}{2}, \qquad \hat{\theta}_\pm = 2\theta_\pm, \quad \tau \in \mathbb{R},\ k \in \mathbb{Z}.$$

On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, define the probability measure $P_{\mathbb{C}}$ by the characteristic transform

$$w_k(\tau,k) = \int\limits_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} \mathrm{e}^{ik\arg z} \mathrm{d}P_{\mathbb{C}} = \sum_{m=1}^{\infty} \frac{a_{\hat{\theta}_+}(m) b_{\hat{\theta}_-}(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R},\ k \in \mathbb{Z}.$$

Theorem 3. *Suppose that* $\sigma > \frac{3}{2}$. *Then* $P_{Q,\mathbb{C}}$ *converges weakly to the measure* $P_{\mathbb{C}}$ *as* $Q \to \infty$.

2 Characteristic Transform of $P_{Q,\mathbb{C}}$

For the proof of Theorem 3, we apply the method of characteristic transforms of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Thus, we start with a formula for the characteristic transform $w_Q(\tau,k)$ of the measure $P_{Q,\mathbb{C}}$. By the definition of $P_{Q,\mathbb{C}}$, we have that

$$\begin{aligned} w_Q(\tau,k) &= \int\limits_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} \mathrm{e}^{ik\arg z} \mathrm{d}P_{Q,\mathbb{C}} \\ &= \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \big|L_E(s,\chi)\big|^{i\tau} \mathrm{e}^{ik\arg L_E(s,\chi)}, \quad \tau \in \mathbb{R},\ k \in \mathbb{Z}. \end{aligned} \tag{2.1}$$

For arbitrary $\delta > 0$, denote by R the half-plane $\{s \in \mathbb{C} : \sigma \geq \frac{3}{2} + \delta\}$. Then, in view of the Hasse estimate

$$\left|\lambda(p)\right| \leq 2\sqrt{p}$$

and (1.1), we have that $L_E(s,\chi) \neq 0$ for $s \in R$. Thus,

$$\left|L_E(s,\chi)\right| = \left(L_E(s,\chi)\overline{L_E(s,\chi)}\right)^{\frac{1}{2}}$$

and

$$\mathrm{e}^{i\arg L_E(s,\chi)} = \left(\frac{L_E(s,\chi)}{\overline{L_E(s,\chi)}}\right)^{\frac{1}{2}}.$$

Using the Euler product (1.1), we find that, for $s \in R$,

$$\begin{aligned}
&\left|L_E(s,\chi)\right|^{i\tau}\mathrm{e}^{ik\arg L(s,\chi)} \\
&\quad = \exp\Bigg\{-\frac{i\tau}{2}\sum_{p|\Delta}\left(\log\left(1-\frac{\lambda(p)\chi(p)}{p^s}\right)+\log\left(1-\frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right) \\
&\qquad - \frac{k}{2}\sum_{p|\Delta}\left(\log\left(1-\frac{\lambda(p)\chi(p)}{p^s}\right)-\log\left(1-\frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right) \\
&\qquad - \frac{i\tau}{2}\sum_{p\nmid\Delta}\left(\log\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)+\log\left(1-\frac{\beta(p)\chi(p)}{p^s}\right)\right) \\
&\qquad - \frac{i\tau}{2}\sum_{p\nmid\Delta}\left(\log\left(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)+\log\left(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right) \\
&\qquad - \frac{k}{2}\sum_{p\nmid\Delta}\left(\log\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)+\log\left(1-\frac{\beta(p)\chi(p)}{p^s}\right)\right) \\
&\qquad + \frac{k}{2}\sum_{p\nmid\Delta}\left(\log\left(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)+\log\left(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right)\Bigg\} \\
&\quad = \prod_{p|\Delta}\exp\left\{-\frac{i\tau}{2}\log\left(1-\frac{\lambda(p)\chi(p)}{p^s}\right)-\frac{k}{2}\log\left(1-\frac{\lambda(p)\chi(p)}{p^s}\right)\right\} \\
&\qquad \times\prod_{p|\Delta}\exp\left\{-\frac{i\tau}{2}\log\left(1-\frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)+\frac{k}{2}\log\left(1-\frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right\} \\
&\qquad \times\prod_{p\nmid\Delta}\exp\Bigg\{-\frac{i\tau}{2}\left(\log\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)+\log\left(1-\frac{\beta(p)\chi(p)}{p^s}\right)\right) \\
&\qquad - \frac{k}{2}\left(\log\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)+\log\left(1-\frac{\beta(p)\chi(p)}{p^s}\right)\right)\Bigg\} \\
&\qquad \times\prod_{p\nmid\Delta}\exp\Bigg\{-\frac{i\tau}{2}\left(\log\left(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)+\log\left(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right) \\
&\qquad + \frac{k}{2}\left(\log\left(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)+\log\left(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)\right)\Bigg\}
\end{aligned}$$

$$= \prod_{p|\Delta}\left(1-\frac{\lambda(p)\chi(p)}{p^s}\right)^{-\theta_+}\prod_{p|\Delta}\left(1-\frac{\lambda(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{-\theta_-}$$
$$\times\prod_{p\nmid\Delta}\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)^{-\theta_+}\left(1-\frac{\beta(p)\chi(p)}{p^s}\right)^{-\theta_+}$$
$$\times\prod_{p\nmid\Delta}\left(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{-\theta_-}\left(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{-\theta_-}. \tag{2.2}$$

Here the multi-valued functions $\log(1-z)$ and $(1-z)^{-w}$, $w\in\mathbb{C}\setminus\{0\}$, in the region $|z|<1$ are defined by continuous variation along any path lying in this region from the values $\log(1-z)|_{z=0}=0$ and $(1-z)^{-w}|_{z=0}=1$, respectively.

In the above notation, we have that, for $|z|<1$,

$$(1-z)^{-\theta_\pm}=\sum_{l=0}^{\infty}d_{\hat{\theta}_\mp}(p^l)z^l.$$

Therefore, equality (2.2) shows that, for $s\in R$,

$$\left|L_E(s,\chi)\right|^{i\tau}\mathrm{e}^{ik\arg L_E(s,\chi)}$$
$$=\prod_{p|\Delta}\sum_{j=0}^{\infty}\frac{d_{\hat{\theta}_+}(p^j)\lambda^j(p)\chi^j(p)}{p^{js}}$$
$$\times\prod_{p|\Delta}\sum_{j=0}^{\infty}\frac{d_{\hat{\theta}_-}(p^j)\lambda^j(p)\overline{\chi}^j(p)}{p^{j\overline{s}}}\prod_{p\nmid\Delta}\sum_{l=0}^{\infty}\frac{d_{\hat{\theta}_+}(p^l)\alpha^l(p)\chi^l(p)}{p^{ls}}$$
$$\times\prod_{p\nmid\Delta}\sum_{r=0}^{\infty}\frac{d_{\hat{\theta}_+}(p^r)\beta^r(p)\chi^r(p)}{p^{rs}}\prod_{p\nmid\Delta}\sum_{l=0}^{\infty}\frac{d_{\hat{\theta}_-}(p^l)\overline{\alpha}^l(p)\overline{\chi}^l(p)}{p^{l\overline{s}}}$$
$$\times\prod_{p\nmid\Delta}\sum_{r=0}^{\infty}\frac{d_{\hat{\theta}_-}(p^r)\overline{\beta}^r(p)\overline{\chi}^r(p)}{p^{r\overline{s}}}. \tag{2.3}$$

Let $\hat{a}_z(m)$ and $\hat{b}_z(m)$ be multiplicative functions with respect to m defined, for $p\nmid\Delta$ and $l\in\mathbb{N}$, by

$$\hat{a}_z(p^l)=\sum_{j=0}^{l}d_z(p^j)\alpha^j(p)\chi^j(p)d_z(p^{l-j})\beta^{l-j}(p)\chi^{l-j}(p) \tag{2.4}$$

and

$$\hat{b}_z(p^l)=\sum_{j=0}^{l}d_z(p^j)\overline{\alpha}^j(p)\overline{\chi}^j(p)d_z(p^{l-j})\overline{\beta}^{l-j}(p)\overline{\chi}^{l-j}(p), \tag{2.5}$$

and, for $p\mid\Delta$ and $l\in\mathbb{N}$, by

$$\hat{a}_z(p^l)=d_z(p^l)\lambda^l(p)\chi^l(p) \tag{2.6}$$

and

$$\hat{b}_z(p^l) = d_z(p^l)\lambda^l(p)\overline{\chi}^l(p). \tag{2.7}$$

For $|\tau| \le c$, with arbitrary $c > 0$, and $l \in \mathbb{N}$, we have that

$$\begin{aligned}\left|d_{\hat{\theta}_\pm}(p^l)\right| &\le \frac{|\theta_\pm|(|\theta_\pm|+1)...(|\theta_\pm|+l-1)}{l!}\\ &\le |\theta_\pm|\prod_{v=1}^{l}\left(1+\frac{|\theta_\pm|}{v}\right) \le (l+1)^{c_1},\end{aligned} \tag{2.8}$$

where c_1 depends on c and k, only. By the definition of the quantities $\alpha(p)$ and $\beta(p)$, we find that $|\alpha(p)| = |\beta(p)| = \sqrt{p}$. Thus, for $p \nmid \Delta$ and $l \in \mathbb{N}$, equalities (2.4) and (2.5) give the bounds

$$\left|\hat{a}_{\hat{\theta}_+}(p^l)\right| \le p^{\frac{l}{2}}\sum_{j=0}^{l}(j+1)^{c_1}(l-j+1)^{c_1} \le p^{\frac{l}{2}}(l+1)^{2c_1+1} \tag{2.9}$$

and

$$\left|\hat{b}_{\hat{\theta}_-}(p^l)\right| \le p^{\frac{l}{2}}(l+1)^{2c_1+1}. \tag{2.10}$$

It is known [6] that, for $p \mid \Delta$, the numbers $\lambda(p)$ take the values 1 or 0. Therefore, in view of (2.6)–(2.8), for $p \mid \Delta$ and $l \in \mathbb{N}$,

$$\left|\hat{a}_{\hat{\theta}_+}(p^l)\right| \le (l+1)^{c_1}, \quad \left|\hat{b}_{\hat{\theta}_-}(p^l)\right| \le (l+1)^{c_1}. \tag{2.11}$$

Now estimates (2.9)–(2.11) and the multiplicativity of the functions $\hat{a}_z(m)$ and $\hat{b}_z(m)$ yield

$$\left|\hat{a}_{\hat{\theta}_+}(m)\right| = \sqrt{m}\prod_{\substack{p^l \mid m\\ p^{l+1}\nmid m}}(l+1)^{2c_1+1} = \sqrt{m}d^{2c_1+1}(m), \tag{2.12}$$

$$\left|\hat{b}_{\hat{\theta}_-}(m)\right| \le \sqrt{m}d^{2c_1+1}(m), \tag{2.13}$$

where $d(m)$ denotes the divisor function. Since

$$d(m) = \mathrm{O}_\varepsilon(m^\varepsilon) \tag{2.14}$$

with every $\varepsilon > 0$, the above estimates show that the series

$$\sum_{m=1}^{\infty}\frac{a_{\hat{\theta}_+}(m)}{m^s} \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{b_{\hat{\theta}_-}(m)}{m^{\overline{s}}}$$

converge absolutely for $|\tau| \le c$, $k \in \mathbb{Z}$ and $s \in R$. Therefore, taking into account (2.3), we obtain that, for $|\tau| \le c$, $k \in \mathbb{Z}$ and and $s \in R$,

$$\begin{aligned}&\left|L_E(s,\chi)\right|^{i\tau}\mathrm{e}^{ik\arg L_E(s,\chi)}\\ &= \prod_{p\mid\Delta}\sum_{j=0}^{\infty}\frac{\hat{a}_{\hat{\theta}_+}(p^j)}{p^{js}}\prod_{p\mid\Delta}\sum_{l=0}^{\infty}\frac{\hat{b}_{\hat{\theta}_-}(p^l)}{p^{l\overline{s}}}\prod_{p\nmid\Delta}\sum_{j=0}^{\infty}\frac{\hat{a}_{\hat{\theta}_+}(p^j)}{p^{js}}\prod_{p\nmid\Delta}\sum_{l=0}^{\infty}\frac{\hat{b}_{\hat{\theta}_-}(p^l)}{p^{l\overline{s}}}\\ &= \sum_{m=1}^{\infty}\frac{\hat{a}_{\hat{\theta}_+}(m)}{m^s}\sum_{n=1}^{\infty}\frac{\hat{b}_{\hat{\theta}_-}(n)}{n^{\overline{s}}}.\end{aligned}$$

Substituting this into (2.1), we find that

$$w_Q(\tau,k)=\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\sum_{m=1}^{\infty}\frac{\hat a_{\hat\theta_+}(m)}{m^s}\sum_{n=1}^{\infty}\frac{\hat b_{\hat\theta_-}(n)}{n^{\bar s}}. \tag{2.15}$$

3 Proof of Theorem 3

For the proof of Theorem 3, we apply the following continuity theorem in terms of characteristic transforms for probability measures on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$. Let P_n, $n\in\mathbb{N}$, and P be probability measures on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$. We say that P_n, as $n\to\infty$, converges weakly in the sense of $\mathbb{C}$ to P if P_n converges weakly to P as $n\to\infty$, and, additionally,

$$\lim_{n\to\infty}P_n\big(\{0\}\big)=P\big(\{0\}\big).$$

Denote by $w_n(\tau,k)$ the characteristic transform of the measure P_n.

Lemma 1. *Suppose that*

$$\lim_{n\to\infty}w_n(\tau,k)=w(\tau,k),\quad \tau\in\mathbb{R},\ k\in\mathbb{Z},$$

where the function $w(\tau,0)$ is continuous at the point $\tau=0$. Then, on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$, there exists a probability measure P such that P_n converges weakly in the sense of $\mathbb{C}$ to P as $n\to\infty$. In this case, $w(\tau,k)$ is the characteristic transform of the measure P.

Proof of the lemma is given in [7], [8].

Proof of Theorem 3. Using formula (2.15), we will obtain the asymptotics for $w_Q(\tau,k)$ as $Q\to\infty$. Let $r=\log Q$. Then estimates (2.12)–(2.14) show that, uniformly in $s\in R$, $|\tau|\le c$, and, for any fixed $k\in\mathbb{Z}$ and $\varepsilon>0$,

$$\sum_{m>r}\frac{\hat a_{\hat\theta_+}(m)}{m^s}=\mathrm{O}_\varepsilon\bigg(\sum_{m>r}\frac{1}{m^{1+\delta-\varepsilon}}\bigg)=\mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big)$$

and

$$\sum_{m>r}\frac{\hat b_{\hat\theta_-}(m)}{m^{\bar s}}=\mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big).$$

Substituting this in (2.15), we obtain that, for $s\in R$, $|\tau|\le c$ and any fixed $k\in\mathbb{Z}$,

$$w_Q(\tau,k)=\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\bigg(\bigg(\sum_{m\le r}\frac{\hat a_{\hat\theta_+}(m)}{m^s}+\mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big)\bigg)$$
$$\bigg(\sum_{n\le r}\frac{\hat b_{\hat\theta_-}(n)}{n^{\bar s}}+\mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big)\bigg)\bigg)$$

$$= \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}}\biggl(\sum_{m\le r}\frac{\hat a_{\hat\theta_+}(m)}{m^s}\sum_{n\le r}\frac{\hat b_{\hat\theta_-}(n)}{n^{\overline{s}}}\biggr)$$
$$+ \mathrm{O}_\varepsilon\biggl(r^{-\delta+\varepsilon}\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}}\biggl(\biggl|\sum_{m\le r}\frac{\hat a_{\hat\theta_+}(m)}{m^s}\biggr|\biggl|\sum_{n\le r}\frac{\hat b_{\hat\theta_-}(n)}{n^{\overline{s}}}\biggr|\biggr)\biggr)$$
$$+ \mathrm{O}_\varepsilon\bigl(r^{-\delta+\varepsilon}\bigr)$$
$$= \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}}\biggl(\sum_{m\le r}\frac{\hat a_{\hat\theta_+}(m)}{m^s}\sum_{n\le r}\frac{\hat b_{\hat\theta_-}(n)}{n^{\overline{s}}}\biggr) + \mathrm{O}_\varepsilon\bigl(r^{-\delta+\varepsilon}\bigr). \tag{3.1}$$

Here we have used the estimates

$$\sum_{m\le r}\frac{\hat a_{\hat\theta_+}(m)}{m^s} = \mathrm{O}\biggl(\sum_{m\le r}\frac{1}{m^{1+\delta-\varepsilon}}\biggr) = \mathrm{O}(1), \quad \sum_{n\le r}\frac{\hat b_{\hat\theta_-}(n)}{n^{\overline{s}}} = \mathrm{O}(1),$$

which are uniform in $s \in R$ and $|\tau| \le c$.

By (2.4)–(2.7), using the multiplicativity of $\hat a_{\hat\theta_+}(m)$ and $\hat b_{\hat\theta_-}(n)$, the complete multiplicativity of Dirichlet characters as well as the notation for $a_{\hat\theta_+}(m)$ and $b_{\hat\theta_-}(n)$, we find that

$$\hat a_{\hat\theta_+}(m) = \prod_{\substack{p^l\mid m\\ p^{l+1}\nmid m,\ p\nmid\Delta}}\sum_{j=0}^{l} d_{\hat\theta_+}\bigl(p^j\bigr)\alpha^j(p)d_{\hat\theta_+}\bigl(p^{l-j}\bigr)\beta^{l-j}(p)\chi\bigl(p^l\bigr)$$
$$\times \prod_{\substack{p^l\mid m\\ p^{l+1}\nmid m,\ p\mid\Delta}} d_{\hat\theta_+}\bigl(p^l\bigr)\lambda^l(p)\chi\bigl(p^l\bigr)$$
$$= \chi(m)\prod_{\substack{p^l\mid m\\ p^{l+1}\nmid m,\ p\nmid\Delta}}\sum_{j=0}^{l} d_{\hat\theta_+}\bigl(p^j\bigr)\alpha^j(p)d_{\hat\theta_+}\bigl(p^{l-j}\bigr)\beta^{l-j}(p)$$
$$\times \prod_{\substack{p^l\mid m\\ p^{l+1}\nmid m,\ p\mid\Delta}} d_{\hat\theta_+}\bigl(p^l\bigr)\lambda^l(p) = a_{\hat\theta_+}(m)\chi(m), \quad \hat b_{\hat\theta_-}(m) = b_{\hat\theta_-}(m)\overline{\chi}(m).$$

Thus, (3.1) can be written in the form

$$w_Q(\tau,k) = \sum_{m\le r}\frac{a_{\hat\theta_+}(m)}{m^s}\sum_{n\le r}\frac{b_{\hat\theta_-}(n)}{n^s}\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n)$$
$$+ \mathrm{O}_\varepsilon\bigl(r^{-\delta+\varepsilon}\bigr). \tag{3.2}$$

First we suppose that $m \neq n$. Then, clearly,

$$\sum_{q \le Q} \sum_{\substack{\chi=\chi(\bmod q)\\ \chi \neq \chi_0}} \chi(m)\overline{\chi}(n) = \sum_{q \le Q} \sum_{\substack{\chi=\chi(\bmod q)\\ \chi \neq \chi_0}} \left|\chi(m)\right|^2$$

$$= M_Q - \sum_{\substack{q|m\\ q \le r}} (q-2) = M_Q + \mathrm{O}\biggl(\sum_{q \le r} q\biggr) = M_Q + \mathrm{O}\bigl(r^2\bigr). \tag{3.3}$$

If $m \neq n$, then, in view of the formula

$$\sum_{\chi=\chi(\bmod q)} \chi(m)\overline{\chi}(n) = \begin{cases} q-1 & \text{if } m \equiv n (\bmod q), \\ 0 & \text{if } m \not\equiv n (\bmod q) \end{cases}$$

with $(m,q)=1$, we find that

$$\sum_{q \le Q} \sum_{\substack{\chi=\chi(\bmod q)\\ \chi \neq \chi_0}} \chi(m)\overline{\chi}(n) = \sum_{q \le Q} \sum_{\substack{\chi=\chi(\bmod q)\\ q|(m-n)}} \chi(m)\overline{\chi}(n)$$

$$+ \sum_{q \le Q} \sum_{\substack{\chi=\chi(\bmod q)\\ q\nmid(m-n)}} \chi(m)\overline{\chi}(n) + \mathrm{O}\biggl(\sum_{q \le Q} 1\biggr)$$

$$= \mathrm{O}\biggl(\sum_{q \le r} q\biggr) + \mathrm{O}\biggl(\frac{Q}{\log Q}\biggr) = \mathrm{O}\left(\frac{Q}{\log Q}\right).$$

Therefore, in view of the estimate [8]

$$M_Q = \frac{Q^2}{2\log Q} + \mathrm{O}\left(\frac{Q^2}{\log^2 Q}\right),$$

we obtain from (3.2) and (3.3) that, uniformly in $s \in R$, $|\tau| \le c$, and any fixed $k \in \mathbb{Z}$

$$w_Q(\tau,k) = \sum_{m=1}^{\infty} \frac{a_{\hat{\theta}_+}(m) b_{\hat{\theta}_-}(m)}{m^{2\sigma}} + \mathrm{o}(1) \tag{3.4}$$

as $Q \to \infty$.

The functions $a_{\hat{\theta}_+}(m)$ and $b_{\hat{\theta}_-}(m)$ are continuous in τ. Therefore, the uniform convergence for $|\tau| \le c$ of the series

$$w(\tau,k) = \sum_{m=1}^{\infty} \frac{a_{\hat{\theta}_+}(m) b_{\hat{\theta}_-}(m)}{m^{2\sigma}}$$

shows that the function $w(\tau,0)$ is continuous at $\tau=0$. Therefore, (3.4) together with Lemma 1 proves the theorem. □

4 Concluding Remarks

Denote by $L(s,\chi)$ the Dirichlet L-functions. E. Stankus in [9] obtained, for $\sigma > \frac{1}{2}$, the weak convergence for

$$P_Q(A) \overset{def}{=} \mu_Q\bigl(L_E(s,\chi) \in A\bigr), \quad A \in \mathcal{B}(\mathbb{C}),$$

as $Q \to \infty$. Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ defined by the characteristic transform

$$w_R(\tau, k) = \sum_{m=1}^{\infty} \frac{d_{\hat{\theta}_+}(m) d_{\hat{\theta}_-}(m)}{m^{2\sigma}}, \quad \sigma > \frac{1}{2},\ \tau \in \mathbb{R},\ k \in \mathbb{Z}_0.$$

Then he proved that P_Q converges weakly to P as $Q \to \infty$. This theorem connects limit theorems for $|L(s,\chi)|$ and $\arg L(s,\chi)$ obtained by P.D.T.A. Elliott in [2] and [3], respectively.

Thus, a limit theorem for Dirichlet L-functions is valid also on the left of the absolute convergence half-plane $\sigma > 1$. For the proof of such a theorem, the convergence, for $\sigma > \frac{1}{2}$, of the series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \chi \neq \chi_0$$

is essentially used. Differently from Dirichlet L-functions, we do not have any information on the convergence of the series for $L_E(s,\chi)$, $\chi \neq \chi_0$, in the half-plane $\sigma > 1$. Therefore, we can prove Theorem 3 only in the half-plane of absolute convergence of the series for $L_E(s,\chi)$. However, we conjecture that Theorem 3 remains also true for $\sigma > 1$.

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