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# Hypothesis on the Solvability of Parabolic Equations with Nonlocal Conditions

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**Abstract**

Numerous and different nonlocal conditions for the solvability of parabolic equations were researched in many articles and reports. The article presented analyzes such conditions imposed, and observes that the existence and uniqueness of the solution of parabolic equation is related mainly to "smallness" of functions, involved in nonlocal conditions. As a consequence the hypothesis has been made, stating the assumptions on functions in nonlocal conditions are related to numerical algorithms of solving parabolic equations, and not to the parabolic equation itself.

**Keywords:** parabolic equation, nonlocal condition, finite difference method

## 1 Statement of the Problem

Let us consider a nonlinear equation of parabolic type

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) - F(x,t,u) \tag{1}$$

with nonlocal conditions

$$u(0,t) = \int_0^1 \alpha_0(x)u(x,t)dx + \mu_0(t), \tag{2}$$

$$u(1,t) = \int_0^1 \alpha_1(x)u(x,t)dx + \mu_1(t) \tag{3}$$

and the initial condition

$$u(x,0) = \varphi(x). \tag{4}$$

Many authors researched parabolic equations imposing different nonlocal conditions. Parabolic equations with a kind of nonlocal conditions were solved in [1] – [6], by numerical methods. The authors of papers [7] – [10] dealt with conditions on the existence and uniqueness of a solution. In most of these articles sufficient conditions for the existence and uniqueness of the solution are related to the "smallness" of functions $\alpha_0(x), \alpha_1(x)$. For example, in [8] – [10] these conditions are

$$\int_0^1 |\alpha_i(x)|dx \leq \mu < 1, \quad i = 1, 2. \tag{5}$$

Authors of [5], [6] use the following conditions

$$\int_0^1 |\alpha_i(x)|^2 dx \leq \mu^2 < 1 \tag{6}$$

or a slightly differing ones.

It is of interest to note, the authors when considering an ordinary differential equation, analogous to equation (1), with a nonlocal condition, get sufficient conditions analogous to that in (5) or (6). It looks like the conditions (5) or (6), defined by the "smallness" of functions $\alpha_i(x)$, $i = 1, 2$, are characteristic to the problem (1) – (4).

The analysis described in this paper raises some doubts to this inference. As a result the hypothesis is posed stating the conditions of type (5) or (6) are associated only with the method considered, but not with the gist of problem (1) – (4).

This paper analyzes three ways of discretization of a differential equation: discretization by spatial coordinate, by the time variable, and by both variables too.

These three cases are studied to find out which additional restrictions on the numerical method are posed by a nonlocal condition.

# 2 Discretization by the time variable

For the first of all we consider a simpler problem with a nonlocal condition:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} - qu - f(x,t), \tag{7}$$

$$u(0,t) = \mu_0(t), \tag{8}$$

$$u(1,t) = \alpha_1 u(\xi,t) + \mu_1(t) \tag{9}$$

and the initial condition (4) ($0 \le x \le 1, 0 \le t \le T$). This is a particular case of the earlier formulated problem (1) – (4). Here $a > 0, q \ge 0, \alpha_1$ are constants, $0 < \xi < 1$. This problem is solved by the line method, i.e., using only the discretization by the variable $t$. We replace problem (7) – (9), (4) by the following system of ordinary differential equations:

$$\frac{u^j(x) - u^{j-1}(x)}{\tau} = a\frac{d^2 u^j(x)}{dx^2} -$$

$$-qu^j(x) - f^j(x), \ j = 1, \dots M \tag{10}$$

$$u^j(0) = \mu_0^j, \tag{11}$$

$$u^j(1) = \alpha_1 u^j(\xi) + \mu_1^j, \tag{12}$$

$$u^0(x) = \varphi(x), \quad 0 \le x \le 1. \tag{13}$$

We denote here

$$u^j(x) = u(x, t_j), \quad t_j = j\tau,$$

$\tau$ is a step in the direction of $t$ axis. Knowing $u^{j-1}(x)$, we can find the function $u^j(x)$ as the solution of the following boundary value problem for the ordinary differential equation with the nonlocal condition:

$$a\frac{d^2u^j}{dx^2} - \big(q + \frac{1}{\tau}\big)u^j = \tilde{f}^j, \tag{14}$$

$$u^j(0) = \mu_0^j, \tag{15}$$

$$u^j(1) = \alpha_1 u^j(\xi) + \mu_1^j. \tag{16}$$

Here $\tilde{f}^j = f^j - u^{j-1}/\tau$. Such a problem has been studied completely in [3]. As $\tau$ is small, the equation (14) turnes to an equation with a small parameter and highest order derivative:

$$\tau\frac{d^2u^j}{dx^2} - \frac{\tau q + 1}{a}u^j = -\frac{\tau}{a}f^j - \frac{u^{j-1}}{a}. \tag{17}$$

The case of a such kind has been studied in many details in [11]. The authors in [3] have proved the boundary value problem for ordinary differential equation with nonlocal condition (14) – (16) is equivalent to the classical boundary value problem, i.e., equation (14) with boundary condition (15) and an additional boundary condition

$$u^j(1) = \lambda^j; \tag{18}$$

here $\lambda^j$ is a mean while unknown finite number.

In other words, one seeks the solution to the boundary value problem with a nonlocal condition in the same functional space as that of the classical boundary value problem.

**Lemma 1** [3]. *To solve the boundary problem with a nonlocal condition (14) – (15) uniquely in the same functional space as the boundary value problem with conditions (15), (18) it is necessary and sufficient that the inequality*

$$1 - \alpha_1 w(\xi) \neq 0 \tag{19}$$

*be valid. Here $w(x)$ is the solution of the following boundary value problem*

$$a\frac{d^2w}{dx^2} - \big(q + \frac{1}{\tau}\big)w = 0, \tag{20}$$

$$w(0) = 0, \quad w(1) = 1. \tag{21}$$

Let us write the solution of problem (20) – (21) as follows:

$$w(x) = \frac{e^{\alpha x} - e^{-\alpha x}}{e^{\alpha} - e^{-\alpha}} = \frac{\operatorname{sh} \alpha x}{\operatorname{sh} \alpha}, \tag{22}$$
$$\alpha = \Big(\frac{1}{a}\big(q + \frac{1}{\tau}\big)\Big)^{1/2}.$$

Since $0 < w(x) < 1$ as $0 < x < 1$, from (19) we obtain a sufficient condition

$$-\infty < \alpha_1 < 1 \tag{23}$$

for single-valued solvability of problem (14) – (16).

The other corollary obtained from Lemma 1 is of importance: there exists only a single value $\alpha_1 = \alpha_1^* > 1$, defined by the equality

$$1 - \alpha_1^* w(\xi) = 0, \tag{24}$$

at which the problem with nonlocal condition (14) – (16) isn't uniquely solved in the same functional space as problem (14), (15), (18).

**Lemma 2**. *For problem (14) – (16), the statement is that*

$$\alpha_1^* \to \infty, \quad as \quad \tau \to 0.$$

P r o o f. If $\tau \to 0$, then $\alpha \to \infty$. Thus, we get from equality (22) that

$$w(x) = \frac{e^{\alpha x} - e^{-\alpha x}}{e^{\alpha} - e^{-\alpha}} = \frac{e^{-\alpha(1-x)} - e^{-\alpha(1+x)}}{1 - e^{-2\alpha}} \to 0$$

as $\alpha \to \infty$ for all $0 \leq x < 1$. So, we obtain from equality (24) that

$$\alpha_1^* \to \infty, \quad as \quad \tau \to 0.$$

The lemma is proved.

Thus, we have obtained the following conditional result.

**Corollary**. If the solution of discrete problem (10) – (13) converges, as $\tau \to 0$, to the solution of the problem with nonlocal condition (7) – (9), (4), then, for the existence and uniqueness of the type (19) is unnecessary, i.e., problem (7) – (9), (4) is solved for all values of $\alpha_1$.

# 3 Discretization by variables $x$ and $t$

Let us consider a finite difference method (an implicit scheme) to solve the problem (7) – (9), (4):

$$\frac{u_i^j - u_i^{j-1}}{\tau} = a\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} - qu_i^j + f_i^j, \qquad (25)$$

$$i = 1, 2, \ldots, N-1;\ j = 1, 2, \ldots, M;$$

$$u_0^j = \mu_0^j, \qquad (26)$$

$$u_N^j = \alpha_1 u_k^j + \mu_1^j, \qquad (27)$$

$$u_0^j = \mu_0^j, \qquad (28)$$

here $h = 1/N, \tau = T/M$. For simplicity, we make an assumption that $\xi = kh$, where $k$ is an integer. If $k$ is non-integer, then equation (28) is replaced by a more complex one [3], and the main results do not change.

Let us make use of the result in [3] for problem (25) – (28):

**Lemma 3**. [3] *In order the system of equations (25) – (27) for any value of* $j = 1, 2, \ldots, M$, *to have a unique solution, the necessary and sufficient condition is the inequality*

$$1 - \alpha_1 w_k \neq 0 \qquad (29)$$

*is valid. Here* $w_i, i = 0, 1, \ldots, N$ *the solution of the system of equations*

$$a\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - (q + \frac{1}{\tau})w_i = 0, \quad i = 1, 2, \ldots, N-1, \qquad (30)$$

$$w_0 = 0, \quad w_1 = 1.$$

We can represent the solution of system (30) in the explicit form

$$w_i = \frac{e^{\alpha i} - e^{-\alpha i}}{e^{\alpha N} - e^{-\alpha N}} = \frac{\operatorname{sh} \alpha i}{\operatorname{sh} \alpha N}, \qquad (31)$$

where

$$\operatorname{ch}\alpha = \frac{2\sigma + (1+\tau q)a^{-1}}{2\sigma}, \quad \sigma = \frac{\tau}{h^2}.$$

By $\alpha_1^*$ we denote, as earlier, a value defined by the equality

$$1 - \alpha_1^* w_k = 0. \tag{32}$$

For the system of equations (25) – (28) the following statement holds.

**Lemma 4**. *If* $\tau \to 0,\ h \to 0,$ *then* $\alpha_1^* \to \infty$.

P r o o f. Assume $\sigma = const \neq 0$, as $\tau \to 0, h \to 0$. Then $\operatorname{ch}\alpha \to (2\sigma + a^{-1})/2\sigma = const$, thus, $\alpha \to const$ and

$$\lim_{\tau\to 0,\ h\to 0} w_k = \lim_{\tau\to 0,\ h\to 0} \frac{e^{\alpha k} - e^{-\alpha k}}{e^{\alpha N} - e^{-\alpha N}} =$$

$$= \lim_{\tau\to 0,\ h\to 0} \frac{e^{-(N-k)\alpha}\left(1 - e^{-2k\alpha}\right)}{1 - e^{-2N\alpha}} = 0.$$

If $\tau$ and $h$ tend to zero so that $\sigma \to 0$, then $\operatorname{ch}\alpha \to \infty$, i.e. $\alpha \to \infty$ and

$$\lim_{\tau\to 0,\ h\to 0} w_k = 0.$$

Analogously, if $\tau$ and $h$ tend to zero so that $\sigma \to \infty$, then $\operatorname{ch}\alpha \to 1$, i.e., $\alpha \to 0$ and

$$\lim_{\tau\to 0,\ h\to 0} w_k = 0.$$

Equality (32) yields the proposition of the lemma.

We will obtain another corollary analogous to that formulated at the end of second part of this paper.

**Corollary**. If the solution of a system of difference equations (25) – (28) converges, as $\tau \to 0, h \to 0$, to the solution of problem with a nonlocal condition (7) – (9), (4), then for the existence and uniqueness of the latter solution, the condition of type (29) is unnecessary, i.e., problem (7) – (9), (4) is solved at all values of $\alpha_1$.

# 4 Discretization by the spatial variable

Let us deal now with the following discretization of problem (7) – (9), (4):

$$\frac{du_i(t)}{dt} = a\frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{h^2} - qu_i(t) - f_i(t), \qquad (33)$$

$$u_0(t) = \mu_0(t), \qquad (34)$$

$$u_N(t) = \alpha_1 u_k(t) + \mu_1(t), \qquad (35)$$

$$u_i(0) = \varphi(x_i); \qquad (36)$$

here $u_i(t) = u(x_i, t); i = 0, 1, \ldots, N; h = 1/N, kh = \xi$.

We rearrange the system of equations (33) – (35) as follows:

$$\begin{aligned} \frac{du_1(t)}{dt} &= \frac{a}{h^2}(-2u_1(t) + u_2(t)) - qu_1(t) - (f_1(t) - \frac{a}{h^2}\mu_0(t)), \\ \frac{du_i(t)}{dt} &= \frac{a}{h^2}(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) - qu_i(t) - f_i(t), \quad (37) \\ & \quad i = 2, 3, \ldots, N-2, \\ \frac{du_{N-1}(t)}{dt} &= \frac{a}{h^2}(u_{N-2}(t) - 2u_{N-1}(t) + \alpha_1 u_k(t)) - qu_N(t) - \\ & \qquad - \left(f_{N-1}(t) - \frac{a}{h^2}\mu_1(t)\right). \end{aligned}$$

Let us write system (37) as follows:

$$\frac{du}{dt} = Au - \tilde{f}, \qquad (38)$$

where $u$ is an $(N-1)$-dimensional vector, $A$ is a given quadratic matrix. For the solution of system (38) the initial condition (36) holds.

**Lemma 5**. *For all values $h > 0$, there exists a unique solution of system (37) that satisfies the initial condition (36).*

P r o o f. The proposition of the lemma follows from the fact that the system (37) is linear. Thus, for each $h > 0$ the Lipschitz condition holds.

Consequently, when solving a parabolic equation with nonlocal condition by the line method (33), the nonlocal condition does not cause any additional restrictions on the existence of the unique solution for all values of $h > 0$ .

# 5 Other nonlocal conditions

Let us take other more general condition, instead of nonlocal conditions (9):

$$u(1,t) = \int_0^1 \rho(\xi)u(\xi,t)d\xi + \mu_1(t), \tag{39}$$

where $\rho(x)$ is a given function in the interval [0,1]. We replace the differential problem (7), (8), (39), (4) by a discrete system (discretization by the variable $t$):

$$a\frac{d^2u^j(x)}{dx^2} - \left(q + \frac{1}{\tau}\right)u^j(x) = f^j(x) - \frac{u^{j-1}(x)}{\tau}, \tag{40}$$

$$u^j(0) = \mu_0^j, \tag{41}$$

$$u^j(1) = \int_0^1 \rho(\xi)u^j(\xi)d\xi + \mu_1^j, \tag{42}$$

$$u^0(x) = \varphi(x), \tag{43}$$

**Lemma 6** [3]. *To solve the boundary value problem with nonlocal condition (40) – (42) for all $j = 1,2,\ldots,M$ uniquely in the same functional space as the boundary value problem with condition (41), (18) the necessary and sufficient condition is that the inequality*

$$1 - \int_0^1 \rho(\xi)w(\xi)d\xi \neq 0 \tag{44}$$

*is valid, where $w(x)$ is a solution of the boundary value problem (20), (21).*

Denote $|\rho(x)| \leq M, \quad 0 \leq x \leq 1$ and let us analyze what conditions should be imposed that the inequality

$$\int_0^1 \rho(x) w(\xi) d\xi < 1$$

must hold. Lemma 2 proves that $w(x) \to 0$ for all $0 \leq x < 1$, as $\tau \to 0$. Thus, $M \to \infty$ as $\tau \to 0$.

Consequently, if we take nonlocal condition (39) instead of condition (9), we obtain the same corollary that was formulated following Lemma 2.

An analogous conclusion is obtained in treating a parabolic type equation with variable coefficients as well as nonlinear differential equation (1). Indeed the results in [3], obtained for a respective ordinary differential equation with nonlocal condition lay the basis for that.

# 6 The Hypothesis

We formulate now a hypothesis as the main result of the paper.

**Hypothesis**. For a boundary value problems of a parabolic equation with the nonlocal condition

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) - F(x,t,u),\\
&u(0,t) = \mu_0(t),\\
&u(1,t) = \int_0^1 \rho(x) u(x,t) dx + \mu_1(t),\\
&u(x,0) = \varphi(x)
\end{aligned}$$

the conditions for the existence of a unique solution are analogous to that on following differential problem (i.e., there are no additional restrictions other than that on an ordinary differential equation):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) - F(x,t,u),$$
$$u(0,t) = \mu_0(t), \quad u(1,t) = \mu_1(t),$$
$$u(x,0) = \varphi(x).$$

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