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# Presheaves on midsymmetrical quantaloids

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## 1. Introduction

In previous paper [2] we studied presheaves and sheaves on an arbitrary quantaloid (category enriched in the category of complete sup-lattices subject to certain laws), where we had choosed the C.J. Mulvey and M. Nawaz theory [3] as a guideline for futher generalizations taking our inspiration in the matrix approach of G. Van den Bossche [1]. In this note we continue those investigations restricting our interest to a special subclass of quantaloids, "midsymmetrical" quantaloids. In this setting it is possible to express the structural matrix of the underlying $Q$-set of a regular and separated presheaf in terms of its "restriction" and "diagonal" (Theorem 3.8). We are going to detail it in a subsequent paper.

## 2. Quantaloids and matrices

We first review some of the basic notions.

DEFINITION 2.1. A *quantaloid* is a locally small category $Q$ such that:

(i) for all $u$,$v$ objects in $Q$, the hom-set $Q(u, v)$ is a complete lattice,

(ii) composition of morphisms of $Q$ (in this paper denoted by &) preserves arbitrary joins in both variables:

$$p\&\bigvee_i q_i = \bigvee_i p\&q_i \text{and} (\bigvee_i p_i)\&q = \bigvee_i p_i\&q$$

for all morphisms $p$,$q$ of $Q$ and for all families $(p_i)$, $(q_i)$ of morphisms of $Q$ (forming respective composable pairs). A quantaloid $Q$ will be called *midsymmetrical* whenever it satisfies

(iii) $p\&(q\&r)\&p' = p\&(r\&q)\&p'$ for all $p \in Q(u, v)$, $q, r \in Q(v, v)$, $p' \in Q(v, v')$.

Note that we use the unconventional left-to-right direction for composition of morphisms. An important example of a one-object midsymmetrical quantaloid is given by the lattice of all closed left-sided (or right-sided) ideals of a non-commutative $C^*$-algebra.

From now $Q$ will be an arbitrary midsymmetrical quantaloid having a small set of objects. Let $Q_0$ denote this set and $Q_1$ the set of morphisms of $Q$. Let $Set/Q_0$ denotes the category whose objects are families $X$ of sets $X_u$ indexed by $u \in Q_0$. An element $x \in X_u$ will be called an element over $u$ and we shall sometimes write $d(x)$ for $u$. Morphisms in $Set/Q_0$ are families of maps $f_u : X_u \to Y_u$.

DEFINITION 2.2. Let $Q$ be a quantaloid and $X$ and $Y$ be two objects of $Set/Q_0$. A *matrix* $M$ from $X$ to $Y$ assigns to each pair $x, y$ of $X \times Y$ an element of $Q_1$:

$$m_{x,y} : d(x) \to d(y).$$

Matrices compose by "matrix multiplication": for $M : X \to Y$, and $N : Y \to Z$, the *composite* $M\&N = L : X \to Z$ has its general element given by

$$l_{x,z} = \bigvee_{y \in Y} m_{x,y} \& n_{y,z}.$$

It is clear that this composition is associative. If $Q$ has units, then the matrix $I$ defined by

$$i_{x,x} = id_{d(x)} \text{ (unit in } Q(d(x), d(x))),$$
$$i_{x,x'} = \bot_{d(x),d(x')} \text{ if } x \neq x' \text{ (bottom element in } Q(d(x), d(x'))),$$

is the unit matrix on $X$. Matrices from $X$ to $Y$ form a partially ordered set for the pointwise partial order. Sets over $Q_0$ with matrices as 1-morphisms and 2-morphisms given by partial order determine a bicategory.

## 3. $Q$-sets and presheaves on a (midsymmetrical) quantaloid $Q$

The notions of a (general) $Q$-set and of a presheaf, which will be given in this section, are taken from [1] and [2].

DEFINITION 3.1. Let $Q$ be a quantaloid. A *$Q$-set* is an object $X$ of $Sets/Q_0$ provided with a matrix $A : X \to X$ satisfying the following:

*Idempotency*: $A\&A = A$.

An element $x \in X$ of a $Q$-set $(X, A)$ will said to be strict provided that it satisfies

*Strictness*: $a_{x,x}\&a_{x,x'} = a_{x,x'}$ and $a_{x',x}\&a_{x,x} = a_{x',x}$ for all $x' \in X$ and a $Q$-set $(X, A)$ itself will be called strict whenever every element $x \in X$ is strict.

A $Q$-set $(X, A)$ will be called regular whenever it satisfies

*Regularity*: $a_{x,x'} = a_{x,x'}\&a_{x',x}\&a_{x,x'}$ for all $x, x' \in X$ .

A $Q$-set $(X, A)$ will be called separated whenever it satisfies

*Separation*: if $a_{x,x''} = a_{x',x''}$ and $a_{x'',x} = a_{x'',x'}$ for all $x'' \in X$ (which is just the condition that $a_{x,x} = a_{x',x'} = a_{x,x'} = a_{x',x}$ when $x$ and $x'$ are strict), then $x = x'$ .

We shall usually write $a_x$ for $a_{x,x}$. Note that regular $Q$-sets are strict. An example of a regular $Q$-set is given in Theorem 3.4 (by (1)).

DEFINITION 3.2. Given a strict $Q$-set $(X, A)$, the restrictable triplet $(p, x, p^{\#})$ (of $Q_1 \times X \times Q_1$) is the element of $Q_1 \times X \times Q_1$ over $d(p, x, p^{\#}) = dom(p)$ such that $dom(p) = cod(p^{\#})$, $cod(p) = dom(p^{\#}) = d(x)$

$$a_x \& p^{\#} \& p \& a_x \leqslant a_x, \quad p \& a_x \leqslant p \& p^{\#} \& p \& a_x, \quad \text{and}$$
$$a_x \& p^{\#} \leqslant a_x \& p^{\#} \& p \& p^{\#},$$

where the last two inequalities are actually equalities owing to the first inequality.

DEFINITION 3.3. By a *presheaf* on $Q$ will be meant a $Q$-set $(X, A)$ together with

*Restriction*: a partial mapping $\Rsh\Lsh$: $Q_1 \times X \times Q_1 \to X$ (more precisely, a "matrix" $\Rsh\Lsh = (\Rsh\Lsh_{u,v})_{v \in Q_0}^{u \in Q_0}$ of partial mappings $\Rsh\Lsh_{u,v}$: $Q(u, v) \times X_v \times Q(v, u) \to X_u$) from the restrictable triplets $(p, x, p^{\#}) \in Q_1 \times X \times Q_1$ to the elements $p \Rsh x \Lsh p^{\#}$ of $X$ satisfying the compatibility conditions that:

$$q \Rsh (p \Rsh x \Lsh p^{\#}) \Lsh q^{\#} = (q \& p) \Rsh x \Lsh (p^{\#} \& q^{\#}), a_x \Rsh x \Lsh a_x = x,$$
$$a_{p \Rsh x \Lsh p^{\#}, x''} = p \& a_{x,x''} \quad \text{and} \quad a_{x'', p \Rsh x \Lsh p^{\#}} = a_{x'',x} \& p^{\#},$$

for all $x, x'' \in X$ and for all restrictable triplets $(p, x, p^{\#}), (q, p \Rsh x \Lsh p^{\#}, q^{\#}) \in Q_1 \times X \times Q_1$. We call a presheaf $(X, A, \Rsh\Lsh)$ on $Q$ regular or separated if the underlying $Q$-set $(X, A)$ is regular or separated, respectively.

Now we are engaged in an explicit description of the underlying $Q$-set of a presheaf.

**Theorem 3.4.** *Given a presheaf $(X, A, \Rsh\Lsh)$ on $Q$, the matrix $A^s = (a^s_{x,x'})_{x' \in X}^{x \in X}$ defined by*

$$a^s_{x,x'} = a_x \& \bigvee \{p \in P_{x,x'} | p \& p^{\#} \Rsh x \Lsh p \& p^{\#} = p \Rsh x' \Lsh p^{\#}\} \& a_{x'} \tag{1}$$

*for all $x, x' \in X$, satisfies Idempotency and Regularity (and thus makes $(X, A^s)$ into a regular $Q$-set), where*

$$\begin{aligned} P_{x,x'} = \{ & p \in Q(d(x), d(x')) | \exists p^{\#} \in Q(d(x'), d(x)) \text{ such that } (p^{\#}, x, p) \text{ and} \\ & (p, x', p^{\#}) \text{ restrictable (i.e., } a_x \& p \& p^{\#} \& a_x \leqslant a_x, \\ & p^{\#} \& a_x = p^{\#} \& p \& p^{\#} \& a_x,\ a_x \& p = a_x \& p \& p^{\#} \& p,\ a_{x'} \& p^{\#} \& p \& a_{x'} \leqslant a_{x'}, \\ & p \& a_{x'} = p \& p^{\#} \& p \& a_{x'}, \text{ and } a_{x'} \& p^{\#} = a_{x'} \& p^{\#} \& p \& p^{\#}) \}. \end{aligned}$$

*Moreover, it is compatible with $A$ and $\Rsh\Lsh$ in the sense that*

$$a^s_{x,x} = a_x, \tag{2}$$
$$a^s_{p \Rsh x \Lsh p^{\#}, x'} = p \& a^s_{x,x'}, \quad \text{and} \quad a^s_{x', p \Rsh x \Lsh p^{\#}} = a^s_{x',x} \& p^{\#}, \tag{3}$$

*for all $x, x' \in X$ and all restrictable triplets $(p, x, p^{\#}) \in Q_1 \times X \times Q_1$ (making the triplet $(X, A^s, \upharpoonleft\urcorner)$ into a regular presheaf on $Q$).*

*Proof.* Firstly, note that the restrictability of a triplet $(p^{\#}, x, p)$ implies the restrictability of the triplet $(p\&p^{\#}, x, p\&p^{\#})$ (used in (1)). For instance, to verify the inequality $a_x\&p\&p^{\#}\&p\&p^{\#}\&a_x \leqslant a_x$ , observe that $a_x\&p\&p^{\#}\&a_x \leqslant a_x$, in view of midsymmetricity, implies that

$$a_x\&p\&p^{\#}\&p\&p^{\#}\&a_x = a_x\&p\&p^{\#}\&a_x\&a_x\&p\&p^{\#}\&a_x \leqslant a_x\&a_x = a_x.$$

Verifying each of the axioms of Definition 3.1 for $A^s$ in turn, we argue as follows:

*Idempotency*: $a^s_{x,x'} = \bigvee_{x''\in X} a^s_{x,x''}\&a^s_{x'',x'}$ for all $x, x' \in X$ , since, on the one hand, we have that

$$\begin{aligned}
&\bigvee_{x''\in X} a_x\&\bigvee\{p \in P_{x,x''}|p\&p^{\#} \upharpoonleft x \urcorner p\&p^{\#} = p \upharpoonleft x'' \urcorner p^{\#}\}\&a_{x''}\\
&\quad \&\bigvee\{p' \in P_{x'',x'}|p'\&p'^{\#} \upharpoonleft x'' \urcorner p'\&p'^{\#} = p' \upharpoonleft x' \urcorner p'^{\#}\}\&a_{x'}\\
&\leqslant \bigvee_{x''\in X} a_x\&\bigvee\{p\&a_{x''}\&p' \in P_{x,x'}|(p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) \upharpoonleft x \urcorner (p\&\\
&\quad a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) = (p\&a_{x''}\&p') \upharpoonleft x' \urcorner (p'^{\#}\&a_{x''}\&p^{\#})\}\&a_{x'}\\
&\leqslant a_x \bigvee\{p'' \in P_{x,x'}|p''\&p''^{\#} \upharpoonleft x \urcorner p''\&p''^{\#} = p'' \upharpoonleft x' \urcorner p''^{\#}\}\&a_{x'},
\end{aligned}$$

because, from the midsymmetricity of the quantaloid, one deduce that

$$\begin{aligned}
&\{p \in P_{x,x''}, p\&p^{\#} \upharpoonleft x \urcorner p\&p^{\#} = p \upharpoonleft x'' \urcorner p^{\#}, p' \in P_{x'',x'}, \quad \text{and}\\
&p'\&p'^{\#} \upharpoonleft x'' \urcorner p'\&p'^{\#} = p' \upharpoonleft x' \urcorner p'^{\#}\} \Rightarrow p\&a_{x''}\&p' \in P_{x,x'}
\end{aligned}$$

(for instance, to verify the inequality:

$$a_x\&(p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\&a_x \leqslant a_x,$$

observe that relations $a_x\&p\&p^{\#}\&a_x \leqslant a_x$, $a_{x''}\&p'\&p'^{\#}\&ax'' \leqslant a_{x''}$, and $p\&p^{\#}\&a_x\&p\&p^{\#} = p\&a_{x''}\&p^{\#}$, imply that

$$\begin{aligned}
a_x\&p\&(a_{x''}\&p'\&p'^{\#}\&a_{x''})\&p^{\#}\&a_x &\leqslant a_x\&(p\&a_{x''}\&p^{\#})\&a_x\\
&= a_x\&p\&p^{\#}\&a_x\&p\&p^{\#}\&a_x \leqslant a_x;
\end{aligned}$$

while, for

$$a_x\&(p\&a_{x''}\&p') = a_x\&(p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\&(p\&a_{x''}\&p'),$$

we calculate as follows:

$$\begin{aligned}
&a_x\&p\&a_{x''}\&(p'\&p'^{\#})\&a_{x''}\&(p^{\#}\&p)\&a_{x''}\&p'\\
&\quad = (a_x\&p\&p^{\#}\&p)\&(a_{x''}\&p'\&p'^{\#}\&p') = a_x\&p\&a_{x''}\&p')
\end{aligned}$$

and because

$$
\begin{aligned}
&\{p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x'' \Lsh p^{\#} \text{ and } p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#} = p' \Rsh x' \Lsh p'^{\#}\}\\
&\Rightarrow \{p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \Rsh p\&p^{\#} \Rsh x \Lsh p\&p^{\#} \Lsh p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#}\\
&= p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \Rsh p \Rsh x'' \Lsh p^{\#} \Lsh p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \quad \text{and}\\
&p\&a_{x''} \Rsh p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#} \Lsh a_{x''}\&p^{\#} = p\&a_{x''} \Rsh p' \Rsh x' \Lsh p'^{\#} \Lsh a_{x''}\&p^{\#}\}\\
&\Rightarrow \{p\&a_{x''}\&p'\&p'^{\#}\&(a_{x''}\&p^{\#}\&p\&p^{\#}) \Rsh x \Lsh (p\&p^{\#}\&p\&a_{x''})\&p'\&p'^{\#}\&a_{x''}\&p^{\#}\\
&= p\&a_{x''}\&(p'\&p'^{\#})\&a_{x''}\&(p^{\#}\&p)\&a_{x''} \Rsh x'' \Lsh a_{x''}\&(p^{\#}\&p)\&a_{x''}\&(p'\&p'^{\#})\&a_{x''}\&p^{\#}\\
&\text{and} \quad p\&a_{x''}\&p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#}\&a_{x''}\&p^{\#} = p\&a_{x''}\&p' \Rsh x' \Lsh p'^{\#}\&a_{x''}\&p^{\#}\}\\
&\Rightarrow \{p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \Rsh x \Lsh p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#}\\
&= (p\&p^{\#}\&p\&a_{x''})\&p'\&p'^{\#}\&a_{x''} \Rsh x'' \Lsh a_{x''}\&p'\&p'^{\#}\&(a_{x''}\&p^{\#}\&p\&p^{\#})\\
&\text{and} \quad p\&a_{x''}\&p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#}\&a_{x''}\&p^{\#} = p\&p' \Rsh x' \Lsh p'^{\#}\&a_{x''}\&p^{\#}\}\\
&\Rightarrow (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) \Rsh x \Lsh (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\\
&= p\&a_{x''}\&p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#}\&a_{x''}\&p^{\#} = (p\&a_{x''}\&p') \Rsh x' \Lsh (p'^{\#}\&a_{x''}\&p^{\#})\\
&\Rightarrow (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) \Rsh x \Lsh (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\\
&= (p\&a_{x''}\&p') \Rsh x' \Lsh (p'^{\#}\&a_{x''}\&p^{\#}),
\end{aligned}
$$

while, on the other hand,

$$
\begin{aligned}
&\bigvee_{x''\in X} a_x\&\bigvee\{p \in P_{x,x''} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x'' \Lsh p^{\#}\}\&a_{x''}\\
&\quad \&a_{x''}\&\bigvee\{p' \in P_{x'',x'} | p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#} = p' \Rsh x' \Lsh p'^{\#}\}\&a_{x'}\\
&\qquad \geqslant a_x\&\bigvee\{p \in P_{x,x'} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_{x'}
\end{aligned}
$$

(putting $x'' = x'$ and $p' = a_{x'}$);

*Regularity*: $a^s_{x,x'} = a^s_{x,x'}\&a^s_{x',x}\&a^s_{x,x'}$ for all $x, x' \in X$, since, in the one direction, we have that

$$a^s_{x,x'}\&a^s_{x',x}\&a^s_{x,x'} \leqslant a^s_{x,x'} \text{ (by Idempotency of } A^s \text{ twice)},$$

while, in the other,

$$
\begin{aligned}
&a_x\&\bigvee\{p \in P_{x,x'} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_{x'}\\
&\&a_{x'}\&\bigvee\{q \in P_{x',x} | q\&q^{\#} \Rsh x' \Lsh q\&q^{\#} = q \Rsh x \Lsh q^{\#}\}\&a_x\\
&\&a_x\&\bigvee\{r \in P_{x,x'} | r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#}\}\&a_{x'}\\
\geqslant\, &a_x\&\bigvee\{p\&p^{\#}\&p\&a_{x'} (= p\&a_{x'}) | p \in P_{x,x'}, p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_{x'}
\end{aligned}
$$

(putting $q = r^{\#} = p^{\#}$ and $q^{\#} = r = p$ and noting that

$$p^{\#}\&p \Rsh x' \Lsh p^{\#}\&p = p^{\#} \Rsh x \Lsh p \Leftrightarrow p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}$$

for all $p \in P_{x,x'}$ ), which completes the verification that the underlying family $X$ of sets of the presheaf $(X, A, \Rsh\Lsh)$ together with $A^s$ is a $Q$-set.

For (2), in the one direction, we have that

$$a_x = a_x \& a_x \& a_x \leqslant a_x \& \bigvee \{p \in P_{x,x} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\} \& a_x,$$

while, in the other,

$$a_x \& \bigvee \{p \in P_{x,x} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\} \& a_x \leqslant a_x,$$

since from the relation $p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x \Lsh p^{\#}$ , it folows that

$$\begin{aligned} a_x \& p \& a_x &= a_x \& a_{p \Rsh x \Lsh p^{\#}, x} = a_x \& a_{p\&p^{\#} \Rsh x \Lsh p\&p^{\#}, x} \\ &= a_x \& p \& p^{\#} \& a_x \leqslant a_x \quad \text{(by the restrictability of } (p, x, p^{\#})). \end{aligned}$$

To prove the first relation in (3), by the observation that

$$q \in P_{p \Rsh x \Lsh p^{\#}, x'} \Rightarrow a_x \& p^{\#} \& q \& a_{x'} \in P_{x,x'}$$

and that

$$\begin{aligned} & q\&q^{\#} \Rsh p \Rsh x \Lsh p^{\#} \Lsh q\&q^{\#} = q \Rsh x' \Lsh q^{\#} \\ & \Rightarrow a_x \& p^{\#} \& q \& a_{x'} \& q^{\#} \Rsh q\&q^{\#} \Rsh p \Rsh x \Lsh p^{\#} \Lsh q\&q^{\#} \Lsh q \& a_{x'} \& q^{\#} \& p \& a_x \\ & = a_x \& p^{\#} \& q \& a_{x'} \& q^{\#} \Rsh q \Rsh x' \Lsh q^{\#} \Lsh q \& a_{x'} \& q^{\#} \& p \& a_x \\ & \Rightarrow a_x \& p^{\#} \& q \& (a_{x'} \& q^{\#} \& q \& q^{\#}) \& p \Rsh x \Lsh p^{\#} \& (q \& q^{\#} \& q \& a_{x'}) \& q^{\#} \& p \& a_x \\ & = a_x \& p^{\#} \& (q \& a_{x'} \& q^{\#} \& q \& a_{x'}) \Rsh x' \Lsh (a_{x'} \& q^{\#} \& q \& a_{x'} \& q^{\#}) \& p \& a_x \\ & \Rightarrow (a_x \& p^{\#} \& q \& a_{x'}) \& (a_{x'} \& q^{\#} \& p \& a_x) \Rsh x \Lsh (a_x \& p^{\#} \& q \& a_{x'}) \\ & \quad \& (a_{x'} \& q^{\#} \& p \& a_x) = (a_x \& p^{\#} \& q \& a_{x'}) \Rsh x' \Lsh (a_{x'} \& q^{\#} \& p \& a_x), \end{aligned}$$

we obtain the following estimations

$$\begin{aligned} & a^s_{p \Rsh x \Lsh p^{\#}, x'} = a_{p \Rsh x \Lsh p^{\#}} \& \bigvee \{q \in P_{p \Rsh x \Lsh p^{\#}, x'} | q\&q^{\#} \Rsh p \Rsh x \Lsh p^{\#} \Lsh q\&q^{\#} \\ & = q \Rsh x' \Lsh q^{\#}\} \& a_{x'} \leqslant p \& \bigvee \{a_x \& p^{\#} \& q \& a_{x'} \in P_{x,x'} | (a_x \& p^{\#} \& q \& a_{x'}) \\ & \quad \& (a_{x'} \& q^{\#} \& p \& e_x) \Rsh x \Lsh (a_x \& p^{\#} \& q \& a_{x'}) \& (a_{x'} \& q^{\#} \& p \& a_x) \\ & = (a_x \& p^{\#} \& q \& a_{x'}) \Rsh x' \Lsh (a_{x'} \& q^{\#} \& p \& a_x)\} \\ & \leqslant p \& \bigvee \{q' \in P_{x,x'} | q' \& q'^{\#} \Rsh x \Lsh q' \& q'^{\#} = q' \Rsh x' \Lsh q'^{\#}\}, \end{aligned}$$

which implies that

$$a^s_{p \Rsh x \Lsh p^{\#}, x'} \leqslant p \& a^s_{x,x'}.$$

For the converse, using the implication

$$\{(p,x,p^{\#}) \text{ is restrictable, } r \in P_{x,x'}, \text{ and } r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#} \\ \text{(from which one has } r\&a_{x'} = r\&r^{\#}\&a_{x,x'})\} \Rightarrow p\&a_x\&r \in P_{p\Rsh x \Lsh p^{\#},x'}$$

and the following series of implications

$$\begin{aligned}
&\{(p,x,p^{\#}) \text{ is restrictable and } r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#}\} \\
&\Rightarrow p\&a_x\&r\&r^{\#} \Rsh r\&r^{\#} \Rsh x \Lsh r\&r^{\#} \Lsh r\&r^{\#}\&a_x\&p^{\#} \\
&= p\&a_x\&r\&r^{\#} \Rsh r \Rsh x' \Lsh r^{\#} \Lsh r\&r^{\#}\&a_x\&p^{\#} \\
&\Rightarrow p\&(a_x\&r\&r^{\#}\&r)\&r^{\#} \Rsh x \Lsh r\&(r^{\#}\&r\&r^{\#}\&a_x)\&p^{\#} \\
&= p\&(a_x\&r\&r^{\#}\&r) \Rsh x' \Lsh (r^{\#}\&r\&r^{\#}\&a_x)\&p^{\#} \\
&\Rightarrow p\&(p^{\#}\&p)\&a_x\&(r\&r^{\#})\&a_x \Rsh x \Lsh a_x\&(r\&r^{\#}) \\
&\quad \&a_x\&(p^{\#}\&p)\&p^{\#} = (p\&a_x\&r) \Rsh x' \Lsh (r^{\#}\&a_x\&p^{\#}) \\
&\Rightarrow (p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#})\&(p\&a_x \Rsh x \Lsh a_x\&p^{\#}) \\
&\&(p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#}) = (p\&a_x\&r) \Rsh x' \Lsh (r^{\#}\&a_x\&p^{\#}) \\
&\Rightarrow (p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#}) \Rsh (p \Rsh x \Lsh p^{\#}) \Lsh (p\&a_x\&r) \\
&\quad \&(r^{\#}\&a_x\&p^{\#}) = (p\&a_x\&r) \Rsh x' \Lsh (r^{\#}\&a_x\&p^{\#}),
\end{aligned}$$

we calculate

$$\begin{aligned}
&p\&a^{s}_{x,x'} = p\&a_x\& \bigvee\{r \in P_{x,x'} | r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#}\}\&a_{x'} \\
&= p\&(p^{\#}\&p)\&a_x\&\bigvee\{a_x\&r | r \in P_{x,x'}, r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#}\}\&a_{x'} \\
&= (p\&a_x\&p^{\#})\&\bigvee\{p\&a_x\&r | r \in P_{x,x'}, r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#}\}\&a_{x'} \\
&\leqslant a_{p\Rsh x\Lsh p^{\#}}\&\bigvee\{p\&a_x\&r \in P_{p\Rsh x\Lsh p^{\#},x'} | (p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#}) \Rsh (p \Rsh x \Lsh p^{\#}) \Lsh \\
&\quad (p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#}) = (p\&a_x\&r) \Rsh x' \Lsh (r^{\#}\&a_x\&p^{\#})\}\&a_{x'} \\
&\leqslant a_{p\Rsh x\Lsh p^{\#}}\&\bigvee\{r' \in P_{p\Rsh x\Lsh p^{\#},x'} | r'\&r'^{\#} \Rsh (p \Rsh x \Lsh p^{\#}) \Lsh r'\&r'^{\#} = r' \Rsh x' \Lsh r'^{\#}\}\&a_{x'} \\
&= a^{s}_{p\Rsh x\Lsh p^{\#},x'},
\end{aligned}$$

which completes the proof of the first relation in (3), while the verification of the rest is similar.

We obtain further properties of the regular $Q$-set $(X, A^s)$ (introduced in Theorem 3.4) in the next few propositions.

**Lemma 3.5.** *Let $(X, A, \Rsh\Lsh)$ be a presheaf on $Q$ and $A^s = (a^s_{x,x'})^{x\in X}_{x'\in X}$ be the structural matrix defined by (1) (in Theorem 3.4). Then the inequality*

$$a^s_{x,x'} \leqslant a_{x,x'} \tag{4}$$

*holds for all* $x, x' \in X$ *,i.e.,* $A^s \leqslant A$ .

*Proof.* Let us consider elements $x, x' \in X$ and $p, p^{\#} \in Q_1$ with the properties that $(p^{\#}, x, p)$ and $(p, x', p^{\#})$ are restrictable (i.e., $p \in P_{x,x'}$ ) and that the equality $p\&p^{\#} \upharpoonleft x \urcorner p\&p^{\#} = p \upharpoonleft x' \urcorner p^{\#}$ holds. Then we infer from (3) that $p\&p^{\#}\&a_{x,x'} = p\&a_{x'}$ . Now we obtain

$$a_x\&p\&a_{x'} = a_x\&p\&p^{\#}\&a_{x,x'} = (a_x\&p\&p^{\#}\&a_x)\&a_{x,x'} \leqslant a_x\&a_{x,x'} = a_{x,x'},$$

whence (4).

**Lemma 3.6.** *Let* $(X, A, \upharpoonleft\urcorner)$ *be a separated presheaf on* $Q$ *(i.e.,* $(X, A)$ *be a separated* $Q$*-set). Let* $(p, x', p^{\#})$ *and* $(p^{\#}, x, p)$ *be restrictable triplets. Then the relation*

$$p\&p^{\#} \upharpoonleft x \urcorner p\&p^{\#} = p \upharpoonleft x' \urcorner p^{\#} \tag{5}$$

*(presented in* (1)*) is just the condition that*

$$p\&p^{\#}\&a_x\&p\&p^{\#} = p\&a_{x'}\&p^{\#} = p\&p^{\#}\&a_{x,x'}\&p^{\#} = p\&a_{x',x}\&p\&p^{\#}. \tag{6}$$

*Proof.* To prove the implication from (5) to (6), assume (5). Then

$$a_{p\&p^{\#}\upharpoonleft x\urcorner p\&p^{\#}} = a_{p\upharpoonleft x'\urcorner p^{\#}} = a_{p\&p^{\#}\upharpoonleft x\urcorner p\&p^{\#},p\upharpoonleft x'\urcorner p^{\#}} = a_{p\upharpoonleft x'\urcorner p^{\#},p\&p^{\#}\upharpoonleft x\urcorner p\&p^{\#}}, \tag{7}$$

which implies (6). Assume now (6). Then it is clear that the relations (7) hold. By Separation, this means that (5) holds.

**Lemma 3.7.** *If a presheaf* $(X, A, \upharpoonleft\urcorner)$ *is separated, then the triplets* $(a_{x',x}, x, a_{x,x'})$ *and* $(a_{x,x'}, x', a_{x',x})$ *are restrictable (i.e.,* $a_{x,x'} \in P_{x,x'}$ *) and*

$$a_{x,x'}\&a_{x',x} \upharpoonleft x \urcorner a_{x,x'}\&a_{x',x} = a_{x,x'} \upharpoonleft x' \urcorner a_{x',x}$$

*for all* $x, x' \in X$ .

*Proof.* In view of Idempotency and Regularity of $A$ , it is clear that $a_{x,x'} \in P_{x,x'}$ for all $x, x' \in X$ . Moreover, we have that

$$a_{x,x'}\&a_{x',x}\&a_x\&a_{x,x'}\&a_{x',x} = a_{x,x'}\&a_{x'}\&a_{x',x} = a_{x,x'}\&a_{x',x}\&a_{x,x'}\&a_{x',x}$$

for all $x, x' \in X$ . By Lemma 3.6, this just says the statement is correct.

**Theorem 3.8.** *If* $(X, A, \upharpoonleft\urcorner)$ *is a regular and separated presheaf, then the* $Q$*-set* $(X, A^s)$ *introduced in Theorem* 3.4 *(by* (1)*) is exactly the underlying* $Q$*-set* $(X, A)$ *, that is,*

$$a_{x,x'} = a_x\&\bigvee\{p \in P_{x,x'} | p\&p^{\#} \upharpoonleft x \urcorner p\&p^{\#} = p \upharpoonleft x' \urcorner p^{\#}\}\&a_{x'}$$

*for all* $x, x' \in X$ .

*Proof.* From Lemma 3.7, it follows that

$$a_{x,x'} = a_x \& a_{x,x'} \& a_{x'} \leqslant a_x \& \bigvee \{p \in P_{x,x'} | p \& p^{\#} \Rsh x \Lsh p \& p^{\#} = p \Rsh x' \Lsh p^{\#}\} \& a_{x'}$$

for all $x, x' \in X$. In view of Lemma 3.5, this concludes the proof.

## References

[1] G. Van den Bossche, Quantaloids and non-commutative ring representations, *Appl. Categ. Structures*,**3**, 305–320 (1995).

[2] R.P. Gylys, Sheaves on quantaloids, *Liet. matem. rink.*, **40**(2), 133–171 (2000).

[3] C.J. Mulvey, M. Nawaz, Quantales: quantal sets, In *Non-Classical Logics and their Applications to Fuzzy Subsets*, eds. U. Höhle, E.P. Klement, Kluwer, Boston, 159-217 (1995).

## Priešpluoštai virš midsimetriškų kvantaloidų

R.P. Gylys

Ankstesniame straipsnyje [2] mes nagrinėjome priešpluoščius ir pluoštus virš laisvojo kvantaloido. Šiame darbe tiriami priešpluoščiai virš midsimetriško kvantaloido. Išvesta priešpluoščio struktūrinės matricos išraiška per jo siaurinį ir diagonalę.