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# On approximation of stochastic integral equations driven by continuous $p$-semimartingales

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## Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$, be a stochastic basis satisfying the usual conditions and let a standard Brownian motion $W$ and a fractional Brownian motion (fBm) $B^H$, with the Hurst index $1/2 < H < 1$, be $\mathbb{F}$-adapted.

A fBm with the Hurst index $0 < H < 1$ is a centered Gaussian process $X = \{X_t, t \geqslant 0\}$ with $X_0 = 0$ and with the covariance

$$\mathrm{Cov}(X_t, X_s) = \frac{1}{2}\,\mathrm{Var}(X_1)\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

for all $t, s \geqslant 0$. If $\mathrm{Var}(X_1) = 1$, we write $X = B^H$. The case $H = 1/2$ corresponds to the standard Brownian motion.

Consider the equation

$$X_t = \xi + \int_0^t f(X_s)\,\mathrm{d}Z_s + \frac{1}{2}\int_0^t ff'(X_s)\,\mathrm{d}s, \quad t \in [0,T], \tag{1}$$

where $Z = W + B^H$, $1/2 < H < 1$. For short, we shall write $ff'(X_s)$ instead of $f(X_s)f'(X_s)$.

If $f \in \mathbb{C}_b^2$ then there exists a unique adapted solution of the equation (1) having almost all sample paths in the space $C\mathcal{W}_q([0,T])$, $2 < q < 1/(1-H)$, where $C\mathcal{W}_q([0,T])$ is the class of all continuous functions defined on $[0,T]$ with a bounded $q$-variation. This result one can easily obtain from [4, 6]. (For definitions see [4, 6].)

Let $\varkappa^n = \{t_k^n\colon 0 \leqslant k \leqslant m(n)\}$ be a sequence of partitions of the interval $[0,T]$, i.e., $0 = t_0^n < t_1^n < \cdots < t_{m(n)}^n = T$, such that $\delta_n = \max_i |t_{i+1}^n - t_i^n|$ tends to 0 as $n \to +\infty$.

Let $Z^n$ be a sequence of linear approximations of a process $Z$, i.e.,

$$Z^n(t) = Z(t_{k-1}^n) + \frac{t - t_{k-1}^n}{t_k^n - t_{k-1}^n}\left(Z(t_k^n) - Z(t_{k-1}^n)\right),$$

for $t \in [t_{k-1}^n, t_k^n]$, $n \in \mathbb{N}$, $1 \leqslant k \leqslant m(n)$. Note that for any $n$ process $Z^n$ has bounded variation.

For partition $\varkappa^n$ define $\rho^n(t) = \max\{t_k^n\colon t_k^n \leqslant t\}$ and $r^n(t) = \max\{k\colon t_k^n \leqslant t\}$, $t \in [0,T]$. For every $x \in D([0,T]) := D([0,T],\mathbb{R})$ the sequence $\{x^{\varkappa^n}\}$ denotes the following discretizations of $x$:

$$x_t^{\varkappa^n} = x(t_k^n) \quad \text{for } t \in [t_k^n, t_{k+1}^n),\ 0 \leqslant k \leqslant m(n),\ n \in \mathbb{N}.$$

Define the approximation

$$X_t^n = \xi + \int_0^t f(X_{s-}^n)\mathrm{d}Z_s^{\varkappa^n} + \frac{1}{2}\int_0^t ff'(X_{s-}^n)\mathrm{d}[Z^{\varkappa^n}]_s, \quad t \in [0,T],\ n \in \mathbb{N}. \tag{2}$$

If $f$ is locally Lipschitz continuous and satisfies linear growth condition then for every $n \in \mathbb{N}$ there exists a unique strong solution to

$$Y^n(t) = \xi + \int_0^t f(Y_s^n)\mathrm{d}Z_s^n, \quad t \in [0,T],\ n \in \mathbb{N}. \tag{3}$$

Now we formulate our results.

**Theorem 1.** *Let $f \in \mathbb{C}_b^2$. Then*

$$\big(X^n, W^{\varkappa^n}, B^{H,\varkappa^n}\big) \xrightarrow{D} (X, W, B^H) \quad as \quad n \to \infty,$$

*where $X$ is the unique solution of the equation (1). By $\xrightarrow{D}$ we denote the weak convergence of corresponding probability measures on $D([0,T],\mathbb{R}^3)$.*

**Theorem 2.** *Assume that $X$ is a solution of (1) and $\{Y^n\}$ is a sequence of solutions of (3). Then*

$$\sup_{t\leqslant T}\big|Y^n(t) - X(t)\big| \xrightarrow{P} 0, \quad as\ n \to \infty.$$

## 1. Auxiliary results and proofs

Since almost all sample paths of the processes $B^H$, $1/2 \leqslant H < 1$, are Hölder continuous then

$$V_r\big(B^H; [s,t]\big) := v_r^{1/r}(B^H; [s,t]) \leqslant L^{H,1/r}\,(t-s)^{1/r}, \tag{4}$$

where $v_r(B^H; [s,t])$ is the $r$-variation of the $B^H$, $s < t$, $r > 1/H$,

$$L^{H,\gamma} = \sup_{\substack{s\neq t\\ s,t\leqslant T}} \frac{\big|B_t^H - B_s^H\big|}{|t-s|^\gamma}, \quad 0 < \gamma < H, \quad \mathbf{E}\big(L^{H,\gamma}\big)^k < \infty, \quad \forall\, k \geqslant 1.$$

Any local martingale is locally of bounded $q$-variation for each $q > 2$. Moreover, for $q > 2$ and $0 < r \leqslant 2$ there are a finite constants $K_{q,r}$, $\ell_r$ such that for continuous martingale $M = \{M(t), 0 \leqslant t \leqslant T\}$

$$\mathbf{E}\{v_q(M;[0,T])\}^{r/q} \leqslant K_{q,r}\mathbf{E}\Big\{\sup_{0\leqslant t\leqslant T}|M(t)|\Big\}^r \leqslant K_{q,r}\ell_r\mathbf{E}\{\langle M\rangle_T\}^{r/2}. \tag{5}$$

**Lemma 3** (see [3, 5]). *Let $\{M^n\}$, $\{A^n\}$, and $\{\widetilde{X}^n\}$ be a sequences of cadlag $\mathbb{F}^n$ adapted processes, where $M^n$ is a local martingale, $A^n$ is a process with p-bounded variation, $1<p<2$, $\widetilde{X}^n$ is a process with q-bounded variation, $q>2$, $q^{-1}+p^{-1}>1$. Assume that*

$$\sup_n \mathbf{E}\sup_{t\leqslant T}|\Delta M_t^n| < +\infty,$$

$\{V_p(A^n;[0,T])\}$, $n\in\mathbb{N}$, *and* $\{V_q(\widetilde{X}^n;[0,T])\}$, $n\in\mathbb{N}$, *are tight in* $\mathbb{R}$. *If*

$$(\widetilde{X}^n, M^n, A^n) \xrightarrow{D} (\widetilde{X}, M, A) \quad \text{in} \quad D([0,T],\mathbb{R}^3),$$

*where $\widetilde{X}$, $M$ and $A$ are continuous processes, then $M$ is a local martingale adapted to the natural filtration $\mathbb{G}$ generated by $(\widetilde{X}, M, A)$, $A$ is a process of p-bounded variation adapted to $\mathbb{G}$, and*

$$\Big(\widetilde{X}^n, M^n, A^n, \int_0^{\cdot} \widetilde{X}^n_{s-}\,\mathrm{d}M^n_s, \int_0^{\cdot} \widetilde{X}^n_{s-}\,\mathrm{d}A^n_s\Big) \xrightarrow{D} \Big(\widetilde{X}, M, A, \int_0^{\cdot} \widetilde{X}_s\,\mathrm{d}M_s, \int_0^{\cdot} \widetilde{X}_s\,\mathrm{d}A_s\Big) \tag{6}$$

*in* $D([0,T],\mathbb{R}^5)$.

Define the approximation

$$\widehat{X}^n_t = \xi + \int_0^t f(\widehat{X}^{n,\varkappa^n}_s)\mathrm{d}Z_s + \frac{1}{2}\int_0^t ff'(\widehat{X}^{n,\varkappa^n}_s)\mathrm{d}s, \quad t\in[0,T],\ n\in\mathbb{N}. \tag{7}$$

**Lemma 4.** *Let $f\in\mathbb{C}^1_b$. Then the sequence $\{\widehat{X}^n\}$ is tight in $C([0,T])$.*

*Proof.* Let $q > 2$, $p > 1/H$, and $q^{-1}+p^{-1} > 1$. First we note that

$$\begin{aligned}\mathbf{E}V_q^{2r}(\widehat{X}^n;[0,T]) \leqslant{}& 4^{2r-1}\frac{1}{(1-\alpha)^{2r}}\Big(K_{q,2r}\ell_{2r}|f|_\infty^{2r}T^r + |f|_\infty^{2r}|f'|_\infty^{2r}T^{2r}\\ &+C^{2r}_{p,q/\alpha}|f|_\infty^{2r}\mathbf{E}V_p^{2r}(B^H;[0,T])\Big)\\ &+4^{2r-1}\mathbf{E}\Big(C_{p,q/\alpha}|f|_\alpha V_p(B^H;[0,T])\Big)^{2r/(1-\alpha)}.\end{aligned} \tag{8}$$

The proof is similar as in Lemma 1 [6].

Now we prove the tightness of the sequence $\{\widehat{X}^n\}$.

At first we will show that there exists an nondecreasing continuous function $F$ and $\beta > 1$ such that for any $s, t \in [0,T]$, $s < t$, $t - s < 1$,

$$\mathbf{E}|\widehat{X}_t^n - \widehat{X}_s^n|^4 \leqslant |F(t) - F(s)|^\beta.$$

By the Love-Young inequality (see [4]), the inequality (4), and Lemma 4.11 [7] we get

$$\begin{aligned}\mathbf{E}|\widehat{X}_t^n - \widehat{X}_s^n|^4 &\leqslant 3^3 \cdot 36|f|_\infty^4(t-s)^2 \\ &\quad +3^3 C_{p,q}^4 \mathbf{E} V_{q,\infty}^4\big(f(\widehat{X}^{n,\varkappa^n}; [0,T]\big) V_p^4(B^H; [s,t]) \\ &\quad +3^3 \cdot 2^{-4}|f|_\infty^4 |f'|_\infty^4 (t-s)^4 \leqslant C(t-s)^2,\end{aligned}$$

where $C$ is the constant not depending on $n$. Thus by Theorem 12.3 in [1] we get the tightness of the sequence $\{\widehat{X}^n\}$ in the space $C([0,T])$.

**Lemma 5.** *Let $f \in \mathbb{C}_b^1$. Then the sequence $\{X^n\}$ is tight in $D([0,T])$.*

*Proof.* Since $X^n(t_i^n) = \widehat{X}^n(t_i^n)$ for $1 \leqslant i \leqslant m(n)$ then

$$\begin{aligned}\sup_{t\leqslant T} |X_t^n - \widehat{X}_t^n| &\leqslant |f|_\infty \sup_{t\leqslant T} |Z(t) - Z^{\varkappa^n}(t)| \\ &\quad +|f|_\infty |f'|_\infty \sum_{i=1}^{m(n)} |W(t_i^n) - W(t_{i-1}^n)| \cdot |B^H(t_i^n) - B^H(t_{i-1}^n)| \\ &\quad +|f|_\infty |f'|_\infty \sum_{i=1}^{m(n)} |B^H(t_i^n) - B^H(t_{i-1}^n)|^2 + |f|_\infty |f'|_\infty \delta_n \\ &\quad +\sup_{t\leqslant T} \left| \sum_{i=1}^{r^n(t)} ff'(\widehat{X}^n(t_{i-1}^n)) \Big[ \big(W(t_i^n) - W(t_{i-1}^n)\big)^2 - (t_i^n - t_{i-1}^n) \Big] \right| = \sum_{i=1}^5 I_i.\end{aligned}$$

We may assume, without loss of generality, that $\delta_n < 1$. Note that

$$\mathbf{E} \sup_{t\leqslant T} |Z_t - Z_t^{\varkappa^n}| \leqslant \mathbf{E}\{L^{1/2,1/q} + L^{H,1/p}\}\delta_n^{1/q},$$

where $q > 2$, $p > 1/H$, and $q^{-1} + p^{-1} > 1$. Further

$$\mathbf{E} I_2 \leqslant C|f|_\infty |f'|_\infty T \delta_n^{H-1/2}, \quad \mathbf{E} I_3 \leqslant |f|_\infty |f'|_\infty T \delta_n^{2H-1}.$$

By the Doob inequality

$$\begin{aligned}\mathbf{E} I_5^2 &\leqslant 4|f|_\infty^2 |f'|_\infty^2 \sum_{i=1}^{m(n)} \mathbf{E}\Big[\big(W(t_i^n) - W(t_{i-1}^n)\big)^2 - (t_i^n - t_{i-1}^n)\Big]^2 \\ &\leqslant 4CT|f|_\infty^2 |f'|_\infty^2 \delta_n.\end{aligned}$$

Therefore $\mathbf{E}\sup_{t\leqslant T}|X_t^n - \widehat{X}_t^n| \longrightarrow 0$, as $n \to \infty$. By Lemma 4 we have that the sequence $\{\widehat{X}^n\}$ is tight. Thus by Lemma 3.31 in Section 6 in [2] we obtain that the sequence $\{X^n\}$ is tight.

**Proof of Theorem 1.** Define $M^n = W^{\varkappa^n}$ and $A^n = B^{H,\varkappa^n}$. The process $(M^n, \mathbb{F}^n)$ is a martingale, where $\mathbb{F}^n = (\mathcal{F}_{\rho^n(t)})$. The process $A^n$ has bounded $p$-variation since $V_p(A^n;[0,T]) \leqslant V_p(B^H;[0,T])$. Note that $M^n \to W$ a.s. and $A^n \to B^H$ a.s. in $C([0,T])$. Moreover,

$$\sup_{t\leqslant T}\big|[M^n]_t - t\big| \xrightarrow{P} 0, \quad n \to \infty,$$

where

$$[M^n]_t = \sum_{k=1}^{m(n)} \big(W(t_k^n \wedge t) - W(t_{k-1}^n \wedge t)\big)^2.$$

By Lemma 5, by Corollary 3.33 in Section 6 in [2], and facts obtained above it follows that the sequence $\{(X^n, M^n, A^n, [M^n], \xi)\}$ is $C$-tight. Thus from every subsequence $\{n'\} \subset \{n\}$ we can choose a further subsequence $\{n''\}$ such that

$$(X^{n''}, M^{n''}, A^{n''}, [M^{n''}], \xi) \xrightarrow{D} (X^\infty, M^\infty, A^\infty, [M^\infty], \xi^\infty),$$

as $n'' \to \infty$, where $(X^\infty, M^\infty, A^\infty, [M^\infty], \xi^\infty)$ is defined on some probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{P}})$ and $\mathcal{L}(\xi^\infty, M^\infty, [M^\infty], A^\infty) = \mathcal{L}(\xi, W, [W], B^H)$. Since

$$\sup_{t\leqslant T}\big|[Z^{\varkappa^{n''}}]_t - [M^{n''}]_t\big| \xrightarrow{P} 0,$$

as $n'' \to \infty$, and functions $f$ and $ff'$ are continuous, then by the continuous mapping theorem

$$\begin{aligned}&(X^{n''}, f(X^{n''}), ff'(X^{n''}), M^{n''}, A^{n''}, [Z^{\varkappa^{n''}}], \xi)\\ &\quad \xrightarrow{D} (X^\infty, f(X^\infty), ff'(X^\infty), M^\infty, A^\infty, [M^\infty], \xi^\infty).\end{aligned}$$

It is evident by the Doob inequality that $\sup_n \mathbf{E}\sup_{t\leqslant T}|\Delta M_t^n| \leqslant 2\mathbf{E}\sup_{t\leqslant T}|W(t)| \leqslant 4\sqrt{T}$. It is not difficult to show that $V_q(X^n;[0,T])$ is tight in $\mathbb{R}$ (see the proof of (8)). Thus the conditions of Lemma 3 are satisfied and

$$\begin{aligned}&\Big(X^{n''}, \int_0^{\cdot} f(X_{s-}^{n''})\mathrm{d}M_s^{n''}, \int_0^{\cdot} f(X_{s-}^{n''})\mathrm{d}A_s^{n''}, \int_0^{\cdot} ff'(X_{s-}^{n''})\mathrm{d}[Z^{\varkappa^{n''}}]_s, \xi\Big)\\ &\quad \xrightarrow{D} \Big(X^\infty, \int_0^{\cdot} f(X_s^\infty)\,\mathrm{d}M_s^\infty, \int_0^{\cdot} f(X_s^\infty)\,\mathrm{d}A_s^\infty, \int_0^{\cdot} ff'(X_s^\infty)\,\mathrm{d}[M^\infty]_s, \xi^\infty\Big).\end{aligned}$$

Thus

$$\sup_{t\leqslant T}\left|X_t^{n''}-\xi-\int_0^t f(X_{s-}^{n''})\mathrm{d}Z_s^{\varkappa^{n''}}-\frac{1}{2}\int_0^t ff'(X_{s-}^{n''})\mathrm{d}[Z^{\varkappa^{n''}}]_s\right|$$

$$\xrightarrow{D}\sup_{t\leqslant T}\left|X_t^{\infty}-\xi^{\infty}-\int_0^t f(X_s^{\infty})\,\mathrm{d}Z_s^{\infty}-\frac{1}{2}\int_0^t ff'(X_s^{\infty})\,\mathrm{d}[M^{\infty}]_s\right|.$$

As a consequence

$$X_t^{\infty}=\xi^{\infty}+\int_0^t f(X_s^{\infty})\mathrm{d}Z_s^{\infty}+\frac{1}{2}\int_0^t ff'(X_s^{\infty})\,\mathrm{d}[M^{\infty}]_s,\quad t\leqslant T.$$

Since on $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbf{P}})$ this equation has a unique solution and $\mathcal{L}(\xi^{\infty},M^{\infty},[M^{\infty}],A^{\infty})=\mathcal{L}(\xi,W,[W],B^H)$ then $\mathcal{L}(X^{\infty},\xi^{\infty},M^{\infty},[M^{\infty}],A^{\infty})=\mathcal{L}(X,\xi,W,[W],B^H)$.

**Proof of Theorem 2.** Since $M^n\to W$ and $A^n\to B^H$ a.s. in $D([0,T])$ then similarly as in [8] one can prove that

$$\sup_{t\leqslant T}\left|X^n(t)-X(t)\right|\xrightarrow{P}0,\quad \text{as } n\to\infty.$$

Since $Y^n(t)=X^n(t)$ for $t\in\varkappa_n$, the proof of Theorem 2 is completed.

## References


[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York (1968).

[2] J. Jacod, A.N. Shiryayev, *Limit Theorems for Stochastic Processes*, Vol. 1, Fiziko-Matematicheskaja Literatura, Moscow (1994).

[3] A. Jakubowski, J. Mémin, G. Pages, Convergence en loi des suites d'intégrales stochastiques sur l'espace $D^1$ de Skorokhod, *Probab. Theory Related Fields*, **81**, 111–137 (1989).

[4] K. Kubilius, The existence and uniqueness of the solution of the integral equation driven by fractional Brownian motion, *Lith. Math. J.* (special issue), **40**, 104–110 (2000).

[5] K. Kubilius, *On Existence and Uniqueness of the Solution of the Integral Equation Driven by a Continuous p-semimartingale*. Preprint (2001).

[6] K. Kubilius, The existence and uniqueness of the solution of the integral equation driven by a $p$-semimartingale of the special type, *Stochastic Processes Appl.* (to appear) (2002).

[7] R.S. Liptser, A.N. Shiryaev, *Statistics of Random Processes*. Nauka, Moscow (1974).

[8] L. Slominski, Stability of strong solutions of stochastic differential equations, *Stochastic Processes Appl.*, **31**, 173–202 (1989).


## Stochastinių integralinių lygčių, valdomų tolydžiųjų $p$-semimartingalų, aproksimacija

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Nagrinėjama integralinių lygčių, valdomų tolydžiųjų $p$-semimartingalų, Vong-Zakai tipo aproksimacija.