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Relation between classical and intuitionistic sequent calculi of temporal logic

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1. Introduction

V. Glivenko proved in [4] that a propositional formula beginning with '$\neg$' is derivable in a classical propositional calculus if and only if it is derivable in an intuitionistic propositional calculus. In [6], classes of sequents are presented such that every sequent which belongs to some of these classes is derivable in a classical predicate sequent calculus if an only if it is derivable in an intuitionistic predicate sequent calculus. In this paper, we consider analogical classes for temporal logic with time gaps, see [3].

From the classical point of view, Glivenko classes give us the classes of constructiveness, i. e., theorems that are of the shape of sequents which belong to a Glivenko class, have a constructive proof.

The main results of the present work are Theorems 4.2 and 4.3.

2. Sequent calculus LB

The sequent calculus LB is obtained from a variant of Gentzen's sequent calculus LK (without structural rules) by adding some rules for temporal operators which are taken from [3] and slightly changed by us.

Axioms: $\Gamma, E \to E, \Delta$; $\Gamma, \mathcal{F} \to \Delta$.

LK rules:

$$\frac{\Gamma, A \to B, \Delta}{\Gamma \to A \supset B, \Delta}\ (\to\supset), \qquad \frac{\Gamma \to A, \Delta; B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ (\supset\to),$$

$$\frac{\Gamma \to A, \Delta; \Gamma \to B, \Delta}{\Gamma \to A \wedge B, \Delta}\ (\to\wedge), \qquad \frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ (\wedge\to),$$

$$\frac{\Gamma \to A, B, \Delta}{\Gamma \to A \vee B, \Delta}\ (\to\vee), \qquad \frac{A, \Gamma \to \Delta; B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee\to),$$

$$\frac{\Gamma \to A(b), \Delta}{\Gamma \to \forall x A(x), \Delta}\ (\to\forall), \qquad \frac{A(t), \forall x A(x), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta}\ (\forall\to),$$

$$\frac{\Gamma \to A(t), \exists x A(x), \Delta}{\Gamma \to \exists x A(x), \Delta}\ (\to\exists), \qquad \frac{A(b), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta}\ (\exists\to).$$

Here: E denotes an atomic formula; A, B denote arbitrary formulae; Γ, Δ denote finite, possibly empty, multisets of formulae; x denotes a bound variable; t denotes a term; in the $(\forall \rightarrow)$, $(\rightarrow \exists)$ rules, $A(x)$ is obtained from $A(t)$ by substituting x for at least one occurrence of t in $A(t)$; b denotes a free variable which does not occur in conclusions of the rules $(\rightarrow \forall)$, $(\exists \rightarrow)$, and $A(x)$ in these rules is obtained by substituting x for every occurrence of b in $A(b)$.

Rules for temporal operators:

$$\frac{\Gamma \rightarrow \Delta}{\Pi, \circ\Gamma \rightarrow \circ\Delta, \Lambda}\ (\circ), \qquad \frac{\Box\Gamma \rightarrow A}{\Pi, \Box\Gamma \rightarrow \Box A, \Lambda}\ (\Box),$$

$$\frac{A, \circ\Box A, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta}\ (\Box \rightarrow), \qquad \frac{\Gamma \rightarrow A, \Delta; \Gamma \rightarrow \circ\Box A, \Delta}{\Gamma \rightarrow \Box A, \Delta}\ (\rightarrow \Box).$$

Here: Γ, Δ, A are as above; if $\Gamma = A_1, \dots, A_n$, then $\sigma\Gamma = \sigma A_1, \dots, \sigma A_n$, where $\sigma \in \{\Box, \circ\}$; Π, Λ denote finite, possibly empty, multisets of formulae. The rule $(\rightarrow \Box)$ corresponds to the weak induction axiom: $(A \wedge \circ\Box A) \supset \Box A$.

The definition of derivation is common (see, e.g., [7]).

A rule is admissible in a sequent calculus $Calc$ if derivability of its premise or premises in $Calc$ implies derivability of its conclusion in $Calc$.

The following properties are correct for LB.

Lemma 2.1. *The structural rule of weakening*

$$\frac{\Gamma \rightarrow \Delta}{\Pi, \Gamma \rightarrow \Delta, \Lambda}\ (w),$$

where Π, Λ *are finite multisets of formulae, is admissible in LB. Moreover, if* $LB \vdash^V \Gamma \rightarrow \Delta$, *then there exists* V', *such that* $LB \vdash^{V'} \Pi, \Gamma \rightarrow \Delta, \Lambda$ *and* $h(V') \leqslant h(V)$. *Here* Π *and* Λ *are finite multisets of formulae.*

Lemma 2.2. *Let* S *be a sequent having the shape of the conclusion of an LB rule* (i), *except* $(\Box \rightarrow)$, $(\circ)$ *and* $(\Box)$. *Let* S' *be the sequent having the shape of a premise of the same rule* (i) *with the restriction that if* $(i) = (\rightarrow \Box)$, *then* S' *is the left premise. If* $LB \vdash^V S$, *then there exists a derivation* V' *such that* $LB \vdash^{V'} S'$ *and* $h(V') \leqslant h(V)$.

Lemma 2.3. *LB rules* $(\Box \rightarrow)$ *and* $(\rightarrow \Box)$ *are invertible in LB. LB rules* $(\circ)$ *and* $(\Box)$ *are invertible in LB if* Π *and* Λ *are empty in these rules.*

Lemma 2.4. *The structural rule of contraction* $(c \rightarrow)$*:*

$$\frac{\Gamma, C, C \rightarrow \Delta}{\Gamma, C \rightarrow \Delta}\ (c \rightarrow),$$

where C *is an arbitrary formula, is admissible in LB.*

Lemma 2.5. *The structural rule of contraction* $(\rightarrow c)$*:*

$$\frac{\Gamma \rightarrow C, C, \Delta}{\Gamma \rightarrow C, \Delta}\ (\rightarrow c),$$

where C is an arbitrary formula, is admissible in LB.

Lemma 2.6. *The cut rule is admissible in LB.*

3. Sequent calculus LBJ

The sequent calculus LBJ is obtained from LJ, which is a variant of the Gentzen's intuitionistic sequent calculus (without structural rules), by adding some rules for temporal operators which are similar to these added to LK in order to get LB.

Axioms: $\Gamma, E \to E$; $\Gamma, \mathcal{F} \to \Delta$.

LJ rules:

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ (\to\supset), \qquad \frac{A \supset B, \Gamma \to A;\ B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ (\supset\to),$$

$$\frac{\Gamma \to A;\ \Gamma \to B}{\Gamma \to A \wedge B}\ (\to \wedge), \qquad \frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ (\wedge \to),$$

$$\frac{\Gamma \to A \text{ or } \Gamma \to B}{\Gamma \to A \vee B}\ (\to \vee), \qquad \frac{A, \Gamma \to \Delta;\ B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee \to),$$

$$\frac{\Gamma \to A(b)}{\Gamma \to \forall x A(x)}\ (\to \forall), \qquad \frac{A(t), \forall x A(x), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta}\ (\forall \to),$$

$$\frac{\Gamma \to A(t)}{\Gamma \to \exists x A(x)}\ (\to \exists), \qquad \frac{A(b), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta}\ (\exists \to).$$

Here: E denotes an atomic formula; A, B denote arbitrary formulae; $\Delta \in \{\emptyset, D\}$ (D is an arbitrary formula); Γ denotes a finite, possibly empty, multiset of formulae; x denotes a bound variable; t denotes a term; in the $(\forall \to)$, $(\to \exists)$ rules, $A(x)$ is obtained from $A(t)$ by substituting x for at least one occurrence of t in $A(t)$; b denotes a free variable which does not occur in conclusions of the rules $(\to \forall)$, $(\exists \to)$, and $A(x)$ in these rules is obtained by substituting x for every occurrence of b in $A(b)$.

Rules for temporal operators:

$$\frac{\Gamma \to A}{\Pi, \circ\Gamma \to \circ A}\ (\circ_1), \qquad \frac{\Gamma \to}{\Pi, \circ\Gamma \to \Delta}\ (\circ_2),$$

$$\frac{\Box\Gamma \to A}{\Pi, \Box\Gamma \to \Box A}\ (\Box), \qquad \frac{A, \circ\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to),$$

$$\frac{\Gamma \to A;\ \Gamma \to \circ\Box A}{\Gamma \to \Box A}\ (\to \Box).$$

Here: Γ, Δ, A are as above; if $\Gamma = A_1, \ldots, A_n$, then $\sigma\Gamma = \sigma A_1, \ldots, \sigma A_n$, where $\sigma \in \{\Box, \circ\}$; Π denotes a finite, possibly empty, multiset of formulae. The rule $(\to \Box)$ corresponds to the weak induction axiom: $(A \wedge \circ\Box A) \supset \Box A$.

The definition of derivation is common (see, e.g., [7]). We usually denote a derivation and a height of the derivation by V and $h(V)$, respectively.

The calculus LBJ is investigated in detail in [1] and [2].

4. Purely Glivenko and Glivenko σ-classes

Let $\odot \in \mathbb{A} = \{\vee, \wedge, \supset, \exists, \forall, \circ, \Box\}$. An occurrence of $\odot$ in a formula or a sequent is called an occurrence of the type $\odot^+$ ($\odot^-$) if $\odot$ is a positive (negative) occurrence in this formula or sequent.

Lemma 4.1. *Let V be a derivation of a sequent S in LB, LBJ* or LBJ. If there are no occurrences of $\odot \in \{\vee, \wedge, \supset, \exists, \forall, \Box\}$ of the type $\odot^+$ ($\odot^-$) in S, then there are applications of the rule $(\to \odot)$ ($(\odot \to)$) in V; also, if there are no occurrences of $\Box$ the type $\Box^+$ in S, then there are no applications of the rule $(\Box)$ in V.*

Following [6], we define σ-classes and purely Glivenko σ-classes. A set $\{U_1^{\alpha_1}, \ldots, U_n^{\alpha_n}\}$, where $U_i \in \mathbb{A}$ (see above) and $\alpha_i \in \{-, +\}$, is called a σ-class. A sequent S belongs to a σ-class $\{U_1^{\alpha_1}, \ldots, U_n^{\alpha_n}\}$ iff there are no occurrences of $U_1, \ldots, U_n$ of the type $U_1^{\alpha_1}, \ldots, U_n^{\alpha_n}$, respectively, in S. A σ-class $\mathfrak{A}$ is contained in a σ-class $\mathfrak{B}$ if every sequent which belongs to $\mathfrak{A}$ also belongs to $\mathfrak{B}$ (note that then $\mathfrak{B} \subseteq \mathfrak{A}$).

A sequent having one formula in succedent is called a singular sequent. A σ-class is called a purely Glivenko σ-class if every singular sequent which belongs to it is derivable in LB iff it is derivable in LBJ.

Theorem 4.2. *A σ-class is a purely Glivenko σ-class iff it is contained in at least one of the following σ-classes:*

1.1. $\{\vee^-, \supset^+\}$;
1.2. $\{\supset^-, \vee^+, \exists^+\}$;
2.1. $\{\circ^+, \circ^-, \supset^+, \forall^+\}$;
2.2. $\{\circ^+, \circ^-, \supset^+, \forall^-\}$;
3.1. $\{\circ^+, \Box^+, \supset^+, \forall^+\}$;
3.2. $\{\circ^+, \Box^+, \supset^+, \forall^-\}$;
4.1. $\{\circ^-, \Box^-, \supset^+, \forall^+\}$;
4.2. $\{\circ^-, \Box^-, \supset^+, \forall^-\}$;
5.1. $\{\circ^+, \Box^-, \supset^+, \forall^+\}$;
5.2. $\{\circ^+, \Box^-, \supset^+, \forall^-\}$.

A σ-class $\mathfrak{A}$ is called a Glivenko σ-class if every sequent with an empty succedent which belongs to $\mathfrak{A}$ is derivable in LB iff it is derivable in LBJ. In the following theorem we give conditions which has to satisfy every Glivenko σ-class.

Theorem 4.3. *A σ-class is a Glivenko σ-class iff it is contained in at least one of the following σ-classes:*

1. $\{\supset^-\}$;
2. $\{\supset^+, \vee^-\}$;
3. $\{\supset^+, \forall^-, \circ^+, \circ^-\}$;
4. $\{\supset^+, \forall^-, \circ^+, \Box^+\}$;
5. $\{\supset^+, \forall^-, \circ^+, \Box^-\}$;

6. $\{\supset^{+}, \forall^{+}, \circ^{+}, \circ^{-}\}$;
7. $\{\supset^{+}, \forall^{+}, \circ^{+}\Box^{-}\}$;
8. $\{\forall^{+}, \circ^{+}, \Box^{+}\}$;
9. $\{\supset^{+}, \forall^{+}, \circ^{-}, \Box^{-}\}$;
10. $\{\supset^{+}, \forall^{-}, \circ^{-}, \Box^{-}\}$.

References

[1] R. Alonderis, Proof-theoretical investigation of temporal logic with time gaps, *Lith. Math. J.*, **40**(3), 255–276 (2000).
[2] R. Alonderis, Specialization of loop rules of a sequent calculus of intuitionistic temporal logic with time gaps, *Lith. Math. J.*, **40**(4), 404–429 (2000).
[3] M. Baaz, A. Leitsch, R. Zach, Completeness of a first-order temporal logic with time gaps, *Theoretical Computer Science*, **160**, 241–270 (1996).
[4] V. Glivenko, Sur quelques points de la logique de M. Brouwer, *Bull. cl. sci. Acad. Roy. Belg.*, ser 5, **15**, 183–188 (1929).
[5] G. Mints, V. Orevkov, Generalization of V. I. Glivenko and G. Kreisel theorems on a class of formulas of predicate calculus, *Doklady Akademii Nauk SSSR*, **152**(3), 553–554 (1963) (in Russian).
[6] V. Orevkov, On Glivenko classes of sequents, *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, **98**, 131–154 (1968).
[7] G. Takeuti, *Proof Theory*, North-Holland, Amsterdam (1975).

Ryšys tarp klasikinių ir intuicionistinių laiko logikos sekvencinių skaičiavimų

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Darbe nagrinėjamas ryšys tarp intuicionistinio ir klasikinio laiko logikos su laiko tarpsniais sekvencinių skaičiavimų LB ir LBJ. Šis ryšys yra nusakomas Glivenko klasėmis. Nurodomos sąlygos, kurias turi tenkinti kiekviena Glivenko ir visiškai Glivenko σ-klasė.