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Initial Boundary-Value Problems for Derivative Nonlinear Schrödinger Equation. Justification of Two-Step Algorithm

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Abstract. We investigate two different initial boundary-value problems for derivative nonlinear Schrödinger equation. The boundary conditions are Dirichlet or generalized periodic ones. We propose a two-step algorithm for numerical solving of this problem. The method consists of Bäcklund type transformations and difference scheme. We prove the convergence and stability in C and H^1 norms of Crank–Nicolson finite difference scheme for the transformed problem. There are no restrictions between space and time grid steps. For the derivative nonlinear Schrödinger equation, the proposed numerical algorithm converges and is stable in C^1 norm.

Keywords: derivative nonlinear Schrödinger equation, initial boundary-value problem, Bäcklund transformations, Crank–Nicolson finite difference scheme, convergence and stability of difference schemes.

AMS classifications: primary 65M06, 65M12; secondary 35A22, 35Q55.

1 Introduction

We consider two different initial boundary-value problems for derivative nonlinear Schrödinger equation. Note that similar derivative dependent nonlinear terms appear in the Korteweg and de Vries (KdV) equation, the Burgers equations, the Navier-Stokes models, and other problems where one must take into account some higher order perturbations.

In this paper, we propose (and justify) the algorithm for solving of the considered problem on the computer. Note that our method is a non-standard one as it consists of two independent steps. The first step, presented in details in Section 4, handles derivative dependent nonlinearities. Also note that the second part of this paper (the second step), which deals with difference scheme for the system of two nonlinear Schrödinger equations, can be considered as independent result. As far as we know, for such initial boundary-value problem (with the boundary conditions being either mixed Dirichlet–Neumann or generalized periodic ones), no numerical method was justified to this time.

The derivative nonlinear Schrödinger equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} + id|u|^2u + ik|u|^4u + \alpha|u|^2\frac{\partial u}{\partial x} + \beta u\frac{\partial |u|^2}{\partial x} \tag{1}$$

(where x and t denote the space and the time coordinates, respectively, while a, d, k, α, β are real constants), is used for modeling of wave processes in different physical systems such as nonlinear optics [1, 20, 22], circular polarized Alfvén waves in plasma [18, 19], Stokes waves in fluids of finite depth, etc. The quantities $\alpha|u|^2\partial_x u$ and $\beta u\partial_x|u|^2$ in equation (1) are called the derivative nonlinear terms.

In nonlinear optics [1, 20, 22], equation (1) can be derived in a systematic way by means of the reductive perturbation scheme as a model for single mode propagation. In the context of waveguides as optical fibres, t usually corresponds to the propagation distance of the electric field envelope u of an optical beam along the fibre, x plays the role of the time, the terms $d|u|^2u$ and $k|u|^4u$ model the nonlinear Kerr effect, while $\alpha|u|^2\partial_x u$ and $\beta u\partial_x|u|^2$ are the nonlinear dispersion contributions.

Note, that equation (1) is a generalization of the standard nonlinear Schrödinger equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} + id|u|^2u, \tag{2}$$

for nonlinearly modulated wave trains modeling in the so called *marginal stable* regime [2].

There are many partial cases of (1), for which, due to the Lax pair formalism, interesting solutions, e. g., solitons can be constructed analytically [3, 13, 15]. Recently, there were computed new classes of symmetry reductions and associated exact solutions of two-dimensional (with the space variables x and y) derivative nonlinear Schrödinger equation [7]. It also appears that these partial cases are gauge equivalent, *i. e.*, can be transformed into each other by some Bäcklund type transformation [16]. One could mention the Kaup–Newell equation, the Chen–Lee–Liu equation and the Gerdjikov–Ivanov equation.

Note that, in [2, 3, 6, 7, 13, 15, 16], the Cauchy problem is dealt with. We examine the boundary problems, that require a different or adopted techniques. It can appear, for example, that a boundary-value problem, for some well-known nonlinear parabolic equations, has no solution at all [12]. Therefore, we have discussed in [17] the well-posedness of the models considered in the present work.

Dealing with (1), the main difficulties are caused by the derivative nonlinear terms. In [6], Hayashi overcomes the so-called derivative loss by reducing the Cauchy problem for the Kaup–Newell equation

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2} \pm \frac{\partial}{\partial x}\left(|u|^2 u\right)$$

to the system of two nonlinear Schrödinger equations. In this paper, we generalize and adopt the transformations applied in [6] (see also [16]) in order to perform the numerical analysis of Dirichlet and periodic initial boundary-value problems. Note also that, in [9], we have used another approach to handle the derivative loss, namely, some parabolic viscosity was introduced.

There is a lot of results on numerical aspects of nonlinear Schrödinger equation (without derivative nonlinearities) [4, 5, 8, 9, 10, 11, 14, 21, 23, 24, 25]. For example, implicit finite difference schemes were justified for many-dimensional Schrödinger models with zero boundary conditions [8, 11]. Similar results are obtained in the case of Neumann boundary conditions [21]. The convergence and stability of an implicit scheme for slightly generalized equation (2) with zero boundary conditions were also proved [10]. A comparison of many difference schemes for the model considered in [10] was presented in [4].

This paper is organized as follows. In Section 2, we state the problem of our interest. Section 3 is for introducing notation and fundamental theorems which our analysis relies on. As the first step of the algorithm, in Section 4 we employ two explicitly invertible transformations (which can be computed numerically, too) to obtain an evolutionary type equation system containing no gradient dependent nonlinearities. The second step of the proposed method are finite difference approximations to the reduced problem which are introduced in Section 5. In Sections 6 and 7, we prove some *a priori* estimates and the convergence of the iterative method applied to nonlinear finite difference schemes. Finally, in Section 8, we prove the convergence and stability of applied difference schemes.

2 Statement of the problem

In this paper, we deal with the derivative nonlinear Schrödinger equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= ia\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + icu + id|u|^2u + ik|u|^4u + \\ &\alpha|u|^2\frac{\partial u}{\partial x} + \beta u\frac{\partial |u|^2}{\partial x},\end{aligned} \tag{3}$$

where $u = u(x,t)$ is an unknown complex function, $a, b, c, d, k, \alpha, \beta$ are given real coefficients, $i = \sqrt{-1}$. This differential equation is studied for $t \in (0, T]$ and for x in a bounded interval $\Omega \subset \mathbf{R}$. For simplicity we take $\Omega = (0,1)$. Note that, in equation (3), by rescaling the time variable t one coefficient can be set to 1. Therefore we further assume that $a = 1$.

We consider (3) together with initial and boundary conditions. Having defined the initial function

$$u(x,0) = u^{(0)}(x), \qquad x \in \Omega, \tag{4}$$

we are going to justify a numerical analysis for the solution $u(x,t)$ satisfying one of the two different type conditions on the boundary. One of the popular ways in numerical modelling of the corresponding Cauchy problem is to truncate a solution outside of some given region Ω, *i. e.*, the zero Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le T. \tag{5}$$

There also exists another approach caused by specific features of some modeled phenomena, namely, the periodic boundary-value problem

$$u(0,t) = \theta u(1,t), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \theta \left.\frac{\partial u}{\partial x}\right|_{x=1}, \qquad 0 \le t \le T \tag{6}$$

with the complex parameter (phase shift) θ such that $|\theta| = 1$.

We assume that, at $t = 0$, the initial function (4) satisfies boundary conditions (5) or (6), respectively.

3 Notation and mathematical preliminaries

Let $Q = \Omega \times (0, T]$. For complex valued functions, let $(\cdot, \cdot)$ and $\| \cdot \|$ denote the inner product and norm in $L_2 = L_2(\Omega)$, and let $H^1 = H^1(\Omega) = \{u \in L_2 : \partial_x u \in L_2\}$ be the standard Sobolev space with the norm

$$\|u\|_{H^1} = \sqrt{\|u\|^2 + \|\partial_x u\|^2}.$$

The space $C^{j,k} = C^{j,k}(Q)$ consists of the functions with continuous in Q jth derivatives with respect to x and continuous kth derivatives with respect to t. Throughout this paper, we denote by g^* the complex conjugate of g.

Denoting difference analogues of domains, derivatives, inner products, and norms we follow the notation accepted in numerical analysis. We introduce the uniform grids with steps τ and h

$$\Omega_h = \{x_j\colon\ x_j = jh,\ j = 1, 2, \dots, N-1\}, \quad \bar{\Omega}_h = \Omega_h \cup \{0, 1\},$$

$$\Omega_h^+ = \Omega_h \cup \{1\},$$

$$\omega_\tau = \{t_k\colon\ t_k = k\tau,\ k = 0, 1, \dots, M-1\}, \qquad \bar{\omega}_\tau = \omega_\tau \cup \{T\},$$

$$Q_h = \Omega_h \times \omega_\tau, \qquad \bar{Q}_h = \bar{\Omega}_h \times \omega_\tau, \qquad Q_h^+ = \Omega_h^+ \times \omega_\tau,$$

where $Nh = 1$, $M\tau = T$.

We denote the difference derivatives

$$p_x = \frac{p(x+h,t) - p(x,t)}{h}, \quad p_{\bar{x}} = \frac{p(x,t) - p(x-h,t)}{h},$$

$$p_{\overset{\circ}{x}} = \frac{p_x + p_{\bar{x}}}{2}, \quad \hat{p} = p(x, t+\tau), \quad \overset{\circ}{p} = \frac{\hat{p} + p}{2}, \quad p_t = \frac{\hat{p} - p}{\tau},$$

and the inner products

$$(p, q)_h = h \sum_{x \in \Omega_h} p(x,t) q^*(x,t), \qquad (p, q]_h = h \sum_{x \in \Omega_h^+} p(x,t) q^*(x,t),$$

$$[p,q]_h = h \sum_{x\in\Omega_h} p(x,t)q^*(x,t) + \frac{h}{2}\Big(p(0,t)q^*(0,t) + p(1,t)q^*(1,t)\Big).$$

The following difference Green formulae are true:

$$(p_{\bar{x}x}, q)_h = -\left(p_{\bar{x}}, q_{\bar{x}}\right]_h + p_{\bar{x}}(1,t)q^*(1,t) - p_x(0,t)q^*(0,t), \tag{7}$$

$$(p_{\bar{x}x}, q]_h = -\left(p_{\bar{x}}, q_{\bar{x}}\right]_h + p_x(1,t)q^*(1,t) - p_x(0,t)q^*(0,t), \tag{8}$$

$$[p_{\bar{x}x}, q]_h = -\left(p_{\bar{x}}, q_{\bar{x}}\right]_h + p_{\overset{\circ}{x}}(1,t)q^*(1,t) - p_{\overset{\circ}{x}}(0,t)q^*(0,t). \tag{9}$$

Note that (8) and (9) require the function p to be defined outside the grid $\bar{\Omega}_h$.

Next, we introduce the norms of the grid functions. We employ the discrete L_2 norms

$$\|p\|_h = \sqrt{(p,p)_h}, \qquad \|p]|_h = \sqrt{(p,p]_h}, \qquad |[p]|_h = \sqrt{[p,p]_h}.$$

It remains to introduce the norm of the discrete space C_h (the grid projection of the continuous function space C)

$$\|p\|_{C,h} = \max_{x\in\bar{\Omega}_h} |p(x)|$$

and the one related to H_h^1 (discretization of H^1)

$$\|p\|_{H^1,h} = \sqrt{|[p]|_h^2 + \|p_{\bar{x}}]|_h^2}.$$

For grid functions $p \in H_h^1$, we should recall the discrete Gagliardo–Nirenberg type estimate

$$\|p\|_{C,h} \le c_G \|p\|_h^{1/2} \|p\|_{H^1,h}^{1/2} \tag{10}$$

and the imbedding theorem $H_h^1 \to C_h$:

$$\|p\|_{C,h} \le \sqrt{2}\,\|p\|_{H^1,h}. \tag{11}$$

The following inequality is true:

$$\frac{1+cx}{1-cx} \le 1 + 4cx, \qquad 0 \le x \le 1/(2c), \qquad c > 0. \tag{12}$$

4 Reduction of nonlinearity

In this section, we simplify (3), (4), (5) and (3), (4), (6) problems by reducing derivative dependent nonlinearities. The transformations introduced in this section represent the first step of a numerical algorithm for derivative nonlinear Schrödinger equation.

For simplicity, we first rearrange derivative nonlinear terms in equation (3).

Define the real function

$$\begin{aligned} q(t) = \int_0^t b|u(0,\tau)|^2 + \left(\frac{\alpha}{2} + \beta\right)|u(0,\tau)|^4 - \\ 2\,\mathrm{Im}\left(\frac{\partial u(0,\tau)}{\partial x}u^*(0,\tau)\right)\,d\tau. \end{aligned} \tag{13}$$

Note that $q(t) \equiv 0$ in the case of boundary conditions (5).

We define the transformed function $v = v(x,t)$ by

$$v = u(x,t)e^{-iA\left(\int_0^x |u(s,t)|^2\,ds + q(t)\right) - i\frac{b}{2}x - i\left(c + \frac{b^2}{4}\right)t}; \tag{14}$$

here A is a real parameter.

Remark 4.1 *Note that, due to* (14), $|u| = |v|$. *Note also, that Bäcklund type transformation* (14) *is explicitly invertible, since*

$$u(x,t) = v(x,t)e^{iA\left(\int_0^x |v(s,t)|^2\,ds + q(t)\right) + i\frac{b}{2}x + i\left(c + \frac{b^2}{4}\right)t} \tag{15}$$

and

$$q(t) = \int_0^t \left(\frac{\alpha}{2} + \beta - 2A\right)|v(0,\tau)|^4 - 2\,\mathrm{Im}\left(\frac{\partial v(0,\tau)}{\partial x}v^*(0,\tau)\right)\,d\tau, \tag{16}$$

and therefore it can be computed numerically.

Proposition 4.2 *Assume that* $u(x,t) \in C^{2,1}$ *is a solution of* (3), (4), (5) *or* (3), (4), (6), *respectively. Then the function* v *defined by* (13), (14) *satisfies*

$$\begin{aligned} \frac{\partial v}{\partial t} = i\frac{\partial^2 v}{\partial x^2} + i\left(d + \frac{\alpha b}{2}\right)|v|^2 v + i\,k_{tr}\,|v|^4 v + \\ (\alpha + \beta - 2A)\,|v|^2\frac{\partial v}{\partial x} + (\beta - 2A)\,v^2\frac{\partial v^*}{\partial x} \end{aligned} \tag{17}$$

for $t > 0$, $x \in \Omega$, and the boundary conditions

$$v(0,t) = v(1,t) = 0, \qquad 0 \le t \le T,$$

or

$$v(0,t) = \theta_1 v(1,t), \qquad \left.\frac{\partial v}{\partial x}\right|_{x=0} = \theta_1 \left.\frac{\partial v}{\partial x}\right|_{x=1}, \qquad 0 \le t \le T,$$

respectively. Here $k_{tr} = k + A\left(A + \frac{\alpha}{2} - \beta\right)$ *and* θ_1 *is a constant such that* $|\theta_1| = 1$.

Proof: Define

$$E(x,t) = e^{iA\left(\int_0^x |u(s,t)|^2\,ds + q(t)\right) + i\frac{b}{2}x + i\left(c + \frac{b^2}{4}\right)t},$$

$$I(x,t) = \frac{\partial}{\partial t}\int_0^x |u(s,t)|^2\,ds.$$

Due to (15) we have

$$\frac{\partial u}{\partial x} = \left(\frac{\partial v}{\partial x} + iA|v|^2 v + i\frac{b}{2}v\right)E(x,t), \tag{18}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} = &\left(\frac{\partial^2 v}{\partial x^2} + ib\frac{\partial v}{\partial x} - \frac{b^2}{4}v - bA|v|^2 v - A^2|v|^4 v + \right.\\ &\left. i2A|v|^2\frac{\partial v}{\partial x} + iAv\frac{\partial |v|^2}{\partial x}\right)E(x,t),\end{aligned} \tag{19}$$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial v}{\partial t} + iAv\,I(x,t) + iAvq'(t) + i\left(c + \frac{b^2}{4}\right)v\right)E(x,t).$$

Multiplying (3) by $2u^*$, integrating, and taking the real part we obtain

$$I(x,t) = i\left(\frac{\partial u}{\partial x}u^* - u\frac{\partial u^*}{\partial x}\right) + b|u|^2 + \left(\frac{\alpha}{2} + \beta\right)|u|^4 - q'(t). \tag{20}$$

By (14) and (18) one gets

$$I(x,t) = i\left(v^*\frac{\partial v}{\partial x} - v\frac{\partial v^*}{\partial x}\right) + \left(\frac{\alpha}{2} + \beta - 2A\right)|v|^4 - q'(t).$$

Therefore,

$$\frac{\partial u}{\partial t} = \left(\frac{\partial v}{\partial t} + i \left(c + \frac{b^2}{4} \right) v + i \left(A \left(\frac{\alpha}{2} + \beta \right) - 2A^2 \right) |v|^4 v - \right.$$

$$\left. 2A|v|^2 \frac{\partial v}{\partial x} + Av \frac{\partial |v|^2}{\partial x} \right) E(x,t).$$

Substituting this expression together with (18) and (19) into (3) we get (17).

The invariance of boundary conditions (5) is trivial. For (6), due to (14) and (18), we obtain

$$\theta_1 = \theta \, e^{i \left(A \|u^{(0)}\|^2 + \frac{b}{2} \right)},$$

since $\|u\| = \|u^{(0)}\|$ (one proves it by taking $x = 1$ in (20)). △

Due to Proposition 4.2, we can neglect some terms in equation (3). Choosing $A = (\alpha + \beta)/2$ and redenoting the coefficients and solution we will further deal with the equation

$$\frac{\partial u}{\partial t} = i \frac{\partial^2 u}{\partial x^2} + id|u|^2 u + ik|u|^4 u + \alpha u^2 \frac{\partial u^*}{\partial x}. \tag{21}$$

Proposition 4.3 *Assume that* $u(x,t) \in C^{3,1}$ *is a solution of* (21), (4), (5) *or* (21), (4), (6), *respectively. We define the function* v *by*

$$v = \frac{\partial u}{\partial x} - i \frac{\alpha}{2} |u|^2 u. \tag{22}$$

Then problem (21), (4), (5) *or* (21), (4), (6), *respectively, is equivalent to the equation system*

$$\begin{aligned} \frac{\partial u}{\partial t} &= i \frac{\partial^2 u}{\partial x^2} + id|u|^2 u + i\tilde{k}|u|^4 u + \alpha u^2 v^*, \\ \frac{\partial v}{\partial t} &= i \frac{\partial^2 v}{\partial x^2} + id \left(2|u|^2 v + u^2 v^* \right) + i\tilde{k} \left(3|u|^4 v + 2|u|^2 u^2 v^* \right) - \\ &\alpha u^* v^2 \end{aligned} \tag{23}$$

for $t > 0$, $x \in \Omega$, $\tilde{k} = k - \alpha^2/2$, *with the boundary conditions*

$$u(0,t) = u(1,t) = 0, \quad \left. \frac{\partial v}{\partial x} \right|_{x=0} = \left. \frac{\partial v}{\partial x} \right|_{x=1} = 0, \quad 0 \le t \le T, \tag{24}$$

or

$$\begin{aligned} &u(0,t) = \theta u(1,t), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \theta \left.\frac{\partial u}{\partial x}\right|_{x=1}, \\ &v(0,t) = \theta v(1,t), \qquad \left.\frac{\partial v}{\partial x}\right|_{x=0} = \theta \left.\frac{\partial v}{\partial x}\right|_{x=1}, \qquad 0 \le t \le T, \end{aligned} \tag{25}$$

respectively.

Proof: The first equation in (23) follows directly by definition (22). We will obtain the second equation. Due to the smoothness of u, (21) implies

$$\frac{\partial^2 u}{\partial x \partial t} = i\frac{\partial^3 u}{\partial x^3} + id\frac{\partial}{\partial x}\left(|u|^2 u\right) + ik\frac{\partial}{\partial x}\left(|u|^4 u\right) + \alpha\frac{\partial}{\partial x}\left(u^2\frac{\partial u^*}{\partial x}\right). \tag{26}$$

By (21) and (22) we also establish the expressions

$$\frac{\partial}{\partial t}\left(|u|^2 u\right) = 2|u|^2\frac{\partial u}{\partial t} + u^2\frac{\partial u^*}{\partial t} =$$

$$i2|u|^2\frac{\partial^2 u}{\partial x^2} - iu^2\frac{\partial^2 u^*}{\partial x^2} + id|u|^4 u + ik|u|^6 u + 2\alpha|u|^2 u^2\frac{\partial u^*}{\partial x} +$$

$$\alpha|u|^4\frac{\partial u}{\partial x} = i2|u|^2\frac{\partial^2 u}{\partial x^2} - iu^2\frac{\partial^2 u^*}{\partial x^2} + id|u|^4 u + i\left(k - \frac{\alpha^2}{2}\right)|u|^6 u +$$

$$2\alpha|u|^2 u^2 v^* + \alpha|u|^4 v,$$

$$\frac{\partial}{\partial x}\left(|u|^2 u\right) = 2|u|^2\frac{\partial u}{\partial x} + u^2\frac{\partial u^*}{\partial x} = 2|u|^2 v + u^2 v^* + i\frac{\alpha}{2}|u|^4 u,$$

$$\frac{\partial}{\partial x}\left(|u|^4 u\right) = 3|u|^4\frac{\partial u}{\partial x} + 2|u|^2 u^2\frac{\partial u^*}{\partial x} = 3|u|^4 v + 2|u|^2 u^2 v^* +$$

$$i\frac{\alpha}{2}|u|^6 u,$$

$$\frac{\partial^2}{\partial x^2}\left(|u|^2 u\right) = \frac{\partial^2 u^2}{\partial x^2}u^* + 2\frac{\partial u^2}{\partial x}\frac{\partial u^*}{\partial x} + u^2\frac{\partial^2 u^*}{\partial x^2} =$$

$$2u^*\left(\frac{\partial u}{\partial x}\right)^2 + 2|u|^2\frac{\partial^2 u}{\partial x^2} + 4u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} + u^2\frac{\partial^2 u^*}{\partial x^2} =$$

$$2u^* v^2 + i2\alpha|u|^4 v - \frac{\alpha^2}{2}|u|^6 u + 2|u|^2\frac{\partial^2 u}{\partial x^2} + 4u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} + u^2\frac{\partial^2 u^*}{\partial x^2}.$$

Therefore by (22) and (26)

$$\frac{\partial v}{\partial t} - i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial t} - i\frac{\partial^3 u}{\partial x^3} - i\frac{\alpha}{2}\frac{\partial}{\partial t}\left(|u|^2 u\right) - \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\left(|u|^2 u\right) =$$

$$id\frac{\partial}{\partial x}\left(|u|^2 u\right) + ik\frac{\partial}{\partial x}\left(|u|^4 u\right) + \alpha\frac{\partial}{\partial x}\left(u^2\frac{\partial u^*}{\partial x}\right) - i\frac{\alpha}{2}\frac{\partial}{\partial t}\left(|u|^2 u\right) -$$

$$\frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\left(|u|^2 u\right) = id\left(2|u|^2 v + u^2 v^*\right) - \frac{\alpha d}{2}|u|^4 u +$$

$$ik\left(3|u|^4 v + 2|u|^2 u^2 v^*\right) - \frac{\alpha k}{2}|u|^6 u + \alpha|u|^2\frac{\partial^2 u}{\partial x^2} - \frac{\alpha}{2}u^2\frac{\partial^2 u^*}{\partial x^2} +$$

$$\frac{\alpha d}{2}|u|^4 u + \left(\frac{\alpha k}{2} - \frac{\alpha^3}{4}\right)|u|^6 u - i\frac{\alpha^2}{2}\left(2|u|^2 u^2 v^* + |u|^4 v\right) +$$

$$2\alpha u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} + \alpha u^2\frac{\partial^2 u^*}{\partial x^2} - \alpha u^* v^2 - i\alpha^2|u|^4 v + \frac{\alpha^3}{4}|u|^6 u -$$

$$\alpha|u|^2\frac{\partial^2 u}{\partial x^2} - 2\alpha u\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} - \frac{\alpha}{2}u^2\frac{\partial^2 u^*}{\partial x^2} =$$

$$id\left(2|u|^2 v + u^2 v^*\right) + i\left(k - \frac{\alpha^2}{2}\right)\left(3|u|^4 v + 2|u|^2 u^2 v^*\right) - \alpha u^* v^2.$$

It remains to prove (24) or (25), respectively. Setting $x = 0$ and $x = 1$ in equation (21) we conclude

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1} = 0, \qquad 0 \le t \le T,$$

or

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = \theta\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1}, \qquad 0 \le t \le T,$$

respectively. Since

$$\frac{\partial v}{\partial x} = \frac{\partial^2 u}{\partial x^2} - i\alpha|u|^2\frac{\partial u}{\partial x} - i\frac{\alpha}{2}u^2\frac{\partial u^*}{\partial x},$$

we obtain (24) or (25). △

5 Finite difference scheme

Due to Proposition 4.2 and Proposition 4.3, we can reduce the derivative nonlinearities. Therefore, the following partial differential equation system will be further considered for $(x,t) \in Q$

$$\begin{aligned} \frac{\partial u}{\partial t} &= i\frac{\partial^2 u}{\partial x^2} + f_1(u,u^*,v,v^*), \\ \frac{\partial v}{\partial t} &= i\frac{\partial^2 v}{\partial x^2} + f_2(u,u^*,v,v^*) \end{aligned} \tag{27}$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{x=0} = \left.\frac{\partial v}{\partial x}\right|_{x=1} = 0, \quad 0 \le t \le T, \tag{28}$$

or

$$\begin{aligned} &u(0,t) = \theta u(1,t), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \theta \left.\frac{\partial u}{\partial x}\right|_{x=1}, \\ &v(0,t) = \theta v(1,t), \qquad \left.\frac{\partial v}{\partial x}\right|_{x=0} = \theta \left.\frac{\partial v}{\partial x}\right|_{x=1}, \qquad 0 \le t \le T, \end{aligned} \tag{29}$$

respectively, here $|\theta| = 1$. The initial data are given by

$$u(x,0) = u^{(0)}(x), \qquad v(x,0) = v^{(0)}(x), \qquad x \in \Omega. \tag{30}$$

We suppose that the functions f_j, $j = 1,2$, and all their partial derivatives up to the second order are continuous. This is satisfied for system (23), since f_j are polynomials. The continuity requirement implies the existence of a continuous nondecreasing function φ such that

$$\begin{aligned} &|f_j(u,u^*,v,v^*)| \le \varphi(y), \qquad |D^m f_j(u,u^*,v,v^*)| \le \varphi(y), \\ &|m| = 1,2, \qquad j = 1,2; \end{aligned} \tag{31}$$

here $y = \max\{|u|,|v|\}$, $D^m = \partial^{|m|}/\partial u^{m_1}\partial u^{*m_2}\partial v^{m_3}\partial v^{*m_4}$, $|m| = m_1+m_2+m_3+m_4$. Another important requirement defines the values of nonlinear functions on the boundary. Suppose that for problem (27), (28), (30)

$$f_1(0,0,v,v^*) \equiv 0, \qquad f_2(0,0,v,v^*) \equiv 0, \tag{32}$$

while, in the case of problem (27), (29), (30), we assume that

$$f_1(\theta u, \theta^* u^*, \theta v, \theta^* v^*) = \theta f_1(u, u^*, v, v^*),$$
$$f_2(\theta u, \theta^* u^*, \theta v, \theta^* v^*) = \theta f_2(u, u^*, v, v^*); \tag{33}$$

here $|\theta| = 1$. Note that the functions f_1 and f_2, defined by the right-hand side of (23), satisfy (32) and (33). The conditions (31), (32), and (33) are satisfied for many physical models.

For problem (27), (30), with corresponding boundary conditions, we are going to apply the finite difference approximations. The solution of (27), (28), (30) can be extended outside the interval $\Omega = (0, 1)$ by defining

$$u(x,t) = u(-x,t), \quad v(x,t) = v(-x,t), \quad -1 \le x \le 0,$$
$$u(x,t) = u(2-x,t), \quad v(x,t) = v(2-x,t), \quad 1 \le x \le 2. \tag{34}$$

It can be easily checked that the first equation of problem (27), (28), (30) is satisfied for $x \in (-1, 2)$, except the points $x = 0$ and $x = 1$, while the solution v is a smooth function for all $x \in (-1, 2)$. Therefore we are allowed to use the approximations of the values $v(-h, t)$ and $v(1 + h, t)$.

In the case of problem (27), (28), we apply the following Crank–Nicolson finite difference scheme:

$$p_t = i \overset{\circ}{p}_{\bar{x}x} + f_1(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*), \qquad (x,t) \in Q_h,$$
$$q_t = i \overset{\circ}{q}_{\bar{x}x} + f_2(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*), \qquad (x,t) \in \bar{Q}_h, \tag{35}$$

$$p(0,t) = p(1,t) = 0,$$
$$q(-h,t) = q(h,t), \qquad q(1-h,t) = q(1+h,t), \qquad t \in \bar{\omega}_\tau. \tag{36}$$

Substituting the boundary conditions for q into the (35) and employing (32) we can eliminate $q(-h)$, $q(1 + h)$. Thus, (35), (36) is equivalent to the scheme

$$p_t = i \overset{\circ}{p}_{\bar{x}x} + f_1(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*),$$
$$q_t = i \overset{\circ}{q}_{\bar{x}x} + f_2(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*), \qquad (x,t) \in Q_h, \tag{37}$$

$$p(0,t) = p(1,t) = 0, \qquad t \in \bar{\omega}_\tau,$$
$$q_t(0,t) = i\frac{2}{h} \overset{\circ}{q}_x (0,t), \qquad q_t(1,t) = -i\frac{2}{h} \overset{\circ}{q}_{\bar{x}} (1,t), \qquad t \in \omega_\tau. \tag{38}$$

According to condition (33), the solution of periodic boundary value problem (27), (29), (30) can be extended outside the domain Ω by

$$u(1+x,t)=\theta^* u(x,t), \quad v(1+x,t)=\theta^* v(x,t), \quad 0\le x\le 1. \tag{39}$$

Therefore, for differential problem (27), (29), we can apply the scheme

$$\begin{aligned} &p_t = i \overset{\circ}{p}_{\bar{x}x} + f_1(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*), \\ &q_t = i \overset{\circ}{q}_{\bar{x}x} + f_2(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*), \qquad (x,t)\in Q_h^+, \end{aligned} \tag{40}$$

$$\begin{aligned} &p(0,t)=\theta p(1,t), \qquad p(h,t)=\theta p(1+h,t), \\ &q(0,t)=\theta q(1,t), \qquad q(h,t)=\theta q(1+h,t), \qquad t\in\bar{\omega}_\tau. \end{aligned} \tag{41}$$

In both cases, the initial functions (30) are approximated by

$$p(x,0)=u^{(0)}(x), \qquad q(x,0)=v^{(0)}(x), \qquad x\in\bar{\Omega}_h. \tag{42}$$

Scheme (37), (38), (42) is implicit and nonlinear. To compute the difference solutions $\hat{p},\hat{q}$ on the upper layer $t+\tau$ we apply the iterations

$$\begin{aligned} &\frac{p^{[s+1]}-p}{\tau} = i\bar{p}^{[s+1]}_{\bar{x}x} + f_1\left(\bar{p}^{[s]}, \bar{p}^{*[s]}, \bar{q}^{[s]}, \bar{q}^{*[s]}\right), \\ &\frac{q^{[s+1]}-q}{\tau} = i\bar{q}^{[s+1]}_{\bar{x}x} + f_2\left(\bar{p}^{[s]}, \bar{p}^{*[s]}, \bar{q}^{[s]}, \bar{q}^{*[s]}\right), \\ &p^{[s+1]}(0)=p^{[s+1]}(1)=0, \\ &\frac{q^{[s+1]}(0)-q(x=0)}{\tau} = \frac{i}{h}\left(q_x^{[s+1]}(0)+q_x(x=0)\right), \\ &\frac{q^{[s+1]}(1)-q(x=1)}{\tau} = -\frac{i}{h}\left(q_{\bar{x}}^{[s+1]}(1)+q_{\bar{x}}(x=1)\right), \\ &s=0,1,\ldots, \qquad x\in\Omega_h, \qquad p^{[0]}=p, \qquad q^{[0]}=q. \end{aligned} \tag{43}$$

We have denoted here

$$\bar{p}^{[s]}=\frac{p^{[s]}+p}{2}, \qquad \bar{q}^{[s]}=\frac{q^{[s]}+q}{2}.$$

For scheme (40), (41), (42), we construct the similar process

$$\begin{aligned}
&\frac{p^{[s+1]}-p}{\tau} = i\bar{p}^{[s+1]}_{\bar{x}x} + f_1\left(\bar{p}^{[s]}, \bar{p}^{*[s]}, \bar{q}^{[s]}, \bar{q}^{*[s]}\right), \\
&\frac{q^{[s+1]}-q}{\tau} = i\bar{q}^{[s+1]}_{\bar{x}x} + f_2\left(\bar{p}^{[s]}, \bar{p}^{*[s]}, \bar{q}^{[s]}, \bar{q}^{*[s]}\right), \\
&p^{[s+1]}(0) = \theta p^{[s+1]}(1), \qquad p^{[s+1]}(h) = \theta p^{[s+1]}(1+h), \\
&q^{[s+1]}(0) = \theta q^{[s+1]}(1), \qquad q^{[s+1]}(h) = \theta q^{[s+1]}(1+h), \\
&s = 0, 1, \ldots, \qquad x \in \Omega_h^+, \qquad p^{[0]} = p, \qquad q^{[0]} = q.
\end{aligned} \tag{44}$$

To find the next iteration defined by (43), we suggest an efficient sweep method. For (44), the modified cycle sweep can be applied.

6 A priori estimates

We consider the auxiliary linear difference scheme

$$w_t = i\,\mathring{w}_{\bar{x}x} + r(x,t) \tag{45}$$

with $r(x,t) \in H_h^1$.

In the common frame, we are going to examine three different boundary problems for scheme (45). The boundary values are set by

$$w(0,t) = w(1,t) = 0, \qquad r(0,t) = r(1,t) = 0, \qquad t \in \bar{\omega}_\tau \tag{46}$$

or

$$\begin{aligned}
&w_t(0,t) = i\frac{2}{h}\,\mathring{w}_x(0,t), \qquad w_t(1,t) = -i\frac{2}{h}\,\mathring{w}_{\bar{x}}(1,t), \qquad t \in \omega_\tau, \\
&r(0,t) = r(1,t) = 0, \qquad t \in \bar{\omega}_\tau,
\end{aligned} \tag{47}$$

or

$$\begin{aligned}
&w(0,t) = \theta w(1,t), \qquad w(h,t) = \theta w(1+h,t), \\
&r(0,t) = \theta r(1,t), \qquad t \in \bar{\omega}_\tau.
\end{aligned} \tag{48}$$

Problems (45), (46) and (45), (47) are stated for $(x,t) \in Q_h$, and (45), (48) for $(x,t) \in Q_h^+$.

We also consider another difference scheme

$$\frac{z}{\tau} = i\frac{z_{\bar{x}x}}{2} + \rho(x) \tag{49}$$

together with the boundary conditions

$$z(0) = z(1) = 0, \qquad \rho(0) = \rho(1) = 0 \tag{50}$$

or

$$\frac{z(0)}{\tau} = \frac{i}{h}z_x(0), \qquad \frac{z(1)}{\tau} = -\frac{i}{h}z_{\bar{x}}(1), \qquad \rho(0) = \rho(1) = 0, \tag{51}$$

or

$$z(0) = \theta z(1), \qquad z(h) = \theta z(1+h), \qquad \rho(0) = \theta\rho(1). \tag{52}$$

The space for problems (49), (50) and (49), (51) is $x \in \Omega_h$, while $x \in \Omega_h^+$ for (49), (52). Here $\rho \in H_h^1$.

We prove the estimates for the solution on the upper layer.

Lemma 6.1 *For the solution of problem* (45), (46) *or* (45), (47), *or* (45), (48) *the following estimates hold:*

$$|[\hat{w}]|_h \leq |[w]|_h + \tau|[r]|_h, \tag{53}$$

$$\|\hat{w}_{\bar{x}}]|_h \leq \|w_{\bar{x}}]|_h + \tau\|r_{\bar{x}}]|_h. \tag{54}$$

Proof: We first prove (53) for the case of boundary conditions (46) or (47). Taking the inner product $(\cdot,\cdot)_h$ on both sides of (45) with $2\tau \overset{\circ}{w}$ we obtain for the real part

$$\mathrm{Re}\,(\hat{w} - w, \hat{w} + w)_h = 2\tau\,\mathrm{Re}\,i\left(\overset{\circ}{w}_{\bar{x}x}, \overset{\circ}{w}\right)_h + 2\tau\,\mathrm{Re}\left(r, \overset{\circ}{w}\right)_h.$$

We have

$$\mathrm{Re}\,(\hat{w} - w, \hat{w} + w)_h = \mathrm{Re}\Big[(\hat{w},\hat{w})_h - (w,w)_h + i2\,\mathrm{Im}\,(\hat{w},w)_h\Big] =$$

$$\|\hat{w}\|_h^2 - \|w\|_h^2.$$

Due to (7), we have

$$i\left(\overset{\circ}{w}_{\bar{x}x}, \overset{\circ}{w}\right)_h = -i\|\overset{\circ}{w}_{\bar{x}}]|_h^2 + i\,\overset{\circ}{w}_{\bar{x}}(1,t)\,\overset{\circ}{w}^*(1,t) - i\,\overset{\circ}{w}_x(0,t)\,\overset{\circ}{w}^*(0,t)$$

and, hence,

$$2\tau\,\mathrm{Re}\,i\left(\overset{\circ}{w}_{\bar{x}x},\overset{\circ}{w}\right)_h=0$$

in the case of boundary conditions (46) while, for (47), we get

$$2\tau\,\mathrm{Re}\,i\left(\overset{\circ}{w}_{\bar{x}x},\overset{\circ}{w}\right)_h=2\tau\,\mathrm{Re}\left[-\frac{h}{2}w_t(1,t)\,\overset{\circ}{w}{}^*(1,t)-\right.$$

$$\left.\frac{h}{2}w_t(0,t)\,\overset{\circ}{w}{}^*(0,t)\right]=-\frac{h}{2}\,\left(|\hat{w}|^2-|w|^2\right)\Big|_{x=1}-\frac{h}{2}\,\left(|\hat{w}|^2-|w|^2\right)\Big|_{x=0}.$$

Therefore, by zero boundary conditions for r, for both schemes (45), (46) and (45), (47) we obtain

$$|[\hat{w}]|_h^2=|[w]|_h^2+2\tau\,\mathrm{Re}\left[r,\overset{\circ}{w}\right]_h. \tag{55}$$

In the case of problem (45), (48), we take the inner product $(\cdot,\cdot]_h$ on both sides of (45) with $2\tau\,\overset{\circ}{w}$ and the real part afterwards. Applying (8) and then (48), we get

$$\mathrm{Re}\,i\left(\overset{\circ}{w}_{\bar{x}x},\overset{\circ}{w}\right]_h=\mathrm{Re}\,i\left[-\|\,\overset{\circ}{w}_{\bar{x}}]|_h^2+\overset{\circ}{w}_x(1,t)\,\overset{\circ}{w}{}^*(1,t)-\right.$$

$$\left.\overset{\circ}{w}_x(0,t)\,\overset{\circ}{w}{}^*(0,t)\right]=0.$$

Due to (48), we obtain (55) in this case, too.

To prove (53) we estimate using the Cauchy–Schwartz inequality

$$\left[|r|,|\,\overset{\circ}{w}\,|\right]_h\le\frac{1}{2}\left[|r|,|\hat{w}|\right]_h+\frac{1}{2}\left[|r|,|w|\right]_h\le\frac{1}{2}|[r]|_h\Big(|[\hat{w}]|_h+|[w]|_h\Big).$$

Substituting this estimate into (55) we have

$$\Big(|[\hat{w}]|_h-|[w]|_h\Big)\Big(|[\hat{w}]|_h+|[w]|_h\Big)\le\tau|[r]|_h\Big(|[\hat{w}]|_h+|[w]|_h\Big),$$

and (53) follows.

It remains to prove (54). In the case of problems (45), (46) and (45), (47), we take the inner product $(\cdot,\cdot)_h$ on both sides of (45) with $-2\tau\,\overset{\circ}{w}_{\bar{x}x}$ to obtain

$$-2\tau\,\mathrm{Re}\left(w_t,\overset{\circ}{w}_{\bar{x}x}\right)_h=-2\tau\,\mathrm{Re}\left(r,\overset{\circ}{w}_{\bar{x}x}\right)_h.$$

By (7) we have

$$-\operatorname{Re}\Big(w_t, \overset{\circ}{w}_{\bar{x}x}\Big)_h = \operatorname{Re}\Big[\Big(w_{t\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h - \overset{\circ}{w}{}^*_{\bar{x}}(1,t)w_t(1,t) +$$

$$\overset{\circ}{w}{}^*_{x}(0,t)w_t(0,t)\Big]$$

and, for problem (45), (46), we get

$$-\operatorname{Re}\Big(w_t, \overset{\circ}{w}_{\bar{x}x}\Big)_h = \operatorname{Re}\Big(w_{t\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h,$$

since $w_t(0,t) = w_t(1,t) = 0$, while, for problem (45), (47),

$$-\operatorname{Re}\Big(w_t, \overset{\circ}{w}_{\bar{x}x}\Big)_h = \operatorname{Re}\Big[\Big(w_{t\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h + i\frac{2}{h}|\overset{\circ}{w}_{\bar{x}}(1,t)|^2 +$$

$$i\frac{2}{h}|\overset{\circ}{w}_{x}(0,t)|^2\Big] = \operatorname{Re}\Big(w_{t\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h.$$

For problems (45), (46) and (45), (47), by condition $r(0,t) = r(1,t) = 0$ we also have

$$-\Big(r, \overset{\circ}{w}_{\bar{x}x}\Big)_h = \Big(r_{\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h.$$

In the case of problem (45), (48), we take the inner product $(\cdot,\cdot]_h$ on both sides of (45) with $-2\tau\,\overset{\circ}{w}_{\bar{x}x}$ and apply (8) in a similar way. Thus, for all considered finite difference boundary problems, we obtain

$$2\tau\operatorname{Re}\Big(w_{t\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h = 2\tau\operatorname{Re}\Big(r_{\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h.$$

After a simple rearrangement we have

$$\|\hat{w}_{\bar{x}}]|_h^2 = \|w_{\bar{x}}]|_h^2 + 2\tau\operatorname{Re}\Big(r_{\bar{x}}, \overset{\circ}{w}_{\bar{x}}\Big]_h.$$

We complete the proof of (54) in a similar way as above estimating (55) by the Cauchy–Schwartz inequality. △

Corollary 6.2 *Suppose that the conditions of Lemma 6.1 are satisfied. Then*

$$\|\hat{w}\|_{H^1,h} \le \|w\|_{H^1,h} + \tau\|r\|_{H^1,h}. \tag{56}$$

Proof: The estimate (56) follows by (53), (54), and the Minkowski inequality. △

Corollary 6.3 *Let w to be a solution of the difference problem*

$$\begin{aligned} &w_t = i\,\overset{\circ}{w}_{\bar{x}x} + r(x,t), \qquad (x,t) \in \bar{Q}_h, \qquad r(0,t) = r(1,t) = 0, \\ &w(-h,t) = w(h,t), \qquad w(1-h,t) = w(1+h,t), \qquad t \in \bar{\omega}_\tau. \end{aligned} \tag{57}$$

Then Lemma 6.1 holds, i. e., (53), (54), and (56) *are satisfied.*

Proof: The problem (57) is equivalent to (45), (47). We prove it by eliminating $w(-h,t)$ and $w(1-h,t)$ in the difference equations. Moreover, the same result can be proved in the way similar to the proof of Lemma 6.1, but using (9). △

Lemma 6.4 *For the solution of problem* (49), (50) *or* (49), (51), *or* (49), (52), *the following estimates hold:*

$$|[z]|_h \leq \tau |[\rho]|_h, \tag{58}$$

$$\|z_{\bar{x}}]|_h \leq \tau \|\rho_{\bar{x}}]|_h. \tag{59}$$

Proof: The proof is almost similar to that of Lemma 6.1. We state the main points only. To prove (58) we take the inner product on both sides of (49) with τz. In the case of boundary conditions (51), a nonzero term appears:

$$\frac{\tau}{2}\operatorname{Re} i\,(z_{\bar{x}x}, z)_h = \frac{\tau}{2}\operatorname{Re}\Big[iz_{\bar{x}}(1)z^*(1) - iz_x(0)z^*(0)\Big] =$$

$$-\frac{h}{2}\Big(|z(0)|^2 + |z(1)|^2\Big).$$

Therefore, in all cases, we obtain

$$|[z]|_h^2 = \tau \operatorname{Re}[\rho, z]_h \leq \tau |[\rho]|_h |[z]|_h,$$

and (58) is proved.

To prove (59) we take the inner product on both sides of (49) with $-\tau z_{\bar{x}x}$. For problem (49), (51), for example, we get

$$-\operatorname{Re}\,(z, z_{\bar{x}x})_h = \|z_{\bar{x}}]|_h^2 + \operatorname{Re}\Big(-z(1)z_{\bar{x}}^*(1) + z(0)z_{\bar{x}}^*(0)\Big) =$$

$$\|z_{\bar{x}}]|_h^2 + \operatorname{Re} i\frac{\tau}{h}\Big(|z_{\bar{x}}(1)|^2 + |z_{\bar{x}}(0)|^2\Big) = \|z_{\bar{x}}]|_h^2.$$

Now we easily obtain the inequality

$$\|z_{\bar{x}}]|_h^2 = \tau \operatorname{Re}(\rho_{\bar{x}}, z_{\bar{x}}]_h \leq \tau \|\rho_{\bar{x}}]|_h \|z_{\bar{x}}]|_h,$$

which is valid for all considered problems. △

Corollary 6.5 *Suppose that the conditions of Lemma 6.4 are satisfied. Then*

$$\|z\|_{H^1,h} \le \tau \|\rho\|_{H^1,h}. \tag{60}$$

Proof: We obtain (60) as a direct conclusion of (58) and (59). △

We next establish some auxiliary estimates necessary to handle the nonlinear terms on the right hand side of the difference equations.

Lemma 6.6 *Suppose that* (31) *is satisfied. Then*

$$\|f_j(\xi, \xi^*, \eta, \eta^*)\|_{H^1,h} \le 2\varphi(c_M)\Big(1 + \|\xi\|_{H^1,h} + \|\eta\|_{H^1,h}\Big), \tag{61}$$

$$|[f_j(\xi, \xi^*, \eta, \eta^*) - f_j(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)]|_h \le 2\varphi(\tilde{c}_M) \Big(|[\xi - \tilde{\xi}]|_h + |[\eta - \tilde{\eta}]|_h\Big), \tag{62}$$

$$\|f_j(\xi, \xi^*, \eta, \eta^*) - f_j(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)\|_{H^1,h} \le 14\varphi(\tilde{c}_M)\,(1 + 4\tilde{c}_H) \Big(\|\xi - \tilde{\xi}\|_{H^1,h} + \|\eta - \tilde{\eta}\|_{H^1,h}\Big), \tag{63}$$

here $j = 1, 2$*, and*

$$c_M = \max\left\{\|\xi\|_{C,h}, \|\eta\|_{C,h}\right\},$$

$$\tilde{c}_M = \max\left\{\|\xi\|_{C,h}, \|\tilde{\xi}\|_{C,h}, \|\eta\|_{C,h}, \|\tilde{\eta}\|_{C,h}\right\},$$

$$\tilde{c}_H = \max\left\{\|\xi\|_{H^1,h}, \|\tilde{\xi}\|_{H^1,h}, \|\eta\|_{H^1,h}, \|\tilde{\eta}\|_{H^1,h}\right\},$$

$\xi, \tilde{\xi}, \eta, \tilde{\eta}$ *are grid* $\bar{\Omega}_h$ *functions.*

Proof: For simplicity, we omit the index j in the proof of the lemma. We start by proving (61). By (31) we have

$$|[f(\xi, \xi^*, \eta, \eta^*)]|_h \le \|f(\xi, \xi^*, \eta, \eta^*)\|_{C,h} \le \varphi(c_M).$$

The Lagrange mean-value theorem and (31) with $|m| = 1$ implies that

$$|f(\xi, \xi^*, \eta, \eta^*) - f(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)| \le 2\varphi\left(\max\{|\xi|, |\tilde{\xi}|, |\eta|, |\tilde{\eta}|\}\right)\left(|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|\right). \tag{64}$$

Applying (64) with $\tilde{\xi} = \xi(x-h)$, $\tilde{\eta} = \eta(x-h)$ and the Minkowski inequality we get

$$\|f_{\bar{x}}(\xi,\xi^*,\eta,\eta^*)]|_h \le 2\varphi(c_M)\sqrt{h\sum_{x\in\Omega_h^+}\Big(|\xi_{\bar{x}}|+|\eta_{\bar{x}}|\Big)^2} \le$$

$$2\varphi(c_M)\left(\|\xi_{\bar{x}}]|_h+\|\eta_{\bar{x}}]|_h\right) \le 2\varphi(c_M)\Big(\|\xi\|_{H^1,h}+\|\eta\|_{H^1,h}\Big).$$

Hence, by the estimates above we have

$$\|f(\xi,\xi^*,\eta,\eta^*)\|_{H^1,h} \le 2\varphi(c_M)\sqrt{1+\Big(\|\xi\|_{H^1,h}+\|\eta\|_{H^1,h}\Big)^2},$$

and (61) follows.

To prove (62) we employ (64) and the Minkowski inequality again:

$$|[f(\xi,\xi^*,\eta,\eta^*)-f(\tilde{\xi},\tilde{\xi}^*,\tilde{\eta},\tilde{\eta}^*)]|_h \le 2\varphi(\tilde{c}_M)\Big(|[\xi-\tilde{\xi}]|_h+|[\eta-\tilde{\eta}]|_h\Big).$$

Estimate (62) is proved.

It remains to prove (63). Denote $\Xi = (\bar{\xi},\bar{\xi}^*,\bar{\eta},\bar{\eta}^*)$, $\bar{y} = \kappa y+(1-\kappa)\tilde{y}$, here $\kappa \in (0,1)$ is some constant, defined by the Lagrange mean value theorem. We have

$$\left|\Big(f(\xi,\xi^*,\eta,\eta^*)-f(\tilde{\xi},\tilde{\xi}^*,\tilde{\eta},\tilde{\eta}^*)\Big)_{\bar{x}}\right| = \left|\Big(\frac{\partial f(\Xi)}{\partial\xi}(\xi-\tilde{\xi})+\right.$$

$$\left.\frac{\partial f(\Xi)}{\partial\xi^*}(\xi^*-\tilde{\xi}^*)+\frac{\partial f(\Xi)}{\partial\eta}(\eta-\tilde{\eta})+\frac{\partial f(\Xi)}{\partial\eta^*}(\eta^*-\tilde{\eta}^*)\Big)_{\bar{x}}\right|.$$

Applying the simple finite differentiation rule

$$\Big(F(x)G(x)\Big)_{\bar{x}} = F_{\bar{x}}(x)G(x)+F(x-h)G_{\bar{x}}(x)$$

and (31), we get

$$\left|\Big(f(\xi,\xi^*,\eta,\eta^*)-f(\tilde{\xi},\tilde{\xi}^*,\tilde{\eta},\tilde{\eta}^*)\Big)_{\bar{x}}\right| \le$$

$$2\max\left\{\left|\frac{\partial f(\Xi)}{\partial\xi}_{\bar{x}}\right|,\left|\frac{\partial f(\Xi)}{\partial\xi^*}_{\bar{x}}\right|,\left|\frac{\partial f(\Xi)}{\partial\eta}_{\bar{x}}\right|,\left|\frac{\partial f(\Xi)}{\partial\eta^*}_{\bar{x}}\right|\right\}$$

$$\Big(|\xi-\tilde{\xi}|+|\eta-\tilde{\eta}|\Big)+2\varphi(\tilde{c}_M)\Big(|(\xi-\tilde{\xi})_{\bar{x}}|+|(\eta-\tilde{\eta})_{\bar{x}}|\Big).$$

We use the Lagrange mean-value theorem and (31) (with $|m| = 2$) again to estimate

$$\max\left\{\left|\frac{\partial f(\Xi)}{\partial \xi}_{\bar{x}}\right|, \left|\frac{\partial f(\Xi)}{\partial \xi^*}_{\bar{x}}\right|, \left|\frac{\partial f(\Xi)}{\partial \eta}_{\bar{x}}\right|, \left|\frac{\partial f(\Xi)}{\partial \eta^*}_{\bar{x}}\right|\right\} \leq$$
$$2\varphi(\tilde{c}_M)\left(|\bar{\xi}_{\bar{x}}| + |\bar{\eta}_{\bar{x}}|\right).$$

Hence, we have

$$\left|\Big(f(\xi, \xi^*, \eta, \eta^*) - f(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)\Big)_{\bar{x}}\right| \leq$$
$$2\varphi(\tilde{c}_M)\left[2\left(|\bar{\xi}_{\bar{x}}| + |\bar{\eta}_{\bar{x}}|\right)\Big(|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|\Big) + |(\xi - \tilde{\xi})_{\bar{x}}| + |(\eta - \tilde{\eta})_{\bar{x}}|\right].$$

Since L_2 norm is the object of our consideration, we proceed with estimating the quantity above squared:

$$\left|\Big(f(\xi, \xi^*, \eta, \eta^*) - f(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)\Big)_{\bar{x}}\right|^2 \leq 4\varphi^2(\tilde{c}_M)$$
$$\Big[16\Big(|\bar{\xi}_{\bar{x}}|^2 + |\bar{\eta}_{\bar{x}}|^2\Big)\Big(|\xi - \tilde{\xi}|^2 + |\eta - \tilde{\eta}|^2\Big) + 2|(\xi - \tilde{\xi})_{\bar{x}}|^2 +$$
$$2|(\eta - \tilde{\eta})_{\bar{x}}|^2 + 4\left(|\bar{\xi}_{\bar{x}}| + |\bar{\eta}_{\bar{x}}|\right)\Big(|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|\Big)$$
$$\Big(|(\xi - \tilde{\xi})_{\bar{x}}| + |(\eta - \tilde{\eta})_{\bar{x}}|\Big)\Big].$$

We use the fact that

$$\left(|\bar{\xi}_{\bar{x}}| + |\bar{\eta}_{\bar{x}}|\right)\Big(|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|\Big)\Big(|(\xi - \tilde{\xi})_{\bar{x}}| + |(\eta - \tilde{\eta})_{\bar{x}}|\Big) \leq$$
$$2\Big(|\bar{\xi}_{\bar{x}}|^2 + |\bar{\eta}_{\bar{x}}|^2\Big)\Big(|\xi - \tilde{\xi}|^2 + |\eta - \tilde{\eta}|^2\Big) + |(\xi - \tilde{\xi})_{\bar{x}}|^2 + |(\eta - \tilde{\eta})_{\bar{x}}|^2$$

to get

$$\left|\Big(f(\xi, \xi^*, \eta, \eta^*) - f(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)\Big)_{\bar{x}}\right|^2 \leq 24\varphi^2(\tilde{c}_M)$$
$$\Big[4\Big(|\bar{\xi}_{\bar{x}}|^2 + |\bar{\eta}_{\bar{x}}|^2\Big)\Big(|\xi - \tilde{\xi}|^2 + |\eta - \tilde{\eta}|^2\Big) + |(\xi - \tilde{\xi})_{\bar{x}}|^2 + |(\eta - \tilde{\eta})_{\bar{x}}|^2\Big].$$

The summation and (11) gives

$$\|(f(\xi, \xi^*, \eta, \eta^*) - f(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*))_{\bar{x}}]|_h^2 \leq 24\varphi^2(\tilde{c}_M)\Big[\|(\xi - \tilde{\xi})_{\bar{x}}]|_h^2 +$$

$$\|(\eta-\tilde{\eta})_{\bar{x}}]|_h^2 + 8\left(\|\xi-\tilde{\xi}\|_{H^1,h}^2 + \|\eta-\tilde{\eta}\|_{H^1,h}^2\right)\left(\|\bar{\xi}\|_{H^1,h}^2 + \|\bar{\eta}\|_{H^1,h}^2\right)\Big].$$

We now recall (62) to obtain

$$\|f(\xi,\xi^*,\eta,\eta^*) - f(\tilde{\xi},\tilde{\xi}^*,\tilde{\eta},\tilde{\eta}^*)\|_{H^1,h}^2 \le 24\varphi^2(\tilde{c}_M)$$

$$\left(\|\xi-\tilde{\xi}\|_{H^1,h}^2 + \|\eta-\tilde{\eta}\|_{H^1,h}^2\right)\left(1+8\left(\|\bar{\xi}\|_{H^1,h}^2 + \|\bar{\eta}\|_{H^1,h}^2\right)\right).$$

This implies

$$\|f(\xi,\xi^*,\eta,\eta^*) - f(\tilde{\xi},\tilde{\xi}^*,\tilde{\eta},\tilde{\eta}^*)\|_{H^1,h} \le \sqrt{192}\varphi(\tilde{c}_M)$$

$$\left(\|\xi-\tilde{\xi}\|_{H^1,h} + \|\eta-\tilde{\eta}\|_{H^1,h}\right)\left(1+\|\bar{\xi}\|_{H^1,h} + \|\bar{\eta}\|_{H^1,h}\right).$$

Finally, by the condition $0<\kappa<1$, (63) follows. $\triangle$

Remark 6.7 *Assume that the functions f_j, $j=1,2$, are of lower smoothness than required by* (31)*, i. e.,* (31) *is satisfied with $|m|=1$ only. Then* (61) *and* (62) *hold anyway, since the case $|m|=2$ of* (31) *was not employed to prove the latter estimates.*

We are going to apply Lemma 6.6 further examining the convergence and stability of the proposed difference methods. The main difficulty faced there is that the estimated constants include H_h^1 norm of a numerical solution. Therefore, we need to establish the following *a priori* estimates.

Lemma 6.8 *Suppose that* (31) *is satisfied with $|m|=1$. Also suppose that the solution of difference problem* (37), (38), (42), (32) *or* (40), (41), (42), (33) *is bounded in C_h norm:*

$$\|p\|_{C,h} \le \gamma < \infty, \qquad \|q\|_{C,h} \le \gamma < \infty, \qquad t\in\bar{\omega}_\tau. \tag{65}$$

Then there exists a constant $\tau_0=\tau_0(\varphi(\gamma))>0$ such that, if $\tau\le\tau_0$, then the following estimates are valid:

$$\|p\|_{H^1,h} \le c_W, \qquad \|q\|_{H^1,h} \le c_W, \qquad t\in\bar{\omega}_\tau. \tag{66}$$

Here $c_W=c_W(T,\varphi(\gamma),\|u^{(0)}\|_{H^1,h},\|v^{(0)}\|_{H^1,h})$ is a constant.

Proof: Note that, respectively, boundary conditions (32) or (33), are satisfied for the functions f_j. Therefore, we can apply Corollary 6.2 to scheme (37), (38) or (40), (41), respectively. We get

$$\|\hat{p}\|_{H^1,h} \leq \|p\|_{H^1,h} + \tau \|f_1(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*)\|_{H^1,h},$$

$$\|\hat{q}\|_{H^1,h} \leq \|q\|_{H^1,h} + \tau \|f_2(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*)\|_{H^1,h}.$$

Due to (31) and (65), we can employ estimate (61) of Lemma 6.6 to obtain

$$\|f_j(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*)\|_{H^1,h} \leq 2\varphi(\gamma)$$

$$\left(1 + \frac{\|\hat{p}\|_{H^1,h} + \|p\|_{H^1,h} + \|\hat{q}\|_{H^1,h} + \|q\|_{H^1,h}}{2}\right),$$

$j = 1, 2$. Denote $Z(t) = \|p(t)\|_{H^1,h} + \|q(t)\|_{H^1,h}$. We have

$$Z(t+\tau) \leq Z(t) + \tau\, 4\varphi(\gamma) \left(1 + \frac{Z(t+\tau) + Z(t)}{2}\right), \qquad t \in \omega_\tau,$$

or

$$Z(t+\tau) \leq \frac{1 + 2\varphi(\gamma)\,\tau}{1 - 2\varphi(\gamma)\,\tau} Z(t) + 8\varphi(\gamma)\,\tau, \quad t \in \omega_\tau, \quad \tau \leq \tau_0.$$

with $\tau_0 = 1/(4\varphi(\gamma))$. Estimating by (12) we write

$$Z(t+\tau) \leq (1 + 8\varphi(\gamma)\,\tau) Z(t) + 8\varphi(\gamma)\,\tau, \quad t \in \omega_\tau, \quad \tau \leq \tau_0.$$

This implies the boundedness of $Z(t)$, $t \in \bar{\omega}_\tau$. △

7 Justification of the iterated approximations

In this section, we prove the convergence of the iterations (43) or (44), respectively, as well as the boundedness of the difference solution on the upper layer.

Lemma 7.1 *Suppose that* (31) *is satisfied and there exists a solution of difference problem* (37), (38), (32) *or* (40), (41), (33) *on the layer* $t = t_k$, *such that*

$$\|p(t)\|_{H^1,h} \leq \sigma, \qquad \|q(t)\|_{H^1,h} \leq \sigma. \tag{67}$$

Then there exists a constant $\tau_0 = \tau_0(\sigma, \varphi(\sigma)) > 0$ *such that, if* $\tau < \tau_0$*, then the iterations* (43) *or* (44)*, respectively, produce the unique sequences* $\{p^{[s]}\}$ *and* $\{q^{[s]}\}$*, convergent in* H_h^1*. The limit functions appear to be a unique solution of* (37), (38) *or* (40), (41)*, respectively, on the layer* $t+\tau$*, and the following estimate holds:*

$$\|\hat{p}\|_{H^1,h} \le 2\sigma, \qquad \|\hat{q}\|_{H^1,h} \le 2\sigma. \tag{68}$$

Proof: Both (43) and (44) are linear algebraic equation systems. To prove the uniqueness of $p^{[s+1]}$ and $q^{[s+1]}$ we consider the correspondent homogeneous problem, *i. e.*, we set $p \equiv q \equiv f_1 \equiv f_2 \equiv 0$. Due to Lemma 6.4,

$$\|p^{[s+1]}\|_{H^1,h} \le 0, \qquad \|q^{[s+1]}\|_{H^1,h} \le 0,$$

i. e., a trivial solution appears to be unique for the homogeneous problem. This proves the correctness of the definition of the iterations.

Now we are going to prove the boundedness of iterated approximations, *i. e.*, that

$$\|p^{[s]}\|_{H^1,h} \le 2\sigma, \qquad \|q^{[s]}\|_{H^1,h} \le 2\sigma, \qquad s = 0, 1, \dots . \tag{69}$$

We apply the mathematical induction. For $s = 0$, (69) is true by (67), since $p^{[0]} = p$, $q^{[0]} = q$. Suppose that (69) holds for $s = l$. Denote $\xi = (p^{[l]} + p)/2$ and $\eta = (q^{[l]} + q)/2$. Then, by the induction assumption and imbedding inequality (11), we get

$$\|\xi\|_{H^1,h} \le 3\sigma/2, \quad \|\eta\|_{H^1,h} \le 3\sigma/2, \quad \|\xi\|_{C,h} \le 3\sigma, \quad \|\eta\|_{C,h} \le 3\sigma.$$

Applying estimate (61) of Lemma 6.6 we see that

$$\|f_j(\xi, \xi^*, \eta, \eta^*)\|_{H^1,h} \le 2\varphi(3\sigma)(1 + 3\sigma) = c_\sigma, \qquad j = 1, 2.$$

Therefore, the solution of (43) or (44), respectively, is bounded by Lemma 6.1:

$$|[p^{[l+1]}]|_h \le \sigma + \tau c_\sigma, \qquad \|p_{\bar{x}}^{[l+1]}]|_h \le \sigma + \tau c_\sigma,$$

or

$$\|p^{[l+1]}\|_{H^1,h} \le \sqrt{2}(\sigma + \tau c_\sigma).$$

In the same way, we obtain

$$\|q^{[l+1]}\|_{H^1,h} \le \sqrt{2}(\sigma + \tau c_\sigma).$$

Hence, (69) is satisfied in the case $s = l+1$ provided that $\tau \le \tau_0' = (\sqrt{2}-1)\sigma/c_\sigma$. The induction step and, therefore, (69) is proved.

Define the operator Λ such that

$$\begin{pmatrix} p^{[s+1]} \\ q^{[s+1]} \end{pmatrix} = \Lambda \begin{pmatrix} p^{[s]} \\ q^{[s]} \end{pmatrix},$$

according to the linear algebraic equation system (43) or (44), respectively. Now we can show that, if τ is small enough, then the operator Λ is a contraction in the Hilbert space H_h^1. This will immediately imply the convergence of the sequences $\{p^{[s]}\}$ and $\{q^{[s]}\}$ in H_h^1 and the uniqueness of the corresponding limits as well as their belonging to H_h^1. Taking $s-1$, instead of s, in (43), one can subtract the obtained equations from (43) in order to get a problem for the differences $p^{[s+1]} - p^{[s]}$ and $q^{[s+1]} - q^{[s]}$. We deal the same with (44). As a result, for the differences above, we have the problems of type (49). Due to estimate (63) of Lemma 6.6, we have

$$\|f_j(\xi, \xi^*, \eta, \eta^*) - f_j(\tilde{\xi}, \tilde{\xi}^*, \tilde{\eta}, \tilde{\eta}^*)\|_{H^1,h} \le$$

$$14\varphi(3\sigma)(1+6\sigma)\left(\|\xi - \tilde{\xi}\|_{H^1,h} + \|\eta - \tilde{\eta}\|_{H^1,h}\right) \le$$

$$7\varphi(3\sigma)(1+6\sigma)\left(\|p^{[s]} - p^{[s-1]}\|_{H^1,h} + \|q^{[s]} - q^{[s-1]}\|_{H^1,h}\right);$$

here $\xi = (p^{[s]}+p)/2$, $\eta = (q^{[s]}+q)/2$, $\tilde{\xi} = (p^{[s-1]}+p)/2$, and $\tilde{\eta} = (q^{[s-1]}+q)/2$, $j = 1, 2$. Therefore, by Corollary 6.5 we get

$$\|p^{[s+1]} - p^{[s]}\|_{H^1,h} + \|q^{[s+1]} - q^{[s]}\|_{H^1,h} \le$$

$$\tau\, 14\varphi(3\sigma)(1+6\sigma)\left(\|p^{[s]} - p^{[s-1]}\|_{H^1,h} + \|q^{[s]} - q^{[s-1]}\|_{H^1,h}\right),$$

Hence, if $\tau < \tau_0$, $\tau_0 = \min\{\tau_0', 1/(14\varphi(3\sigma)(1+6\sigma))\}$, we prove the contractibility of Λ. By taking limit $s \to \infty$ in (69) we get (68). $\triangle$

Remark 7.2 *Suppose that the functions f_j, $j = 1, 2$, are of lower smoothness than required by* (31)*, i. e.,* (31) *is satisfied with $|m| = 1$ only. Then Lemma 7.1 holds with the convergence of the iterated approximations in C_h. We conclude this by taking into account Remark 6.7. The use of* (63) *can be avoided in the proof by estimating*

$$|[p^{[s+1]} - p^{[s]}]|_h + |[q^{[s+1]} - q^{[s]}]|_h \leq$$

$$\tau\, c \left(|[p^{[s]} - p^{[s-1]}]|_h + |[q^{[s]} - q^{[s-1]}]|_h\right)$$

and further applying the Gagliardo–Nirenberg multiplicative estimate (10)*. The boundedness of the solution on the upper layer in H_h^1 follows by Lemma 6.8.*

Remark 7.3 *Due to the imbedding theorem $H_h^1 \to C_h$* (11) *and Lemma 7.1, the convergence of* (43) *or* (44)*, respectively, in C_h follows.*

8 Convergence and stability

Now we are able to prove the convergence and stability of the difference schemes. Taking the grid projections of the differential solutions u and v we estimate the approximation error of applied schemes.

Proposition 8.1 *Assume that there exist the unique solutions $u, v \in C^{4,3}$ of system* (27), (28), (30) *or* (27), (29), (30)*, respectively. Suppose that nonlinear functions f_j, $j = 1, 2$, satisfy* (31) *with $|m| = 1$ and* (32) *or* (33)*, respectively. Let* (35), (36), (42) *to be the finite difference scheme related to problem* (27), (28), (30)*, while* (40), (41), (42) *deals with* (27), (29), (30)*. Then the approximation error Ψ can be estimated by*

$$\|\Psi\|_{C,h} = O(\tau^2 + h^2), \qquad \tau, h \to 0, \qquad t \in \bar{\omega}_\tau, \tag{70}$$

for both schemes.

Proof: We begin with estimating the approximation error of difference equations (35) and (40). Substituting the differential solution into them, using its Taylor expansion in the neighbourhood of the point $(x, t + \tau/2)$, and employing the

Taylor expansion of functions f_j, $j = 1, 2$ together with (31) in the case $|m| = 1$, we get (70).

The boundary conditions are approximated exactly in all cases. This fact is trivial for Dirichlet boundary conditions. Due to the extension (34), we prove an exact approximation of von Neumann boundary conditions for the function v. In the case of periodic problem, we see that, due to (33), the differential solutions u and v satisfy (41), *i. e.,* the approximation on the boundary is exact.

The approximation (42) is just the exact projection of initial functions (30) to the grid. $\triangle$

Proposition 8.2 *Assume that for problem* (27), (29), (30) *problem the conditions of Proposition 8.1 are satisfied with differential solutions* $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x} \in C^{4,3}$ *and that* (31) *holds. Then the approximation error* Ψ *of scheme* (40), (41), (42) *can be estimated by*

$$\|\Psi\|_{H^1,h} = O(\tau^2 + h^2), \qquad \tau, h \to 0, \qquad t \in \bar{\omega}_\tau. \tag{71}$$

Proof: We operate in the way similar to the proof of Proposition 8.1. One needs to write the difference equations at the points $x = 0$ and $x = 1$ and to substitute the differential solution into them as well. The values of the differential solutions outside the domain Ω are defined by extension (39). Note also, that, in the case of periodic problem (27), (29), (30), the extended solutions $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x} \in C^{4,3}$, $-1 \leq x \leq 2$.

Taking the first order difference derivative with respect to x of the expression for approximation error we estimate $|\Psi_{\bar{x}}|$, $x \in \Omega_h^+$. We expand u and v in the neighbourhood of the point $(x - h/2, t + \tau/2)$. $\triangle$

Let u and v be the discretizated solutions of the corresponding differential problem. Denote the errors $\varepsilon = u - p$, $\delta = v - q$. Here p and q are the difference solutions. In the case of differential problem (27), (28) and of the approximating it finite difference scheme (35), (36), we have

$$\begin{aligned} &\varepsilon_t = i \overset{\circ}{\varepsilon}_{\bar{x}x} + f_1(\overset{\circ}{u}, \overset{\circ}{u}{}^*, \overset{\circ}{v}, \overset{\circ}{v}{}^*) - f_1(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*) + \Psi_1, \quad (x,t) \in Q_h, \\ &\delta_t = i \overset{\circ}{\delta}_{\bar{x}x} + f_2(\overset{\circ}{u}, \overset{\circ}{u}{}^*, \overset{\circ}{v}, \overset{\circ}{v}{}^*) - f_2(\overset{\circ}{p}, \overset{\circ}{p}{}^*, \overset{\circ}{q}, \overset{\circ}{q}{}^*) + \Psi_2, \quad (x,t) \in \bar{Q}_h, \end{aligned} \tag{72}$$

$$\begin{aligned} &\varepsilon(0,t) = \varepsilon(1,t) = 0, \\ &\delta(-h,t) = \delta(h,t), \qquad \delta(1-h,t) = \delta(1+h,t), \qquad t \in \bar{\omega}_\tau. \end{aligned} \tag{73}$$

Moreover, for problem (27), (29) and its approximation (40), (41) scheme, we get

$$\begin{aligned} \varepsilon_t &= i \overset{\circ}{\varepsilon}_{\bar{x}x} + f_1(\overset{\circ}{u}, \overset{\circ}{u}^*, \overset{\circ}{v}, \overset{\circ}{v}^*) - f_1(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*) + \Psi_1, \\ \delta_t &= i \overset{\circ}{\delta}_{\bar{x}x} + f_2(\overset{\circ}{u}, \overset{\circ}{u}^*, \overset{\circ}{v}, \overset{\circ}{v}^*) - f_2(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*) + \Psi_2, \quad (x,t) \in Q_h^+, \end{aligned} \tag{74}$$

$$\begin{aligned} &\varepsilon(0,t) = \theta\varepsilon(1,t), \qquad \varepsilon(h,t) = \theta\varepsilon(1+h,t), \\ &\delta(0,t) = \theta\delta(1,t), \qquad \delta(h,t) = \theta\delta(1+h,t), \qquad t \in \bar{\omega}_\tau. \end{aligned} \tag{75}$$

In both cases, Ψ_1 and Ψ_2 are the approximation errors. Due to Proposition 8.1, $\Psi_1, \Psi_2 \to 0$, $\tau, h \to 0$.

On the first layer $t = 0$, the initial functions $u^{(0)}(x)$ and $v^{(0)}(x)$ are approximated exactly by (42) for both differential problems. Therefore,

$$\varepsilon(x,0) = 0, \qquad \delta(x,0) = 0, \qquad x \in \bar{\Omega}_h. \tag{76}$$

Theorem 8.3 *Assume that the conditions of Proposition 8.1 are satisfied. Then there exist constants $\tau_0, h_0 > 0$ such that, if $\tau \le \tau_0$, $h \le h_0$, then there exists the unique solution p, q of finite difference scheme which converges to the solution u, v of the corresponding differential problem. In both cases* (72), (73), (76) *and* (74), (75), (76)*, the error is estimated by*

$$\|\varepsilon\|_{C,h} + \|\delta\|_{C,h} = O(\tau + h), \qquad \tau, h \to 0, \qquad t \in \bar{\omega}_\tau. \tag{77}$$

Proof: Suppose that

$$\|p\|_{C,h} + \|q\|_{C,h} \le 2\left(\|u\|_{C,h} + \|v\|_{C,h}\right) = c_D, \qquad t \in \bar{\omega}_\tau. \tag{78}$$

We will prove estimate (78) later.

Applying (53) and Corollary 6.2 we get

$$|[\hat{\varepsilon}]|_h \le |[\varepsilon]|_h + \tau |[f_1(\overset{\circ}{u}, \overset{\circ}{u}^*, \overset{\circ}{v}, \overset{\circ}{v}^*) - f_1(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*)]|_h + \tau |[\Psi_1]|_h,$$

$$|[\hat{\delta}]|_h \le |[\delta]|_h + \tau |[f_2(\overset{\circ}{u}, \overset{\circ}{u}^*, \overset{\circ}{v}, \overset{\circ}{v}^*) - f_2(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*)]|_h + \tau |[\Psi_2]|_h.$$

To deal with nonlinearities we use (62) in Lemma 6.6:

$$|[f_j(\overset{\circ}{u}, \overset{\circ}{u}^*, \overset{\circ}{v}, \overset{\circ}{v}^*) - f_j(\overset{\circ}{p}, \overset{\circ}{p}^*, \overset{\circ}{q}, \overset{\circ}{q}^*)]|_h \le$$

$$\varphi(c_D)\left(|[\hat{\varepsilon}]|_h + |[\varepsilon]|_h + |[\hat{\delta}]|_h + |[\delta]|_h\right),$$

$j = 1, 2$. Denote $Z(t) = |[\varepsilon]|_h + |[\delta]|_h$ and $\Phi = \max_{t\in\bar{\omega}_\tau} \max\{\|\Psi_1\|_{C,h}, \|\Psi_2\|_{C,h}\}$. Then

$$Z(t+\tau) \le Z(t) + \tau\, 2\varphi(c_D)[Z(t+\tau) + Z(t)] + \tau\, 2\Phi, \qquad t \in \omega_\tau,$$

or

$$Z(t+\tau) \le \frac{1 + 2\varphi(c_D)\,\tau}{1 - 2\varphi(c_D)\,\tau} Z(t) + \tau\, 4\Phi, \quad t \in \omega_\tau, \quad \tau \le \tau_0'.$$

with $\tau_0' = 1/(4\varphi(c_D))$. By (12) we have

$$Z(t+\tau) \le (1 + 8\varphi(c_D)\,\tau) Z(t) + \tau\, 4\Phi, \quad t \in \omega_\tau, \quad \tau \le \tau_0'. \tag{79}$$

Since $Z(0) = 0$, this implies

$$Z(t) \le c_Z\, \Phi, \qquad t \in \bar{\omega}_\tau, \qquad \tau \le \tau_0'; \tag{80}$$

here $c_Z = c_Z(T, \varphi(c_D))$ is a constant.

While (78) holds, it follows by Lemma 6.8 that, if $\tau \le \tau_0'' = \tau_0''(\varphi(c_D))$, then the difference solution is bounded in H_h^1 norm by some constant

$$\tilde{c}_W = \tilde{c}_W(T, \varphi(c_D), \|u^{(0)}\|_{H^1,h}, \|v^{(0)}\|_{H^1,h}).$$

Since the differential solution is smooth, we can assume that it is bounded in H_h^1 by the same constant.

Now, due to Gagliardo–Nirenberg multiplicative estimate (10) and Proposition 8.1, we obtain for $\tau \le \tau_0', \tau_0''$

$$\|\varepsilon\|_{C,h} + \|\delta\|_{C,h} \le 4 c_G {c_Z}^{1/2} {\tilde{c}_W}^{1/2} {c_\Phi}^{1/2} (\tau^2 + h^2)^{1/2}, \quad t \in \bar{\omega}_\tau,$$

here c_Φ is the constant defined in Proposition 8.1.

To complete the proof it remains to show estimate (78). We use the mathematical induction method. Estimate (78) is satisfied for $t = 0$, since $p(0,t) = u^{(0)}(x)$, $q(0,t) = v^{(0)}(x)$. Suppose that it holds for layers $t_s = s\tau$, $s = 0, 1, \ldots, j$. Then by Lemma 6.8 it follows that

$$\|p(t_j)\|_{H^1,h} \le c_W, \qquad \|q(t_j)\|_{H^1,h} \le c_W, \qquad \tau \le \tau_0'' = \tau_0''(\varphi(c_D)).$$

Here $c_W = c_W(T, \varphi(c_D), \|u^{(0)}\|_{H^1,h}, \|v^{(0)}\|_{H^1,h})$. Applying Lemma 7.1 and Remark 7.2 we see that there exists a unique difference solution on the upper layer t_{j+1} and that

$$\|p(t_{j+1})\|_{H^1,h} \le 2c_W, \quad \|q(t_{j+1})\|_{H^1,h} \le 2c_W, \quad \tau \le \tau_0''',$$

here $\tau_0''' = \tau_0'''(c_W, \varphi(c_W), \tau_0'')$. By the imbedding $H_h^1 \to C_h$ inequality (11) we get

$$\|p(t_{j+1})\|_{C,h} \le 2\sqrt{2}\, c_W, \quad \|q(t_{j+1})\|_{C,h} \le 2\sqrt{2}\, c_W, \quad \tau \le \tau_0'''.$$

Therefore, in a similar way as above one can obtain

$$Z(t_{j+1}) \le Z(t_j) + \tau\, 2\varphi(\tilde{c})[Z(t_{j+1}) + Z(t_j)] + \tau\, 2\Phi, \qquad \tau \le \tau_0''';$$

here $\tilde{c} = \tilde{c}(c_D, c_W)$ is a constant. The same arguments as above imply

$$Z(t_{j+1}) \le (1 + 8\varphi(\tilde{c})\,\tau) Z(t_j) + \tau\, 4\Phi, \qquad \tau \le \tau_0'''' = 1/(4\varphi(\tilde{c})), \tau_0'''.$$

Estimating $Z(t_j)$ by (80) we have

$$Z(t_{j+1}) \le \tilde{\tilde{c}}\,\Phi, \qquad \tau \le \tau_0''', \tau_0'''', \qquad \tilde{\tilde{c}} = \tilde{\tilde{c}}(c_Z, \varphi(\tilde{c})).$$

By multiplicative estimate (10) it follows that

$$\|\varepsilon(t_{j+1})\|_{C,h} + \|\delta(t_{j+1})\|_{C,h} \le \tilde{\tilde{\tilde{c}}}(\tau^2 + h^2)^{1/2}, \qquad \tau \le \tau_0''', \tau_0'''',$$

$\tilde{\tilde{\tilde{c}}} = \tilde{\tilde{\tilde{c}}}(c_G, c_W, c_\Phi, \tilde{\tilde{c}})$. Since

$$\|p(t_{j+1})\|_{C,h} + \|q(t_{j+1})\|_{C,h} \le \|u(t_{j+1})\|_{C,h} + \|v(t_{j+1})\|_{C,h} +$$

$$\|\varepsilon(t_{j+1})\|_{C,h} + \|\delta(t_{j+1})\|_{C,h},$$

it is sufficient to choose $\tau_0 > 0$ and $h_0 > 0$ such that $\tau_0 \le \min\{\tau_0', \tau_0'', \tau_0''', \tau_0''''\}$ and $\tilde{\tilde{\tilde{c}}}(\tau_0^2 + h_0^2)^{1/2} \le c_D/2$ to prove

$$\|p(t_{j+1})\|_{C,h} + \|q(t_{j+1})\|_{C,h} \le c_D, \quad \tau \le \tau_0, \quad h \le h_0,$$

and, hence, we get (78). $\triangle$

Theorem 8.4 *Assume that the conditions of Theorem 8.3 are satisfied. Then, for both initial boundary-value problems, we have for $t \in \bar{\omega}_\tau$*

$$\|\varepsilon\|_{H^1,h} + \|\delta\|_{H^1,h} = O\left(\max_{t\in\bar{\omega}_\tau}\max_{j=1,2}\|\Psi_j\|_{H^1,h}\right), \quad \tau, h \to 0. \tag{81}$$

Proof: The proof is similar to that of Theorem 8.3. We first note, that by Theorem 8.3, if $\tau \le \tau_0'$, $h \le h_0$, then (78) is satisfied. Here $\tau_0', h_0 > 0$ are some constants. Then it follows by Lemma 6.8 that, if $\tau \le \tau_0'' = \tau_0''(\varphi(c_D))$, then the difference solution is bounded in H_h^1 norm by some constant

$$c_W = c_W(T, \varphi(c_D), \|u^{(0)}\|_{H^1,h}, \|v^{(0)}\|_{H^1,h}).$$

Since the differential solution is smooth, we can assume that it is bounded in H_h^1 by the same constant.

Due to Corollary 6.3, we can apply (56) to get

$$\|\hat{\varepsilon}\|_{H^1,h} \le \|\varepsilon\|_{H^1,h} + \tau\|f_1(\mathring{u}, \mathring{u}^*, \mathring{v}, \mathring{v}^*) - f_1(\mathring{p}, \mathring{p}^*, \mathring{q}, \mathring{q}^*)\|_{H^1,h} +$$

$$\tau\|\Psi_1\|_{H^1,h},$$

$$\|\hat{\delta}\|_{H^1,h} \le \|\delta\|_{H^1,h} + \tau\|f_2(\mathring{u}, \mathring{u}^*, \mathring{v}, \mathring{v}^*) - f_2(\mathring{p}, \mathring{p}^*, \mathring{q}, \mathring{q}^*)\|_{H^1,h} +$$

$$\tau\|\Psi_2\|_{H^1,h}.$$

One can employ inequality (63) in Lemma 6.6 to estimate the nonlinear error

$$\|f_j(\mathring{u}, \mathring{u}^*, \mathring{v}, \mathring{v}^*) - f_j(\mathring{p}, \mathring{p}^*, \mathring{q}, \mathring{q}^*)\|_{H^1,h} \le 7\varphi(c_D)\,(1 + 4c_W)$$

$$\left(\|\hat{\varepsilon}\|_{H^1,h} + \|\varepsilon\|_{H^1,h} + \|\hat{\delta}\|_{H^1,h} + \|\delta\|_{H^1,h}\right), \quad j = 1, 2.$$

Denote $Z(t) = \|\varepsilon\|_{H^1,h} + \|\delta\|_{H^1,h}$ and $\Phi = \max_{t\in\bar{\omega}_\tau}\max\{\|\Psi_1\|_{H^1,h}, \|\Psi_2\|_{H^1,h}\}$. Then for $t \in \omega_\tau$

$$Z(t+\tau) \le Z(t) + \tau\, 14\varphi(c_D)\,(1 + 4c_W)\,[Z(t+\tau) + Z(t)] + \tau\, 2\Phi.$$

Since $Z(0) = 0$, by the same arguments we used proving (80) one obtains

$$Z(t) \le c_Z\,\Phi, \qquad t \in \bar{\omega}_\tau, \qquad \tau \le \tau_0''' = \tau_0'''(\varphi(c_D), c_W);$$

here $c_Z = c_Z(T, \varphi(c_D), c_W)$ is a constant. We define $\tau_0 = \min\{\tau_0', \tau_0'', \tau_0'''\}$. $\triangle$

Remark 8.5 *Assume that the conditions of Proposition 8.2 are satisfied. Then (for periodic boundary-value problem) Theorem 8.4 implies that*

$$\|\varepsilon\|_{H^1,h} + \|\delta\|_{H^1,h} = O(\tau^2 + h^2), \qquad \tau, h \to 0, \qquad t \in \bar{\omega}_\tau,$$

and, due to the imbedding $H_h^1 \to C_h$ inequality (11), *we improve the ratio of convergence:*

$$\|\varepsilon\|_{C,h} + \|\delta\|_{C,h} = O(\tau^2 + h^2), \qquad \tau, h \to 0, \qquad t \in \bar{\omega}_\tau.$$

We have proved the convergence of difference schemes. It remains to consider the stability.

Let p_1, q_1 be the solution of finite difference scheme (35), (36), (42) or (40), (41), (42), respectively, with the initial functions $u_1^{(0)}(x)$ and $v_1^{(0)}(x)$. Let also p_2, q_2 be the solution of the same difference problem with another initial functions $u_2^{(0)}(x)$ and $v_2^{(0)}(x)$.

Theorem 8.6 *Assume that nonlinear functions f_j, $j = 1, 2$, satisfy* (31) *with $|m| = 1$ and* (32) *or* (33), *respectively. Then, for both difference problems* (35), (36), (42) *and* (40), (41), (42), *there exist constants $\tau_0, h_0 > 0$ such that*

$$\|p_1 - p_2\|_{C,h} + \|q_1 - q_2\|_{C,h} \le c_S \left(|[u_1^{(0)} - u_2^{(0)}]|_h + |[v_1^{(0)} - v_2^{(0)}]|_h \right)^{1/2}, \tag{82}$$

if $\tau \le \tau_0$, $h \le h_0$, $t \in \bar{\omega}_\tau$. The constant c_S does not depend on the grid steps τ and h.

Proof: We denote $Z(t) = |[p_1 - p_2]|_h + |[q_1 - q_2]|_h$. Similarly as in the proof of Theorem 8.3, we come to the inequality analogous to (79):

$$Z(t + \tau) \le (1 + 8\varphi(c_D)\,\tau) Z(t), \quad t \in \omega_\tau, \quad \tau \le \tau_0.$$

This implies

$$|[p_1 - p_2]|_h + |[q_1 - q_2]|_h \le c_Z \left(|[u_1^{(0)} - u_2^{(0)}]|_h + |[v_1^{(0)} - v_2^{(0)}]|_h \right),$$

here c_Z is a constant and $t \in \bar{\omega}_\tau$.

To complete the proof we employ Gagliardo–Nirenberg multiplicative estimate (10). △

Theorem 8.7 *Suppose that nonlinear functions f_j, $j = 1, 2$, satisfy* (31) *and* (32) *or* (33)*, respectively. Then, for both difference problems* (35), (36), (42) *and* (40), (41), (42)*, there exist constants $\tau_0, h_0 > 0$ such that*

$$\begin{aligned} &\|p_1 - p_2\|_{H^1,h} + \|q_1 - q_2\|_{H^1,h} \leq \\ &c_S \left(\|u_1^{(0)} - u_2^{(0)}\|_{H^1,h} + \|v_1^{(0)} - v_2^{(0)}\|_{H^1,h} \right), \end{aligned} \tag{83}$$

if $\tau \leq \tau_0$, $h \leq h_0$, $t \in \bar{\omega}_\tau$. The constant c_S does not depend on the grid steps τ and h.

Proof: We denote $Z(t) = \|p_1 - p_2\|_{H^1,h} + \|q_1 - q_2\|_{H^1,h}$. Similarly as in the proof of Theorems 8.4 and 8.6, we get

$$Z(t) \leq c_S Z(0), \qquad \tau \leq \tau_0, \qquad t \in \bar{\omega}_\tau.$$

△

Remark 8.8 *Assume that the conditions of Theorem 8.7 are satisfied. Then due to the imbedding $H_h^1 \to C_h$ inequality* (11)*, the correspondent scheme is stable in C_h.*

Remark 8.9 *Due to Proposition 4.2 and Proposition 4.3, we have proposed and justified the algorithm for the numerical solution of derivative nonlinear Schrödinger equation* (3)*. Note that, while the convergence and stability of difference schemes is proved in C norm, by the relation*

$$v = \frac{\partial u}{\partial x} - i\frac{\alpha}{2}|u|^2 u, \qquad 0 \leq x \leq 1,$$

(see (22) *in Proposition 4.3) it follows that the whole method converges and is stable in C^1 norm, for both initial boundary-value problems of* (3)*.*

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