

On weak solutions of Stratonovich integral equation driven by a continuous p -semimartingales

Kęstutis KUBILIUS (MII, VGTU)

e-mail: kubilius@ktl.mii.lt

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, be a stochastic basis satisfying the usual conditions.

Definition 1. (see [7]). For $p \in [1, 2)$, an $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ adapted cadlag stochastic process Z is called a p -semimartingale if there exist stochastic processes M and A such that

$$Z - Z(0) = M + A \quad \text{almost surely,}$$

where $M(0) = A(0) = 0$, M is an $\mathbb{F}$ local martingale and A is an $\mathbb{F}$ -adapted process with locally bounded p -variation, i.e., for any fixed $T > 0$, the process $A = \{A_t, 0 \leq t \leq T\}$ has bounded p -variation.

Let Y, Z be two p -semimartingales with continuous trajectories and let $Z = M + A$, where its summands are continuous processes. Then the Stratonovich integral $(S) \int Y(s) dZ(s)$ is defined by the formula

$$(S) \int_0^t Y(s) dZ(s) := \int_0^t Y(s) dZ(s) + \frac{1}{2} [Y, Z](t), \quad t \geq 0.$$

The first integral we understand as a sum of two integrals

$$(SI) \int_0^t f(Y_s) dM_s \quad \text{and} \quad (RS) \int_0^t f(Y_s) dA_s,$$

where the symbol SI denotes the usual stochastic integral and the symbol RS denotes the Riemann–Stieltjes integral.

Consider the equation

$$X_t = \xi + (S) \int_0^t f(X_s) dZ_s, \quad t \geq 0,$$

or equivalent equation

$$X_t = \xi + \int_0^t f(X_s) dZ_s + \frac{1}{2} \int_0^t f f'(X_s) d[M]_s, \quad t \geq 0. \quad (1)$$

For short, we shall write $ff'(X_s)$ instead of $f(X_s)f'(X_s)$.

The purpose of this paper is to find conditions when the weak solution of the equation (1) exists.

Definition 2. We say that the equation (1) has a weak solution if there exists a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ and an $\tilde{\mathbb{F}}$ adapted processes $\tilde{X}, \tilde{\xi}, \tilde{Z}, [\tilde{M}]$ such that $\mathcal{L}(\tilde{\xi}, \tilde{Z}, [\tilde{M}]) = \mathcal{L}(\xi, Z, [M])$ and (1) holds for $\tilde{X}, \tilde{\xi}, \tilde{Z}, [\tilde{M}]$ in place of $X, \xi, Z, [M]$.

For $0 < \alpha \leq 1$, $C^\alpha(\mathbb{R})$ is the space of bounded Hölder functions g with the norm

$$\|g\|_\alpha := |g|_\infty + |g|_\alpha = \sup_x |g(x)| + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\alpha} < \infty.$$

The main result of this paper is the following theorem.

Theorem 3. Let $f \in C^1(\mathbb{R})$ and $\mathbf{E} \sup_{t \leq T} |M_t| < \infty$ for every $T > 0$. Then there exists a weak solution of equation (1).

1. Basic notions and auxiliary results

All facts mentioned below on the p -variation are taken from [1]

The p -variation, $0 < p < \infty$, of a real-valued function f on $[a, b]$ is defined as

$$v_p(f; [a, b]) = \sup_{\kappa} \sum_{k=1}^n |f(x_k) - f(x_{k-1})|^p,$$

where the supremum is taken over all subdivisions $\kappa = \{x_i: i = 0, \dots, n\}$ of $[a, b]$ such that $a = x_0 < x_1 < \dots < x_n = b$. If $v_p(f; [a, b]) < \infty$, f is said to have bounded p -variation on $[a, b]$. Let

$$\mathcal{W}_p([a, b]) := \{f: [a, b] \rightarrow \mathbb{R}: v_p(f; [a, b]) < \infty\}.$$

Define $V_p(f) := V_p(f; [a, b]) = v_p^{1/p}(f)$, which is a seminorm on $\mathcal{W}_p([a, b])$ provided $p \geq 1$ and $V_p(f)$ is 0 if and only if f is a constant.

Let $f \in \mathcal{W}_q([a, b])$ and $h \in \mathcal{W}_p([a, b])$ with $1 \leq p < \infty$, $q > 0$, $1/p + 1/q > 1$. If f and h have no common discontinuities then the RS integral $\int_a^b f dh$ exists and the Love–Young inequality

$$\left| \int_a^b f dh - f(y)[h(b) - h(a)] \right| \leq C_{p,q} V_q(f; [a, b]) V_p(h; [a, b]), \quad (2)$$

holds for any $y \in [a, b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$, $\zeta(s) = \sum_{n \geq 1} n^{-s}$. If, moreover, the function h is continuous, i.e., $h \in C\mathcal{W}_p([a, b])$, then the indefinite integral $\int_a^y f dh$, $y \in [a, b]$, is a continuous function.

Let τ and σ be a stopping times such that $\sigma < \tau \leq T$. Define $v_p(Y; [\sigma, \tau]) := v_p(Y^\tau - Y^\sigma; [0, T])$, where $Y^\tau = \{Y_{t \wedge \tau}, t \geq 0\}$.

Any local martingale is locally of bounded q -variation for each $q > 2$ (see [6] and [8]). Moreover, for $q > 2$ and $1 \leq r < \infty$ there is a finite constants $K_{q,r}$ such that for every r -integrable martingale $M = \{M(t), 0 \leq t \leq T\}$, $T > 0$,

$$\mathbf{E}\{V_q(M; [0, T])\}^r \leq K_{q,r} \mathbf{E}\left\{\sup_{0 \leq t \leq T} |M(t)|\right\}^r.$$

Moreover, if M is a continuous martingale then by the Burkholder–Davis–Gundy inequality we get

$$\mathbf{E}\{V_q(M; [0, T])\}^r \leq K_{q,r} \ell_r \mathbf{E}\langle M \rangle_T^{r/2},$$

where ℓ_r is the constant from the Burkholder–Davis–Gundy inequality.

2. Proofs

Let $\varkappa^i = \{t_k^i: k \geq 0\}$ be a sequence of partitions of $[0, \infty)$, i.e., $0 = t_0^i < t_1^i < t_2^i < \dots$, $\lim_{k \rightarrow \infty} t_k^i = \infty$, such that for every $T > 0$ we have $\max_{k \leq r^i(T)} |t_{k+1}^i - t_k^i| \rightarrow 0$ as $i \rightarrow +\infty$, where $r^i(T) = \max\{k: t_k^i \leq T\}$. For every $x \in D(\mathbb{R})$ and $\varkappa^i$ the sequence $\{x^{\varkappa^i}\}$ denotes the following discretizations of x :

$$x_t^{\varkappa^i} = x(t_k^i) \quad \text{for } t \in [t_k^i, t_{k+1}^i), \quad k \in \mathbb{N} \cup \{0\}, \quad i \in \mathbb{N}.$$

Define the approximations

$$X_t^n = \xi + \int_0^t f(X_{s-}^n) dZ_s^n + \frac{1}{2} \int_0^t f f'(X_{s-}^n) d[Z^{\varkappa^n}]_s, \quad t \geq 0, \quad n \in \mathbb{N},$$

and

$$\widehat{X}_t^n = \xi + \int_0^t f(\widehat{X}_{s-}^{n,\kappa^n}) dZ_s + \frac{1}{2} \int_0^t f f'(\widehat{X}_{s-}^{n,\kappa^n}) d\langle M \rangle_s, \quad t \geq 0, n \in \mathbb{N}.$$

Lemma 4. Let $f \in C^1(\mathbb{R})$ and let $q > 2$ be such that $\frac{1}{q} + \frac{1}{p} > 1$, where $1 \leq p < 2$. Then the sequence $\{V_q(\widehat{X}^n; [0, T])\}$ is tight in $\mathbb{R}$ for every $T > 0$ and the sequence $\{\widehat{X}^n\}$ is C -tight.

Proof. Denote

$$\gamma_N = \inf\{t > 0: \langle M \rangle_t > N, V_p(A; [0, t]) > N\},$$

and

$$\widehat{X}_t^{n,N} = \xi + \int_0^t f(\widehat{X}_{s-}^{n,N,\kappa^n}) dZ_s^N + \frac{1}{2} \int_0^t f f'(\widehat{X}_{s-}^{n,N,\kappa^n}) d\langle M \rangle_s^N, \quad t \geq 0, n \in \mathbb{N},$$

where $Y^{n,N}(t) = Y^n(t \wedge \gamma_N)$.

Similarly as in [4] Lemma 1 one can get

$$\begin{aligned} \mathbf{E}V_q(\widehat{X}^{n,N}; [0, T]) &\leq \frac{1}{1-\alpha} \{K_{q,1}\ell_1|f|_\infty \mathbf{E}\sqrt{\langle M \rangle_T^N} + C_{p,q/\alpha}|f|_\infty \mathbf{E}V_p(A^N; [0, T]) \\ &\quad + |f|_\infty |f'|_\infty \mathbf{E}\langle M \rangle_T^N\} + \mathbf{E}\{C_{p,q/\alpha}|f|_\alpha V_p(A^N; [0, T])\}^{1/(1-\alpha)} \\ &\leq \frac{1}{1-\alpha} \{K_{q,1}\ell_1|f|_\infty \sqrt{N} + C_{p,q/\alpha}|f|_\infty N \\ &\quad + |f|_\infty |f'|_\infty N + \{C_{p,q/\alpha}|f|_\alpha N\}^{1/(1-\alpha)}\}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{P}(V_q(\widehat{X}^n; [0, T]) > K) &\leq \mathbf{P}(\gamma_N < T) + K^{-1} \mathbf{E}V_p(\widehat{X}^{n,N}; [0, T]) \\ &\leq \mathbf{P}(\langle M \rangle_T > N, V_p(A; [0, T]) > N) + K^{-1} \mathbf{E}V_q(\widehat{X}^{n,N}; [0, T]), \end{aligned}$$

and the sequence $V_q(\widehat{X}^n; [0, T])$ is tight in $\mathbb{R}$.

Now we prove the tightness of $\{\widehat{X}^n\}$. We use the well known Aldous criterion. Let $\tau^n, n \geq 1$, be stopping times such that $\tau^n \leq T$. Then by inequality (2)

$$\begin{aligned} \sup_{t \leq \delta} |\widehat{X}_{\tau^n+t}^n - \widehat{X}_{\tau^n}^n| &\leq \sup_{t \leq \delta} \left| \int_{\tau^n}^{\tau^n+t} f(\widehat{X}_{s-}^{n,\kappa^n}) dM_s \right| \\ &\quad + C_{p,q} \{ |f'|_\infty V_q(\widehat{X}^n; [0, T]) + |f|_\infty \} V_p(A; [\tau^n, \tau^n + \delta]) \\ &\quad + |f|_\infty |f'|_\infty (\langle M \rangle_{\tau^n+\delta} - \langle M \rangle_{\tau^n}). \end{aligned}$$

By the Lengart–Rebolledo inequality for every ε, η we have

$$\begin{aligned} & \mathbf{P} \left(\sup_{t \leq \delta} \left| \int_{\tau^n}^{\tau^n+t} f(\widehat{X}_{s-}^{n, \mathbf{x}^n}) dM_s \right| > \varepsilon \right) \\ & \leq \varepsilon^{-2} \mathbf{E} \left\{ \left(\int_{\tau^n}^{\tau^n+\delta} f^2(\widehat{X}_{s-}^{n, \mathbf{x}^n}) d\langle M \rangle_s \right) \wedge \eta \right\} + \mathbf{P} \left(\int_{\tau^n}^{\tau^n+\delta} f^2(\widehat{X}_{s-}^{n, \mathbf{x}^n}) d\langle M \rangle_s \geq \eta \right) \\ & \leq \varepsilon^{-2} \mathbf{E} \left\{ |f|_\infty^2 (\langle M \rangle_{\tau^n+\delta} - \langle M \rangle_{\tau^n}) \wedge \eta \right\} + \mathbf{P} \left(|f|_\infty^2 (\langle M \rangle_{\tau^n+\delta} - \langle M \rangle_{\tau^n}) \geq \eta \right). \end{aligned}$$

Thus we get the tightness of $\{\widehat{X}^n\}$.

Lemma 5. Let $f \in C^1(\mathbb{R})$ and let $q > 2$ be such that $\frac{1}{q} + \frac{1}{p} > 1$, where $1 \leq p < 2$. Then the sequence $\{V_q(X^n; [0, T])\}$ is tight in $\mathbb{R}$ for every $T > 0$ and the sequence $\{X^n\}$ is tight in $D([0, \infty))$.

Proof. The proof of the tightness of the sequence $\{V_q(X^n; [0, T])\}$ is similar as in previous lemma.

Since $X^n(t_i^n) = \widehat{X}^n(t_i^n)$ for all $i \geq 0$ then for every $T > 0$ we get

$$\begin{aligned} & \sup_{t \leq T} |X_t^n - \widehat{X}_t^n| \\ & \leq |f|_\infty \sup_{t \leq T} |Z(t) - Z^{\mathbf{x}^n}(t)| \\ & \quad + |f|_\infty |f'|_\infty \sum_{i=1}^{r^n(T)} |M(t_i^n) - M(t_{i-1}^n)| \cdot |A(t_i^n) - A(t_{i-1}^n)| \\ & \quad + |f|_\infty |f'|_\infty \sum_{i=1}^{r^n(T)} |A(t_i^n) - A(t_{i-1}^n)|^2 + |f|_\infty |f'|_\infty \sup_{t \leq T} |\langle M \rangle(t) - \langle M \rangle^{\mathbf{x}^n}(t)| \\ & \quad + \sup_{t \leq T} \left| \sum_{i=1}^{r^n(t)} f f'(\widehat{X}^n(t_{i-1}^n)) \left[(M(t_i^n) - M(t_{i-1}^n))^2 - (\langle M \rangle(t_i^n) - \langle M \rangle(t_{i-1}^n)) \right] \right|. \end{aligned}$$

Therefore $\sup_{t \leq T} |X_t^n - \widehat{X}_t^n| \xrightarrow{\mathbf{P}} 0$, as $n \rightarrow \infty$. By Lemma ?? we have that the sequence $\{\widehat{X}^n\}$ is tight. Thus by Lemma 3.31 in Section 6 in [2] we obtain that the sequence $\{X^n\}$ is tight.

Proof of Theorem 1. Define $M^n = M^{\mathbf{x}^n}$ and $A^n = A^{\mathbf{x}^n}$. The process $(M^n, \mathbb{F}^n)$ is a martingale, where $\mathbb{F}^n = (\mathcal{F}_{\rho^n(t)})$, $\rho^n(t) = \max\{t_k^n: t_k^n \leq t\}$. The process A^n has locally bounded p -variation since $V_p(A^n; [0, T]) \leq V_p(A; [0, T])$. By this inequality it follows that the sequence $\{v_p(A^n; [0, T])\}$ is tight in $\mathbb{R}$. Note that $M^n \rightarrow M$ a.s. and $A^n \rightarrow A$ a.s. in $C([0, T])$. Moreover,

$$\sup_{t \leq T} |[M^n]_t - \langle M \rangle_t| \xrightarrow{\mathbf{P}} 0, \quad n \rightarrow \infty,$$

where

$$[M^n]_t = \sum_{k=1}^{r^n(t)} (M(t_k^n) - M(t_{k-1}^n))^2.$$

By Lemma 5, by Corollary 3.33 in Section 6 in [2], and facts obtained above it follows that the sequence $\{(X^n, M^n, A^n, [M^n], \xi)\}$ is C -tight. Thus from every subsequence $\{n'\} \subset \{n\}$ we can choose a further subsequence $\{n''\}$ such that

$$(X^{n''}, M^{n''}, A^{n''}, [M^{n''}], \xi) \xrightarrow{D} (\tilde{X}, \tilde{M}, \tilde{A}, [\tilde{M}], \tilde{\xi}),$$

as $n'' \rightarrow \infty$, where $(\tilde{X}, \tilde{M}, \tilde{A}, [\tilde{M}], \tilde{\xi})$ is defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and $\mathcal{L}(\tilde{\xi}, \tilde{M}, [\tilde{M}], \tilde{A}) = \mathcal{L}(\xi, M, [M], A)$. Since

$$\sup_{t \leq T} |[Z^{\mathbf{x}^{n''}}]_t - [M^{n''}]_t| \xrightarrow{\mathbf{P}} 0,$$

as $n'' \rightarrow \infty$, and functions f and ff' are continuous, then by the continuous mapping theorem

$$\begin{aligned} & (X^{n''}, f(X^{n''}), ff'(X^{n''}), M^{n''}, A^{n''}, [Z^{\mathbf{x}^{n''}}]_s, \xi) \\ & \xrightarrow{D} (\tilde{X}, f(\tilde{X}), ff'(\tilde{X}), \tilde{M}, \tilde{A}, [\tilde{M}]_s, \tilde{\xi}). \end{aligned}$$

By Lemma 5, we get the tightness of the sequence $\{v_q(f(X^n); [0, T])\}$, $q > 2$, $T > 0$, in $\mathbb{R}$. Note that

$$\sup_n \mathbf{E} \sup_{t \leq T} |\Delta M_t^n| \leq 2 \mathbf{E} \sup_{t \leq T} |M(t)|.$$

Thus conditions of Lemma 3 in [5] are satisfied and

$$\begin{aligned} & \left(X^{n''}, \int_0^\cdot f(X_{s-}^{n''}) dM_s^{n''}, \int_0^\cdot f(X_{s-}^{n''}) dA_s^{n''}, \int_0^\cdot ff'(X_{s-}^{n''}) d[Z^{\mathbf{x}^{n''}}]_s, \xi \right) \\ & \xrightarrow{D} \left(\tilde{X}, \int_0^\cdot f(\tilde{X}_s) d\tilde{M}_s, \int_0^\cdot f(\tilde{X}_s) d\tilde{A}_s, \int_0^\cdot ff'(\tilde{X}_s) d[\tilde{M}]_s, \tilde{\xi} \right). \end{aligned}$$

Therefore

$$\begin{aligned} & \sup_{t \leq T} \left| X_t^{n''} - \xi - \int_0^t f(X_{s-}^{n''}) dZ_s^{\mathbf{x}^{n''}} - \frac{1}{2} \int_0^t ff'(X_{s-}^{n''}) d[Z^{\mathbf{x}^{n''}}]_s \right| \\ & \xrightarrow{D} \sup_{t \leq T} \left| \tilde{X}_t - \tilde{\xi} - \int_0^t f(\tilde{X}_s) d\tilde{Z}_s - \frac{1}{2} \int_0^t ff'(\tilde{X}_s) d[\tilde{M}]_s \right|. \end{aligned}$$

As a consequence

$$\tilde{X}_t = \tilde{\xi} + \int_0^t f(\tilde{X}_s) d\tilde{Z}_s + \frac{1}{2} \int_0^t f f'(\tilde{X}_s) d[\tilde{M}]_s, \quad t \leq T.$$

References

- [1] R.M. Dudley, R. Norvaiša, An introduction to p -variation and Young integrals, *Lecture Notes*, **1**, Aarhus University (1998).
- [2] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, Vol. 1, Fiziko-Matematicheskaja Literatura, Moscow (1994.)
- [3] K. Kubilius, On approximation of stochastic integral equations driven by continuous p -semimartingales, *Lith. Math. J.* (special issue), **41**, 165–170, (2001).
- [4] K. Kubilius, The existence and uniqueness of the solution of the integral equation driven by a p -semimartingale of the special type, *Stochastic Processes Appl.*, **98**, 289–315 (2002).
- [5] K. Kubilius, On weak solutions of an integral equation driven by a p -semimartingale of special type (2002).
- [6] D. Lépine, La variation d'ordre p des semi-martingales, *Z. Wahrs. Verw. Gebiete*, **36**, 295–316 (1976).
- [7] R. Norvaiša, Quadratic variation, p -variation and integration with applications to stock price modeling. The Fields Institute, Toronto (available at <http://xxx.lanl.gov>) (2001).
- [8] G. Pisier, Q. Xu, The strong p -variation of martingales and orthogonal series, *Probab. Th. Rel. Fields*, **77**, 497–514 (1988).

Apie silpnus Stratanovičiaus integralinės lygties sprendinius, valdomus tolydžiu p -semimartingalu

K. Kubilius

Nagrinėjamas silpno Stratanovičiaus integralinės lygties sprendinio, valdomo tolydaus p -semimartingalo, egzistavimas.