Decision procedure for a fragment of quantified branching temporal logic

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1. Introduction

Temporal logic has had strong impact on a number of field, including computer science and artificial intelligence, especially as a tool for reasoning about programs. Branching-time logic is more convenient for describing sets of execution behaviours of parallel computation (see, e.g., [1]) than the linear temporal logic. In this paper first-order branching-time logic (in short FBL) is considered. FBL is close to dynamic logic, and is incomplete, in general.

Here we present deduction-based decision procedure for a miniscoped fragment of FBL. The main characteristic peculiarity of the proposed procedure is a verification of a loop property.

2. Preliminaries, infinitary sequent calculus $BL_\omega$

Let us consider first-order branching-time logic. Contrary to linear time logic where time is considered to be a linear sequence, in FBL time is considered as a tree structured time, allowing some instants to have more than a single successor. The language of FBL is based upon a set of predicate variables $P, Q, P_1, Q_1, \ldots$, the set of logical connectives $\exists, \land, \lor, \rightarrow, \forall, \exists$, and two temporal operators, namely "always"($\Box$) and "next"($\Diamond$). The modalities have the following intuitive meaning: let $T$ be a tree, $s$ be a node in $T$, and $A$ be a formula, then $\Box A$ means that A holds at s (in T) iff A is true at all nodes of the subtree rooted at s (including s) and $\Diamond A$ means that A holds at s (in T) iff A is true at every immediate successor of s in the subtree rooted at s.

Formulas are defined inductively, as usual. A sequent is an expression of the form $\Gamma \rightarrow \Delta$, where $\Gamma, \Delta$ are arbitrary finite multisets of formulas.

It is known that FBL is not finitary axiomatizable, but it becomes $\omega$-complete when $\omega$-like rule is added.

Let us consider infinitary sequent calculus $BL_\omega$.

Axiom is defined as usual and logical rules are traditional invertible rules for $\exists, \land, \lor, \rightarrow, \forall, \exists$.

Rules for temporal operators:

\[
\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, \Diamond A; \ldots; \Gamma \rightarrow \Delta, \Diamond^k A; \ldots \quad (\rightarrow \Box_{\omega}),
\]

\[
\Gamma \rightarrow \Delta, \Box A
\]
where $\circ^k A$ means $\circ \ldots \circ A$.

\[
\begin{align*}
    &A, \circ \square A, \Gamma \rightarrow \Delta \quad (\square \rightarrow), \\
    &\circ A, \Gamma \rightarrow \Delta
\end{align*}
\]

\[
\begin{align*}
    &A_1, \ldots, A_m \rightarrow B^* \\
    &\circ A_1, \ldots, \circ A_m, \Gamma \rightarrow \Delta, \circ B^* (\circ),
\end{align*}
\]

where $\Gamma \neq \Gamma_1, \circ \Gamma_2$, i.e., $\Gamma$ does not contain formulas of the shape $\circ A$; $\circ B^* \in \{\emptyset, \circ B\}$, and if $\circ B^* = \emptyset$ and $\Delta \neq \Delta_1, \circ \Delta_2$, then $B^* = \emptyset$ and $m \geq 1$, otherwise $B^* = B$ and $m \geq 0$.

**Theorem 1.** Let $A$ be an arbitrary formula in FBL. Then $\forall M \vdash A$ iff $BL_\omega \vdash \rightarrow A$, i.e., the calculus $BL_\omega$ is sound and $\omega$-complete.

**Remark 1.** As follows from the rule (\(\circ\)) FBL is some intuitionistic variant of the first-order linear temporal logic. It is known that in the linear temporal logic the formula $\neg \circ A \supset \circ \neg A$ is valid, however it becomes invalid in the branching temporal logic, though the formula $\circ \neg A \supset \neg \circ A$ is valid in both logics. So in FBL a normal form where negation is "slid into" through the logical operators and modality $\square$ can not be constructed.

**Definition 1.** A sequent $S$ is miniscoped sequent if $S$ satisfies the following miniscoped condition: all negative (positive) occurrences of $\forall (\exists$, correspondingly) in $S$ occurs only in formulas of the shape $Qx E$, where $E$ is an atomic formula; this formula is called an quasi-atomic formula. Atomic formula is a special case of quasi-atomic formula, if $Qx = \emptyset$.

**Definition 2.** A miniscoped sequent $S$ is *MR-sequent* if $S$ satisfies the following regularity condition: let a formula $\square A$ occur negatively in $S$, then the formula $A$ does not contain positive occurrences of formulas $\sigma B$ in $S$ where $\sigma \in \{\square, \circ\}$.

A *MR*-sequent $S$ is an induction-free *MR*-sequent if $S$ does not contain positive occurrences of formulas $\square A$. Otherwise a *MR*-sequent $S$ is non-induction-free one.

### 3. Calculi $BL_\omega^*$, $BL$, $KG$

The calculus $BL_\omega^*$ is obtained from $BL_\omega$ by dropping the rules $(\forall \rightarrow), (\rightarrow \exists)$ and adding the following axioms:

1. $\Gamma, E(t_1, \ldots, t_n) \rightarrow \Delta, \exists x_1 \ldots x_n E(x_1, \ldots, x_n);$
2. $\Gamma, \forall x_1 \ldots x_n E(x_1, \ldots, x_n) \rightarrow \Delta, E(t_1, \ldots, t_n);$
3. $\Gamma, \forall x_1 \ldots x_n E(t_1(x_1), \ldots, t_n(x_n)) \rightarrow \Delta, \exists y_1 \ldots y_n E(p_1(y_1), \ldots, p_n(y_n)),$

where $E$ is a predicate symbol, $\forall i (1 \leq i \leq n)$ terms $t_i(x_i)$ and $p_i(y_i)$ are unifiable.
Theorem 2. Let $S$ be a MR-sequent, then $BL^*_\omega \vdash S \iff BL^*_\omega \vdash S$.

The calculus $BL^*$ is obtained from $BL^*_\omega$ by dropping the rule $(\rightarrow \square\omega)$.

The calculus $KG$ is obtained from $BL^*_\omega$ by dropping the rules for temporal operators, i.e., $(\rightarrow \square\omega)$, $(\square \rightarrow \sqcap)$ and $(\square \rightarrow)$.

Definition 3. A MR-sequent $S$ is primary one if $S = \Sigma_1, \bigcirc^k \Pi_1, \square \Omega_1 \rightarrow \Sigma_2, \bigcirc^l \Pi_2, \square \Omega_2$, and a MR-sequent $S$ is $N$-primary one if $S = \Sigma_1, \bigcirc^k \Pi_1, \bigcirc^n \square \Omega_1 \rightarrow \Sigma_2, \bigcirc^l \Pi_2, \bigcirc^n \square \Omega_2$, where $k \geq 1, l \geq 1, m \geq 1, n \geq 1$ and $\Sigma_i = \emptyset (i \in \{1, 2\})$ or consists of quasi-atomic formulas; $\Pi_i = \emptyset (i \in \{1, 2\})$ or consists of formulas not containing the operator $\square; \square \Omega_i = \emptyset (i \in \{1, 2\})$ or consists of formulas of the shape $\square A$, where $A$ is an arbitrary formula.

Now we present rules by means of which reduction of a MR-sequent to $N$-primary MR-sequents is carried out.

Definition 4. The following rules will be called reduction rules (all these rules will be applied in the bottom-up manner):
1) all logical rules of the calculus $BL^*_\omega$;
2) the rule $(\square \rightarrow)$ and the following rule:

$$\dfrac{\Gamma \rightarrow \Delta, A; \Gamma \rightarrow \Delta, \bigcirc A \rightarrow \square \square}{\Gamma \rightarrow \Delta, \square A}$$

Lemma 1. The rule of inference $(\rightarrow \square \square)$ is admissible and invertible in $BL^*_\omega$.

Proof. Follows from the fact that $BL^*_\omega \vdash \square A \equiv A \land \bigcirc \square A$ and the admissibility of (cut) in $BL^*_\omega$.

Let $\{i\}$ denotes a set of reduction rules. Then reduction of a sequent $S$ to a set of sequents $S_1, \ldots, S_n$ (denoted by $R(S)\{i\} \Rightarrow \{S_1, \ldots, S_n\}$ or briefly by $R(S)$), is defined to be a tree of sequents with the root $S$ and leaves $S_1, \ldots, S_n$, and, possibly, axioms of the calculus, such that each sequent in $R(S)$, different from $S$, is the premise of the rule from $\{i\}$ whose conclusion also belongs to $R(S)$.

Theorem 3. Let $S$ be a MR-sequent. Then there exists the following reduction of the sequent $S : R(S)\{i\} \Rightarrow \{S_1, \ldots, S_n\}$, where $\{i\}$ is the set of reduction rules and $S_i$ ($1 \leq i \leq n$) is a $N$-primary MR-sequent. Moreover, if $BL^*_\omega \vdash S$, then $BL^*_\omega \vdash S_i$ ($1 \leq i \leq n$).

4. Decidability of calculi $BL^*, KG$

To prove the decidability of considered calculi let us introduce a separation rule.
**Definition 5.** Let $S = \Sigma_1, O^k \Pi_1, O^m \Box \Omega_1 \rightarrow \Sigma_2, O^{l} \Pi_2, O^n \Box \Omega_2$ be $N$-primary sequent. Then a separation rule has a following shape:

$$S = \Sigma_1, O^k \Pi_1, O^m \Box \Omega_1 \rightarrow \Sigma_2, O^{l} \Pi_2, O^n \Box \Omega_2 \quad (\text{Sep}),$$

where $S_1 = \Sigma_1 \rightarrow \Sigma_2$; $S_2 = O^{k-j} \Pi_1, O^{m-j} \Box \Omega_1 \rightarrow O^{l-j} B_j$, if $\Pi_2 = B_1, \ldots, B_k$ and $1 \leq j \leq k$; $S_3 = O^{k-j} \Pi_1, O^{m-j} \Box \Omega_1 \rightarrow O^{n-j} \Box C_i$ if $\Box \Omega_2 = \Box C_1, \ldots, \Box C_i$ and $1 \leq i \leq l$.

**Theorem 4 (disjunctive invertibility of the rule (Sep)).** Let $S$ be an $N$-primary sequent. Let $S$ be a conclusion and $S_i$ ($1 \leq i \leq 3$) be premises of the rule (Sep) and $BL^*_\omega \vdash S$. Then either $KG \vdash S_1$, or there exists such $j$ ($1 \leq j \leq k$), that $BL \vdash S_2$, or there exists such $j$ ($1 \leq j \leq l$), that $BL^*_\omega \vdash S_3$.

The calculus $BL^*$ is obtained from the calculus $BL$ replacing the rule $(\Box)$ by the rule (Sep) which is applied bottom-up. It is evident that only induction-free MR-sequent can be derivable in the calculi $BL$ and $BL^*$ because these calculi have no rule of the shape $(\rightarrow \Box)$. 

**Theorem 5.** Let $S$ be an induction-free MR-sequent. Then $BL \vdash S$ iff $BL^* \vdash S$.

Using the invertibility of the rules, regularity condition, and Theorem 4 we get

**Theorem 6.** Calculi $BL^*$ and $KG$ are decidable with respect to induction-free MR-sequents.

### 5. Decidable procedure for MR-sequents

Before a description of the decidable procedure for MR-sequents some concepts must be introduced.

**Definition 6.** Formulas $A, A^*$ are called parametrically identical formulas (in symbols $A \approx A^*$) if either $A = A^*$ or $A$ and $A^*$ are congruent, or $A$ and $A^*$ differ only by corresponding occurrences of eigen-variables of the rules $(\rightarrow \forall), (\exists \rightarrow)$.

We say that the MR-sequents $S$ and $S^*$ are parametrically identical (in symbols $S \approx S^*$) if the sequents $S, S^*$ differ only by parametrically identical formulas.

**Definition 7.** Let us introduce the following structural rule:

$$\frac{\Gamma \rightarrow \Delta}{\Pi, \Gamma^* \rightarrow \Delta^*, \theta} (W^*), \quad \text{where} \quad \Gamma \rightarrow \Delta \approx \Gamma^* \rightarrow \Delta^*.$$
We say that a $MR$-sequent $S_1$ subsumes a $MR$-sequent $S_2$ or $S_2$ is subsumed by $S_1$ (in symbols $S_1 \supseteq S_2$) if $S_2$ is a conclusion of an application of the rule ($W^*$) to $S_1$ (in a special case $S_1 \approx S_2$).

**Decidable procedure for $MR$-sequents**

Let $S$ be an arbitrary $MR$-sequent. For the sake of simplicity let the $S$ contains only one positive occurrence of the formula of the shape $\Box A$.

1. Using Theorem 3 let us reduce $S$ to a set of $N$-primary $MR$-sequents $\{S_1, \ldots, S_n\}$.

2. For each $i$ ($1 \leq i \leq n$) let us apply the rule ($Sep$). According to Theorems 4, 5 and 6 we can check the provability of all induction-free $MR$-sequents. So if for every $i$ ($1 \leq i \leq n$) at least one induction-free $MR$-sequent is proved in calculi $KG$ or $BL^*$, then $BL^*_\omega \vdash S$.

3. In opposite case there exists $i$ such that applying the rule ($Sep$) to $S_i$ we get non-induction-free $MR$-sequent $S_i^3$.

4. Let us continue the process of applying the rule ($Sep$) to $S_i^3$ and its descendants until provable induction-free $MR$-sequents are constructed or primary $MR$-sequent of the shape $S_i^+ = \Sigma_1, \Diamond^k \Pi_1, \Box \Omega_1 \rightarrow \Box A$ is obtained.

5. Let us repeat the procedure from step 1 for each obtained primary $MR$-sequent. If we get only provable in calculi $KG$ or $BL^*$ induction-free $MR$-sequents, then $BL^*_\omega \vdash S$. If in all leaves we get primary $MR$-sequents $S_j$ such that for each $j$ there exists $i$ such that $S_i \supseteq S_j$ (where $S_i$ is below than $S_j$, but not necessary in the same branch), i.e., loop property holds, then $BL^*_\omega \vdash S$.

6. In opposite case $BL^*_\omega \nvdash S$.

**Remark 2.** If the $MR$-sequent $S$ includes more than one positive occurrence of the formula of the shape $\Box A$, decision procedure must be applied looking over all these occurrences of modality $\Box$.

From Theorem 3 and finiteness of the set of parametrically different sequents we get

**Theorem 7.** The decidable procedure for $BL^*_\omega$ is sound and complete.

**References**


**Išsprendžiamoji procedūra kvantorinės skaidaus laiko logikos fragmentui**

A. Pliuškevičienė

Pasiūlyta dedukciją pagrįsta išsprendžiamoji procedūra miniskopizuotam pirmos eilės kvantorinės skaidaus laiko logikos fragmentui. Pasiūlyta išsprendžiamoji procedūra yra korektiška ir pilna.