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Decision procedure for a fragment of quantified branching temporal logic

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1. Introduction

Temporal logic has had strong impact on a number of field, including computer science and artificial intelligence, especially as a tool for reasoning about programs. Branching-time logic is more convenient for describing sets of execution behaviours of parallel computation (see, e.g., [1]) than the linear temporal logic. In this paper first-order branching-time logic (in short FBL) is considered. FBL is close to dynamic logic, and is incomplete, in general.

Here we present deduction-based decision procedure for a miniscoped fragment of FBL. The main characteristic peculiarity of the proposed procedure is a verification of a loop property.

2. Preliminaries, infinitary sequent calculus BL_ω

Let us consider first-order branching-time logic. Contrary to linear time logic where time is considered to be a linear sequence, in FBL time is considered as a tree structured time, allowing some instants to have more than a single successor. The language of FBL is based upon a set of predicate variables $P, Q, P_1, Q_1, \ldots$, the set of logical connectives $\supset, \wedge, \vee, \neg, \forall, \exists$, and two temporal operators, namely "always"($\Box$) and "next"($\bigcirc$). The modalities have the following intuitive meaning: let T be a tree, s be a node in T, and A be a formula, then $\Box A$ means that A holds at s (in T) iff A is true at all nodes of the subtree rooted at s (including s) and $\bigcirc A$ means that A holds at s (in T) iff A is true at every immediate successor of s in the subtree rooted at s.

Formulas are defined inductively, as usual. A sequent is an expression of the form $\Gamma \to \Delta$, where Γ, Δ are arbitrary finite multisets of formulas.

It is known that FBL is not finitary axiomatizable, but it becomes ω-complete when ω-like rule is added.

Let us consider infinitary sequent calculus BL_ω.

Axiom is defined as usual and logical rules are traditional invertible rules for $\supset, \wedge, \vee, \neg, \forall, \exists$.

Rules for temporal operators:

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, \bigcirc A; \ldots ; \Gamma \to \Delta, \bigcirc^k A; \ldots}{\Gamma \to \Delta, \Box A}\ (\to \Box_\omega),$$

where $\bigcirc^k A$ means $\overbrace{\bigcirc\ldots\bigcirc}^{k\text{ times}} A$.

$$\frac{A, \bigcirc\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta} \ (\Box \to),$$

$$\frac{A_1, \ldots, A_m \to B^*}{\bigcirc A_1, \ldots, \bigcirc A_m, \Gamma \to \Delta, \bigcirc B^*} \ (\bigcirc),$$

where $\Gamma \neq \Gamma_1, \bigcirc\Gamma_2$, i.e., Γ does not contain formulas of the shape $\bigcirc A$; $\bigcirc B^* \in \{\varnothing, \bigcirc B\}$, and if $\bigcirc B^* = \varnothing$ and $\Delta \neq \Delta_1, \bigcirc\Delta_2$, then $B^* = \varnothing$ and $m \geqslant 1$, otherwise $B^* = B$ and $m \geqslant 0$.

Theorem 1. *Let A be an arbitrary formula in FBL. Then $\forall M \models A$ iff $BL_\omega \vdash\!\!\to A$, i.e., the calculus BL_ω is sound and ω-complete.*

REMARK 1. As follows from the rule $(\bigcirc)$ FBL is some intuitionistic variant of the first-order linear temporal logic. It is known that in the linear temporal logic the formula $\neg\bigcirc A \supset \bigcirc\neg A$ is valid, however it becomes invalid in the branching temporal logic, though the formula $\bigcirc\neg A \supset \neg\bigcirc A$ is valid in both logics. So in FBL a normal form where negation is "slid into" through the logical operators and modality $\Box$ can not be constructed.

DEFINITION 1. A sequent S is *miniscoped sequent* if S satisfies the following *miniscoped condition*: all negative (positive) occurrences of $\forall$ ($\exists$, correspondingly) in S occurs only in formulas of the shape QxE, where E is an atomic formula; this formula is called an quasi-atomic formula. Atomic formula is a special case of quasi-atomic formula, if $Qx = \varnothing$.

DEFINITION 2. A miniscoped sequent S is *MR-sequent* if S satisfies the following *regularity condition*: let a formula $\Box A$ occur negatively in S, then the formula A does not contain positive occurrences of formulas σB in S where $\sigma \in \{\Box, \bigcirc\}$.

A MR-sequent S is *an induction-free MR-sequent* if S does not contain positive occurrences of formulas $\Box A$. Otherwise a MR-sequent S is *non-induction-free one.*

3. Calculi BL^*_ω, BL, KG

The calculus BL^*_ω is obtained from BL_ω by dropping the rules $(\forall \to)$, $(\to \exists)$ and adding the following axioms:

1) $\Gamma, E(t_1, \ldots, t_n) \to \Delta, \exists x_1 \ldots x_n E(x_1, \ldots, x_n)$;

2) $\Gamma, \forall x_1 \ldots x_n E(x_1, \ldots, x_n) \to \Delta, E(t_1, \ldots, t_n)$;

3) $\Gamma, \forall x_1 \ldots x_n E(t_1(x_1), \ldots, t_n(x_n)) \to \Delta, \exists y_1 \ldots y_n E(p_1(y_1), \ldots, p_n(y_n))$, where E is a predicate symbol, $\forall i$ $(1 \leqslant i \leqslant n)$ terms $t_i(x_i)$ and $p_i(y_i)$ are unifiable.

Theorem 2. *Let S be a MR-sequent, then $BL^*_\omega \vdash S \iff BL_\omega \vdash S$.*

The calculus BL is obtained from BL^*_ω by dropping the rule $(\rightarrow \Box_\omega)$.

The calculus KG is obtained from BL^*_ω by dropping the rules for temporal operators, i.e., $(\rightarrow \Box_\omega)$, $(\Box \rightarrow)$ and $(\bigcirc)$.

DEFINITION 3. A MR-sequent S is *primary* one if $S = \Sigma_1, \bigcirc^k\Pi_1, \Box\Omega_1 \rightarrow \Sigma_2, \bigcirc^l\Pi_2, \Box\Omega_2$, and a MR-sequent S is *N-primary* one if $S = \Sigma_1, \bigcirc^k\Pi_1, \bigcirc^m\Box\Omega_1 \rightarrow \Sigma_2, \bigcirc^l\Pi_2, \bigcirc^n\Box\Omega_2$, where $k \geqslant 1$, $l \geqslant 1, m \geqslant 1, n \geqslant 1$ and $\Sigma_i = \varnothing$ $(i \in \{1,2\})$ or consists of quasi-atomic formulas; $\Pi_i = \varnothing$ $(i \in \{1,2\})$ or consists of formulas not containing the operator $\Box$; $\Box\Omega_i = \varnothing$ $(i \in \{1,2\})$ or consists of formulas of the shape $\Box A$, where A is an arbitrary formula.

Now we present rules by means of which reduction of a MR-sequent to N-primary MR-sequents is carried out.

DEFINITION 4. The following rules will be called *reduction rules* (all these rules will be applied in the bottom-up manner):

1) all logical rules of the calculus BL_ω;

2) the rule $(\Box \rightarrow)$ and the following rule:

$$\frac{\Gamma \rightarrow \Delta, A;\ \Gamma \rightarrow \Delta, \bigcirc\Box A}{\Gamma \rightarrow \Delta, \Box A}\ (\rightarrow \bigcirc\Box)$$

Lemma 1. *The rule of inference $(\rightarrow \bigcirc\Box)$ is admissible and invertible in BL_ω.*

Proof. Follows from the fact that $BL_\omega \vdash \Box A \equiv A \wedge \bigcirc\Box A$ and the admissibility of (cut) in BL_ω.

Let $\{i\}$ denotes a set of reduction rules. Then *reduction* of a sequent S to a set of sequents $S_1, \ldots, S_n$ (denoted by $R(S)\{i\} \Rightarrow \{S_1, \ldots, S_n\}$ or briefly by $R(S)$), is defined to be a tree of sequents with the root S and leaves $S_1, \ldots, S_n$, and, possibly, axioms of the calculus, such that each sequent in $R(S)$, different from S, is the premise of the rule from $\{i\}$ whose conclusion also belongs to $R(S)$.

Theorem 3. *Let S be a MR-sequent. Then there exists the following reduction of the sequent S : $R(S)\{i\} \Rightarrow \{S_1, \ldots, S_n\}$, where $\{i\}$ is the set of reduction rules and S_i $(1 \leqslant i \leqslant n)$ is a N-primary MR-sequent. Moreover, if $BL_\omega \vdash S$, then $BL_\omega \vdash S_i$ $(1 \leqslant i \leqslant n)$.*

4. Decidability of calculi BL^*, KG

To prove the decidability of considered calculi let us introduce a separation rule.

DEFINITION 5. Let $S = \Sigma_1, \bigcirc^k\Pi_1, \bigcirc^m\Box\Omega_1 \rightarrow \Sigma_2, \bigcirc^l\Pi_2, \bigcirc^n\Box\Omega_2$ be N-primary sequent. Then a separation rule has a following shape:

$$\frac{S_1 \ \textit{or} \ S_2 \ \textit{or} \ S_3}{S = \Sigma_1, \bigcirc^k\Pi_1, \bigcirc^m\Box\Omega_1 \rightarrow \Sigma_2, \bigcirc^l\Pi_2, \bigcirc^n\Box\Omega_2} \ (Sep),$$

where $S_1 = \Sigma_1 \rightarrow \Sigma_2$; $S_2 = \bigcirc^{k-1}\Pi_1, \bigcirc^{m-1}\Box\Omega_1 \rightarrow \bigcirc^{l-1}B_j$, if $\Pi_2 = B_1, \ldots, B_k$ and $1 \leqslant j \leqslant k$; $S_3 = \bigcirc^{k-1}\Pi_1, \bigcirc^{m-1}\Box\Omega_1 \rightarrow \bigcirc^{n-1}\Box C_i$ if $\Box\Omega_2 = \Box C_1, \ldots, \Box C_l$ and $1 \leqslant i \leqslant l$.

Theorem 4 (disjunctive invertability of the rule (Sep)). *Let S be an N-primary sequent. Let S be a conclusion and S_i $(1 \leqslant i \leqslant 3)$ be premises of the rule (Sep) and $BL^*_\omega \vdash S$. Then either $KG \vdash S_1$, or there exists such j $(1 \leqslant j \leqslant k)$, that $BL \vdash S_2$, or there exists such j $(1 \leqslant j \leqslant l)$, that $BL^*_\omega \vdash S_3$.*

The calculus BL^* is obtained from the calculus BL replacing the rule $(\bigcirc)$ by the rule (Sep) which is applied bottom-up. It is evident that only induction-free MR-sequent can be derivable in the calculi BL and BL^* because these calculi have no rule of the shape $(\rightarrow \Box_\omega)$.

Theorem 5. *Let S be an induction-free MR-sequent. Then $BL \vdash S$* **iff** *$BL^* \vdash S$.*

Using the invertibility of the rules, regularity condition, and Theorem 4 we get

Theorem 6. *Calculi BL^* and KG are decidable with respect to induction-free MR-sequents.*

5. Decidable procedure for MR-sequents

Before a description of the decidable procedure for MR-sequents some concepts must be introduced.

DEFINITION 6. Formulas A, A^* are called *parametrically identical formulas* (in symbols $A \approx A^*$) if either $A = A^*$ or A and A^* are congruent, or A and A^* differ only by corresponding occurrences of eigen-variables of the rules $(\rightarrow \forall)$, $(\exists \rightarrow)$.

We say that the MR-sequents S and S^* are parametrically identical (in symbols $S \approx S^*$) if the sequents S, S^* differ only by parametrically identical formulas.

DEFINITION 7. Let us introduce the following structural rule:

$$\frac{\Gamma \rightarrow \Delta}{\Pi, \Gamma^* \rightarrow \Delta^*, \theta} \ (W^*), \quad \text{where} \quad \Gamma \rightarrow \Delta \approx \Gamma^* \rightarrow \Delta^*.$$

We say that a MR-sequent S_1 subsumes a MR-sequent S_2 or S_2 is subsumed by S_1 (in symbols $S_1 \succcurlyeq S_2$) if S_2 is a conclusion of an application of the rule (W^*) to S_1 (in a special case $S_1 \approx S_2$).

Decidable procedure for MR-sequents

Let S be an arbitrary MR-sequent. For the sake of simplicity let the S contains only one positive occurrence of the formula of the shape $\Box A$.

1. Using Theorem 3 let us reduce S to a set of N-primary MR-sequents $\{S_1, \dots, S_n\}$.

2. For each i $(1 \leqslant i \leqslant n)$ let us apply the rule (Sep). According to Theorems 4, 5 and 6 we can check the provability of all induction-free MR-sequents. So if for every i $(1 \leqslant i \leqslant n)$ at least one induction-free MR-sequent is proved in calculi KG or BL^*, then $BL^*_\omega \vdash S$.

3. In opposite case there exists i such that applying the rule (Sep) to S_i we get non-induction-free MR-sequent S_i^3.

4. Let us continue the process of applying the rule (Sep) to S_i^3 and its descendants until provable induction-free MR-sequents are constructed or primary MR-sequent of the shape $S_i^+ = \Sigma_1, \bigcirc^k\Pi_1, \Box\Omega_1 \to \Box A$ is obtained.

5. Let us repeat the procedure from step 1 for each obtained primary MR-sequent. If we get only provable in calculi KG or BL^* induction-free MR-sequents, then $BL^*_\omega \vdash S$. If in all leaves we get primary MR-sequents S_j such that for each j there exists i such that $S_i \succcurlyeq S_j$ (where S_i is below than S_j, but not necessary in the same branch), i.e., loop property holds, then $BL_\omega \vdash S$.

6. In opposite case $BL_\omega \nvdash S$.

REMARK 2. If the MR-sequent S includes more than one positive occurrence of the formula of the shape $\Box A$, decision procedure must be applied looking over all these occurrences of modality $\Box$.

From Theorem 3 and finiteness of the set of parametrically different sequents we get

Theorem 7. *The decidable procedure for BL_ω is sound and complete.*

References

[1] M. Ben-Ari, A. Pnueli, Z. Manna, The temporal logic of branching time, *Acta Informatica*, **20**, 207–226 (1983).

Išsprendžiamoji procedūra kvantorinės skaidaus laiko logikos fragmentui

A. Pliuškevičienė

Pasiūlyta dedukcija pagrįsta išsprendžiamoji procedūra miniskopizuotam pirmos eilės kvantorinės skaidaus laiko logikos fragmentui. Pasiūlyta išsprendžiamoji procedūra yra korektiška ir pilna.