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An inequality for the modified Selberg zeta-function

Igoris Belovas

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Abstract We consider the absolute values of the modified Selberg zeta-function at places symmetric with respect to the critical line. We prove an inequality for the modified Selberg zeta-function, in a different way re-proving and extending the result of R. Garunkštis and A. Grigutis and concluding the result of I. Belovas and L. Sakalauskas.

Keywords Selberg zeta-function · Clausen function · inequalities

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1 Introduction

Let $s = \sigma + it$ be a complex variable and $\zeta(s)$ be the Riemann zeta-function. The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\sigma > 1$, and by analytic continuation elsewhere except for a simple pole at $s = 1$. T. S. Trudgian [9] obtained, that

$$|\zeta(1-s)| > |\zeta(s)| \tag{1}$$

except at the zeros of $\zeta(s)$, with $|t| \geqslant 6.29073$ and $\sigma > 1/2$. By well-known functional equation for the Riemann zeta-function,

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s), \tag{2}$$

I. Belovas
Vilnius University Institute of Data Science and Digital Technologies
Akademijos 4, LT-04812 Vilnius, Lithuania
E-mail: Igoris.Belovas@mif.vu.lt

$\zeta(1-s)$ and $\zeta(s)$ have the same zeros when $0 < \sigma < 1$. Inequalities of (1) type are of great interest, since a necessary and sufficient condition for the Riemann hypothesis is the inequality (1) valid without exceptions [3]. It was R. Spira [7], who started to investigate inequalities of (1) type for the Riemann zeta-function (note that he also obtained the inequality for the Ramanujan τ-Dirichlet series [8]).

Garunkštis and Grigutis examined, whether Selberg zeta-function, associated to the compact Riemann surface of genus $g \geqslant 2$ and the modified Selberg zeta-function satisfy the inequalities of (1) type (see Theorems 1 and 2 in [3]; the result was improved by Belovas in [1])). The Selberg zeta-function, associated with the modular group $\mathrm{PSL}(2,\mathbb{Z})$ satisfies (cf. (2)) [5] the functional equation

$$Z_{\mathrm{PSL}(2,\mathbb{Z})}(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(1-s)\frac{\zeta(2s)}{\zeta(2(1-s))}\frac{\Gamma(2s)}{\Gamma(2(1-s))}(2\pi)^{1-2s}\times$$
$$\times\exp\left(\frac{\pi}{3}\int_0^{s-1/2} v\tan\pi v dv - \frac{\pi}{2}\int_0^{s-1/2}\frac{dv}{\cos\pi v} - \frac{4\pi}{3\sqrt{3}}\int_0^{s-1/2}\frac{\cos\pi v/3}{\cos\pi v}\mathrm{d}v\right).$$

Garunkštis and Grigutis have proved the following theorem for the modified Selberg zeta-function

$$W(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)/\zeta(2s). \tag{3}$$

Theorem 1 (Garunkštis and Grigutis [3])
If $1/2 < \sigma < 1$ *and* $t \geqslant 6.053$*, then*

$$|W(1-s)| > |W(s)|.$$

Belovas and Sakalauskas, using a new inequality for the modulus of the ratio of two complex gamma functions (see Lemma 3 in [2]), in a different way re-proved and extended the result of Garunkštis and Grigutis for the modified Selberg zeta-function.

Theorem 2 (Belovas and Sakalauskas [2])
For $1/2 < \sigma < 1$ *and* $t \in [0, t_1) \cup (t_2, \infty)$ *we have*

$$|W(1-s)| > |W(s)|.$$

Here $t_1 = 1.740440...$ *and* $t_2 = 6.088036...$ *are the roots of the function*

$$L_1(t) = \log\left(\frac{1+t^2}{\pi\cosh\log\left(2\sinh\frac{\pi t}{6}\right)}\right).$$

In the present paper we show, that it is possible to circumvent the restriction, imposed by the inequality of Lemma 3 in [2], and obtain the result (see Theorem 3 in the next section), which does not rely on the estimation, thus concluding the study (the region by t, given in Theorem 3 can not be improved).

2 An inequality for the modified Selberg zeta-function

Theorem 3 *For* $1/2 < \sigma < 1$ *and* $t \in [0, t_1) \cup (t_2, \infty)$ *we have*

$$|W(1-s)| > |W(s)|. \tag{4}$$

Here $t_1 = 2.233...$ *and* $t_2 = 5.967...$ *are the roots of the function*

$$\Psi_1(t) = \frac{1}{2}\log(1+4t^2) + \log\frac{t}{2\pi} - \log\cosh\log\left(2\sinh\frac{\pi t}{6}\right). \tag{5}$$

By the definition of the modified Selberg zeta-function (3) we have

$$\begin{aligned}\left|\frac{W(s)}{W(1-s)}\right| &= \left|\frac{\Gamma(2s)}{\Gamma(2(1-s))}(2\pi)^{1-2s}e^{Q(s)}\right| = \\ &= \left|\frac{\Gamma(2s)}{\Gamma(2(1-s))}\right|(2\pi)^{1-2\sigma}e^{\Re(Q(s))},\end{aligned} \tag{6}$$

here

$$Q(s) = \int_0^{s-1/2} \frac{\pi}{3}v\tan\pi v - \frac{\pi}{2\cos\pi v} - \frac{4\pi}{3\sqrt{3}}\frac{\cos\pi v/3}{\cos\pi v}\mathrm{d}v. \tag{7}$$

Evaluating the integral using triangular contour with vertices at $A(0,0)$, $B(\sigma - 1/2, t)$ and $C(\sigma - 1/2, 0)$ (note that $\int_{AC} + \int_{CB} + \int_{BA} = 0$), we obtain (cf. formula (3.20) in [2]) that

$$\begin{aligned}\Re(Q(s)) =&(1-2\sigma)\frac{\log 2}{12} - \frac{1}{6\pi}\mathrm{Cl}_2(2\pi\sigma) \\ &+\frac{\pi}{3}\sin 2\pi\sigma\int_0^t \frac{\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}\mathrm{d}\theta \\ &-\frac{2\sigma-1}{12}\log(\cosh 2\pi t - \cos 2\pi\sigma) \\ &+\frac{1}{4}\log\frac{\cosh\pi t + \cos\pi\sigma}{\cosh\pi t - \cos\pi\sigma} \\ &+\frac{1}{3}\log\frac{\cosh\frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}}{\cosh\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}}.\end{aligned} \tag{8}$$

Here $\mathrm{Cl}_2(x)$ stands for the Clausen function of order 2,

$$\mathrm{Cl}_2(x) = -\int_0^x \log\left|2\sin\frac{t}{2}\right|\mathrm{d}t. \tag{9}$$

Let us denote

$$L(\sigma, t) = \log\left|\frac{W(s)}{W(1-s)}\right|. \tag{10}$$

Hence (cf. (6) and (8)),

$$\begin{aligned}L(\sigma,t) =&\log\left|\frac{\Gamma(2s)}{\Gamma(2(1-s))}\right| + (1-2\sigma)\log(2\pi) + \Re(Q(s))\\ =&\Theta(\sigma,t) + (1-2\sigma)\left(\log(2\pi) + \frac{\log 2}{12}\right) - \frac{\mathrm{Cl}_2(2\pi\sigma)}{6\pi}\\ &+\frac{\pi}{3}\sin 2\pi\sigma\int_0^t \frac{\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}\mathrm{d}\theta\\ &-\frac{2\sigma-1}{12}\log(\cosh 2\pi t - \cos 2\pi\sigma)\\ &+\frac{1}{4}\log\frac{\cosh \pi t + \cos \pi\sigma}{\cosh \pi t - \cos \pi\sigma}\\ &+\frac{1}{3}\log\frac{\cosh \frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}}{\cosh \frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}}.\end{aligned} \tag{11}$$

Here

$$\Theta(\sigma,t) = \log\left|\frac{\Gamma(2s)}{\Gamma(2(1-s))}\right|. \tag{12}$$

Next we will establish several lemmas concerning the behaviour of the function $L(\sigma,t)$. Note that the function $L(\sigma,t)$ is even by t, thus, it suffices to consider positive t values.

3 The derivatives of the function $L(\sigma,t)$

Lemma 1 *For $1/2<\sigma<1$ and fixed $t>t_0=1.578...$ the function $L(\sigma,t)$ (11) is convex by σ.*

Proof By Lobachevsky's formula (cf. 8.326.1 in [4]),

$$\log|\Gamma(2s)| = \log|\Gamma(2\sigma)| - \frac{1}{2}\sum_{n=0}^{\infty}\log\left(1+\frac{4t^2}{(2\sigma+n)^2}\right). \tag{13}$$

Hence, for $1/2<\sigma<1$,

$$\begin{aligned}\Theta(\sigma,t) =& \log\Gamma(2\sigma) - \log\Gamma(2-2\sigma)\\ &-\frac{1}{2}\sum_{n=0}^{\infty}\log\left(1+\frac{4t^2}{(2\sigma+n)^2}\right)\\ &+\frac{1}{2}\sum_{n=0}^{\infty}\log\left(1+\frac{4t^2}{(2-2\sigma+n)^2}\right).\end{aligned} \tag{14}$$

Taking into account that the second derivative of the log-gamma function is the trigamma function (or a special case of the Hurwitz zeta-function),

$$\psi_1(z) := \frac{d^2}{dz^2}\log\Gamma(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}, \tag{15}$$

we can calculate the second derivative of $\Theta(\sigma,t)$ with respect to the variable σ,

$$\begin{aligned}\Theta''_{\sigma\sigma}(\sigma,t) = 4\sum_{n=0}^{\infty}\left(\frac{1}{(2\sigma+n)^2+4t^2}-\frac{1}{(2-2\sigma+n)^2+4t^2}\right)\\ +4\sum_{n=0}^{\infty}\left(\frac{8t^2}{((2-2\sigma+n)^2+4t^2)^2}-\frac{8t^2}{((2\sigma+n)^2+4t^2)^2}\right).\end{aligned} \tag{16}$$

Hence,

$$\Theta''_{\sigma\sigma}(\sigma,t) = \underbrace{16(2\sigma-1)}_{>0}\,\Omega(\sigma,t), \tag{17}$$

here

$$\begin{aligned}\Omega(\sigma,t) &= \sum_{n=0}^{\infty}\frac{8t^2(n+1)((n+2\sigma)^2+(n+2-2\sigma)^2+8t^2)}{((n+2\sigma)^2+4t^2)^2((n+2-2\sigma)^2+4t^2)^2}\\ &-\sum_{n=0}^{\infty}\frac{n+1}{((n+2\sigma)^2+4t^2)((n+2-2\sigma)^2+4t^2)}\\ &=\underbrace{\sum_{n=0}^{\infty}\frac{8t^2(n+1)}{((n+2\sigma)^2+4t^2)^2((n+2-2\sigma)^2+4t^2)}}_{=F_1(\sigma,t)}\\ &+\underbrace{\sum_{n=0}^{\infty}\frac{8t^2(n+1)}{((n+2\sigma)^2+4t^2)((n+2-2\sigma)^2+4t^2)^2}}_{=F_2(\sigma,t)}\\ &-\underbrace{\sum_{n=0}^{\infty}\frac{n+1}{((n+2\sigma)^2+4t^2)((n+2-2\sigma)^2+4t^2)}}_{=F_3(\sigma,t)}.\end{aligned} \tag{18}$$

Let us denote (cf. 5.1.25.4 and 5.1.25.25 in [6])

$$h_1(t) = \sum_{n=0}^{\infty}\frac{1}{n^2+4t^2} = \frac{1}{8t^2}+\frac{\pi}{4t}\coth 2\pi t, \tag{19}$$

$$h_2(t) = \sum_{n=0}^{\infty}\frac{1}{(n^2+4t^2)^2} = \frac{1}{32t^4}+\frac{\pi}{32t^3}\coth 2\pi t+\frac{\pi^2}{16t^2\sinh^2 2\pi t}, \tag{20}$$

$$h_3(t) = \sum_{n=0}^{\infty}\frac{n}{(n^2+4t^2)^2}. \tag{21}$$

The lower bound of the function $F_1(\sigma, t)$ from (18) is

$$\begin{aligned} F_1(\sigma,t) &\geqslant \sum_{n=0}^{\infty} \frac{8t^2(n+1)}{((n+2)^2+4t^2)^2((n+2-2\sigma)^2+4t^2)} \\ &\geqslant 8t^2 \sum_{n=0}^{\infty} \frac{n+1}{((n+2)^2+4t^2)^2((n+1)^2+4t^2)} \\ &= \frac{8t^2}{(16t^2+1)^2}(f_{11}(t)+f_{12}(t)-f_{13}(t)), \end{aligned} \tag{22}$$

where

$$\begin{aligned} f_{11}(t) &= \sum_{n=1}^{\infty} \frac{(1-16t^2)n+16t^2}{n^2+4t^2}, \\ f_{12}(t) &= \sum_{n=1}^{\infty} \frac{(16t^2-1)n+16t^2-2}{(n+1)^2+4t^2}, \\ f_{13}(t) &= \sum_{n=1}^{\infty} \frac{(16t^2+1)n+128t^4+40t^2+2}{((n+1)^2+4t^2)^2}. \end{aligned} \tag{23}$$

The sum $f_{10}(t) = f_{11}(t) + f_{12}(t)$ equals

$$\begin{aligned} f_{10}(t) &= \underbrace{\frac{1}{1+4t^2} + \sum_{n=2}^{\infty} \frac{(1-16t^2)n+16t^2}{n^2+4t^2}}_{=f_1(t)} \\ &+ \underbrace{\sum_{n=2}^{\infty} \frac{(16t^2-1)n-1}{n^2+4t^2}}_{f_2(t)} \\ &= \frac{1}{1+4t^2} + (16t^2-1)\sum_{n=2}^{\infty} \frac{1}{n^2+4t^2} \\ &= \frac{1}{1+4t^2} + (16t^2-1)\left(h_1(t) - \frac{1}{4t^2} - \frac{1}{1+4t^2}\right). \end{aligned} \tag{24}$$

Taking into account (19) we have

$$f_{10}(t) = \frac{1}{1+4t^2} + (16t^2-1)\left(-\frac{1}{8t^2} + \frac{\pi}{4t}\coth 2\pi t - \frac{1}{1+4t^2}\right). \tag{25}$$

The function $f_{13}(t)$ (23) equals

$$\begin{aligned} f_{13}(t) &= \sum_{n=2}^{\infty} \frac{(16t^2+1)n + 128t^4 + 24t^2 + 1}{(n^2+4t^2)^2} \\ &= (16t^2+1)\left(\underbrace{\sum_{n=1}^{\infty} \frac{n}{(n^2+4t^2)^2}}_{=h_3(t)} - \frac{1}{(1+4t^2)^2}\right) \\ &+ (128t^4+24t^2+1)\left(\underbrace{\sum_{n=0}^{\infty} \frac{1}{(n^2+4t^2)^2}}_{=h_2(t)} - \frac{1}{16t^4} - \frac{1}{(1+4t^2)^2}\right). \end{aligned} \tag{26}$$

Taking into account (19) we have

$$\begin{aligned} f_{13}(t) &= (16t^2+1)\left(h_3(t) - \frac{1}{(1+4t^2)^2}\right) \\ &+ (128t^4+24t^2+1) \\ &\times \left(-\frac{1}{32t^4} + \frac{\pi}{32t^3}\coth 2\pi t + \frac{\pi^2}{16t^2\sinh^2 2\pi t} - \frac{1}{(1+4t^2)^2}\right). \end{aligned} \tag{27}$$

Hence, by (25) and (27),

$$\begin{aligned} f_{10}(t) - f_{13}(t) &= 6 + \frac{7}{8t^2} + \frac{1}{32t^4} - \left(1 + \frac{1}{32t^2}\right)\frac{\pi}{t}\coth 2\pi t \\ &- (16t^2+1)h_3(t) - \left(8t^2 + \frac{3}{2} + \frac{1}{16t^2}\right)\frac{\pi^2}{\sinh^2 2\pi t}. \end{aligned} \tag{28}$$

Thus,

$$\begin{aligned} F_1(\sigma,t) &\geqslant \frac{1}{(16t^2+1)^2}\left(48t^2 + 7 + \frac{1}{4t^2} - \left(8t^2 + \frac{1}{4}\right)\frac{\pi}{t}\coth 2\pi t\right) \\ &- \frac{1}{(16t^2+1)^2}\left((16t^2+1)8t^2 h_3(t) + \frac{\pi^2(64t^4 + 12t^2 + 1/2)}{\sinh^2 2\pi t}\right). \end{aligned} \tag{29}$$

Next, consider the function $F_2(\sigma,t)$,

$$\begin{aligned} F_2(\sigma,t) &\geqslant \sum_{n=0}^{\infty} \frac{8t^2(n+1)}{((n+2)^2+4t^2)((n+2-2\sigma)^2+4t^2)^2} \\ &\geqslant 8t^2 \sum_{n=0}^{\infty} \frac{n+1}{((n+2)^2+4t^2)((n+1)^2+4t^2)^2} \\ &= \frac{8t^2}{(16t^2+1)^2}(f_{21}(t) + f_{22}(t) + f_{23}(t)). \end{aligned} \tag{30}$$

Here

$$\begin{aligned} f_{21}(t) &= \sum_{n=1}^{\infty} \frac{(-3-16t^2)n - 16t^2 - 4}{(n+1)^2 + 4t^2}, \\ f_{22}(t) &= \sum_{n=1}^{\infty} \frac{(16t^3+3)n - 16t^2 - 2}{n^2 + 4t^2}, \\ f_{23}(t) &= \sum_{n=1}^{\infty} \frac{(16t^2+1)n + 128t^4 + 8t^2}{(n^2+4t^2)^2}. \end{aligned} \tag{31}$$

The sum $f_{20}(t) = f_{21}(t) + f_{22}(t)$ equals

$$f_{20}(t) = 4 - (16t^2+3) \sum_{n=1}^{\infty} \frac{1}{n^2+4t^2}. \tag{32}$$

Taking into account (19) we have

$$f_{20}(t) = 4 - (16t^2+3)\left(-\frac{1}{8t^2} + \frac{\pi}{4t}\coth 2\pi t\right). \tag{33}$$

The function $f_{23}(t)$ (31) is

$$\begin{aligned} f_{23}(t) = (16t^2+1) \underbrace{\sum_{n=1}^{\infty} \frac{n}{(n^2+4t^2)^2}}_{=h_3(t)} \\ + (128t^4+8t^2)\left(\underbrace{\sum_{n=0}^{\infty} \frac{1}{(n^2+4t^2)^2}}_{=h_2(t)} - \frac{1}{16t^4}\right). \end{aligned} \tag{34}$$

Taking into account (19) we have

$$\begin{aligned} f_{23}(t) = (16t^2+1)h_3(t) + (128t^4+8t^2) \\ \times \left(-\frac{1}{32t^4} + \frac{\pi}{32t^3}\coth 2\pi t + \frac{\pi^2}{16t^2\sinh^2 2\pi t}\right). \end{aligned} \tag{35}$$

Hence, by (33) and (35),

$$\begin{aligned} f_{20}(t) + f_{23}(t) &= 6 + \frac{3}{8t^2} - (16t^2+3)\frac{\pi}{4t}\coth 2\pi t \\ &+ (16t^2+1)h_3(t) + (16t^2+1) \\ &\times \left(-\frac{1}{4t^2} + \frac{\pi}{4t}\coth 2\pi t + \frac{\pi^2}{2\sinh^2 2\pi t}\right) \\ &= 2 + \frac{1}{8t^2} - \frac{\pi}{2t}\coth 2\pi t \\ &+ (16t^2+1)h_3(t) + (16t^2+1)\frac{\pi^2}{2\sinh^2 2\pi t}. \end{aligned} \tag{36}$$

Thus,

$$F_2(\sigma,t) \geqslant \frac{1}{(16t^2+1)^2}\big((16t^2+1) - 4\pi t \coth 2\pi t\big) + \frac{1}{(16t^2+1)^2}\left((16t^2+1)8t^2 h_3(t) + \frac{4\pi^2 t^2(16t^2+1)}{\sinh^2 2\pi t}\right). \tag{37}$$

Next consider the function $F_3(\sigma,t)$ (18),

$$\begin{aligned} F_3(\sigma,t) &\leqslant \sum_{n=0}^{\infty} \frac{n+1}{((n+1)^2+4t^2)((n+2-2\sigma)^2+4t^2)} \\ &\leqslant \sum_{n=0}^{\infty} \frac{n+1}{((n+1)^2+4t^2)(n^2+4t^2)} \\ &= \sum_{n=0}^{\infty} \frac{1}{16t^2+1}\left(\frac{n+1-8t^2}{(n+1)^2+4t^2} + \frac{-n+1+8t^2}{n^2+4t^2}\right) \\ &= \frac{2}{16t^2+1} + \frac{1}{16t^2+1}\underbrace{\sum_{n=0}^{\infty}\frac{1}{n^2+4t^2}}_{=h_1(t)} \\ &= \frac{1}{16t^2+1}\left(2 + \frac{1}{8t^2} + \frac{\pi}{4t}\coth 2\pi t\right). \end{aligned} \tag{38}$$

Now we can give the lower bound of the function $\Omega(\sigma,t)$ (18). By (29), (37) and (38) we have

$$\begin{aligned} \Omega(\sigma,t) &\geqslant \frac{1}{8t^2} - \frac{32t^2+1}{2t(16t^2+1)^2}\pi \coth 2\pi t \\ &\quad - \frac{\pi^2}{2(16t^2+1)\sinh^2 2\pi t} \\ &\geqslant \left(\frac{1}{8} - \frac{c_2\pi^2}{32}\right)\frac{1}{t^2} - \frac{c_1\pi}{16t^3}, \end{aligned} \tag{39}$$

for $t \geqslant t_0$. Here $t_0 = 1.578$, $c_1 = \coth 2\pi t_0$, $c_2 = \sinh^{-2} 2\pi t_0$. Hence (cf. 17),

$$\Theta''_{\sigma\sigma}(\sigma,t) \geqslant (2\sigma-1)\left(\left(2 - \frac{c_2\pi^2}{2}\right)\frac{1}{t^2} - \frac{c_1\pi}{t^3}\right). \tag{40}$$

Let us calculate partial derivatives of the function $L(\sigma,t)$ with respect to the variable σ. The first derivative is

$$\begin{aligned} L'_\sigma(\sigma,t) &= \Theta'_\sigma(\sigma,t) + 2\log\frac{1}{2\pi} + \frac{-\pi t}{3}\,\frac{\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma} \\ &\quad + \frac{\pi}{6}(2\sigma-1)\frac{-\sin 2\pi\sigma}{\cosh 2\pi t - \cos 2\pi\sigma} + \frac{-\pi \sin \pi\sigma \cosh \pi t}{\cosh 2\pi t - \cos 2\pi\sigma} \\ &\quad + \frac{\pi}{3\sqrt{3}}\,\frac{1 - 2\cosh\frac{2\pi t}{3}\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}{\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}\right)}. \end{aligned} \tag{41}$$

The second derivative is

$$\begin{aligned}
L''_{\sigma\sigma}(\sigma,t) &= \Theta''_{\sigma\sigma}(\sigma,t)+\\
&+\underbrace{\frac{-\pi}{3}\frac{\sin 2\pi\sigma+\pi(2\sigma-1)\cos 2\pi\sigma+3\pi\cos\pi\sigma\cosh\pi t}{\cosh 2\pi t-\cos 2\pi\sigma}}_{=G_1(\sigma,t)}\\
&+\underbrace{\frac{\pi^2}{3}\sin 2\pi\sigma\frac{(2\sigma-1)\sin 2\pi\sigma+6\sin\pi\sigma\cosh\pi t+2t\sinh 2\pi t}{(\cosh 2\pi t-\cos 2\pi\sigma)^2}}_{=G_2(\sigma,t)}\\
&+\underbrace{\frac{2\pi^2}{9\sqrt{3}}\frac{\sin\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)}{\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi(\sigma-1)}{3}\right)^2\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi\sigma}{3}\right)^2}}_{=G_3(\sigma,t)}\\
&\times\underbrace{\left(2\cos\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)-\cosh\frac{2\pi t}{3}\left(\frac{7}{2}+\cos\left(\frac{4\pi\sigma}{3}-\frac{2\pi}{3}\right)\right)+2\cosh^3\frac{2\pi t}{3}\right)}_{=G_4(\sigma,t)}.
\end{aligned} \tag{42}$$

Let us give a lower bounds of the functions $G_k(\sigma,t)$, which are defined in (42). Denote

$$v_1(\sigma,t)=\cosh 2\pi t-\cos 2\pi\sigma,$$
$$u_1(\sigma,t)=\sin 2\pi\sigma+\pi(2\sigma-1)\cos 2\pi\sigma+3\pi\cos\pi\sigma\cosh\pi t.$$

Note that $v_1(\sigma,t)$ is positive for $t\in\mathbb{R}$ and $1/2<\sigma<1$. Next, $u_1(\sigma,t)$ is negative for $t\in\mathbb{R}$ and $1/2<\sigma\leqslant 3/4$. For $3/4<\sigma<1$, we have $0<\cos 2\pi\sigma<1$ and $\cos\pi\sigma<0$. Hence,

$$u_1(\sigma,t)\leqslant\underbrace{\sin 2\pi\sigma+\pi(2\sigma-1)+3\pi\cos\pi\sigma}_{=w_1(\sigma)}.$$

The function $w_1(\sigma)$ is convex, because

$$w_1''(\sigma)=-4\pi^2\underbrace{\sin 2\pi\sigma}_{\leqslant 0}-3\pi^3\underbrace{\cos\pi\sigma}_{\leqslant 0}>0,$$

for $3/4<\sigma<1$. With $w_1(3/4)<0$ and $w_1(1)<0$ it yields us $u_1(\sigma,t)<0$ for $t\in\mathbb{R}$ and $1/2<\sigma<1$. Thus,

$$G_1(\sigma,t)=-\frac{\pi}{3}\frac{u_1(\sigma,t)}{v_1(\sigma,t)}>0 \tag{43}$$

for $t\in\mathbb{R}$ and $1/2<\sigma<1$.

The function

$$\begin{aligned}
G_2(\sigma,t) &> \frac{2\pi^2}{3}\sin 2\pi\sigma\frac{3\cosh\pi t+t\sinh 2\pi t}{(\cosh 2\pi t-1)^2}\\
&\geqslant\frac{d_1\pi^2}{6}\delta(\sigma)\frac{2t+3d_2}{\sinh^2\pi t}
\end{aligned} \tag{44}$$

for $t \geqslant t_0$ and $1/2 < \sigma < 1$. Here $d_1 = \coth \pi t_0$, $d_2 = 1/\sinh \pi t_0$ and

$$\delta(\sigma) = \begin{cases} \pi(1-2\sigma), & \text{for } 1/2 < \sigma \leqslant 3/4, \\ 2\pi(\sigma - 1), & \text{for } 3/4 < \sigma < 1. \end{cases}$$

The function $G_3(\sigma, t)$ is positive for $1/2 < \sigma < 1$.

Next let us show that $G_4(\sigma, t)$ is increasing by σ and by t. Indeed, derivatives

$$\begin{aligned}(G_4)'_\sigma &= -\frac{4\pi}{3}\sin\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right) + \frac{4\pi}{3}\cosh\frac{2\pi t}{3}\sin\left(\frac{4\pi\sigma}{3} - \frac{2\pi}{3}\right) \\ &= \frac{4\pi}{3}\underbrace{\sin\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}_{>0}\left(\underbrace{2\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}_{\in(1,2)}\underbrace{\cosh\frac{2\pi t}{3}}_{\geqslant 1} - 1\right) > 0\end{aligned}$$

and

$$(G_4)'_t = \frac{2\pi}{3}\sinh\frac{2\pi t}{3}\left(6\cosh^2\frac{2\pi t}{3} - \left(\frac{7}{2} + \cos\left(\frac{4\pi\sigma}{3} - \frac{2\pi}{3}\right)\right)\right) > 0$$

for $t > 0$. Thus $G_4(\sigma, t) > G_4(1/2, 1/2) > 0$.

Let us consider the second partial derivative $L''_{\sigma\sigma}$ (42). We have shown that the function $G_1(\sigma, t)$ (43) and the product $G_3(\sigma, t)G_4(\sigma, t)$ are positive for $t > 0$. Hence, using (40) and (44), we obtain

$$L''_{\sigma\sigma}(\sigma, t) \geqslant \underbrace{(2\sigma - 1)\left(\left(2 - \frac{c_2\pi^2}{2}\right)\frac{1}{t^2} - \frac{c_1\pi}{t^3}\right) + \frac{d_1\pi^2}{6}\delta(\sigma)\frac{2t + 3d_2}{\sinh^2 \pi t}}_{=D(\sigma,t)}.$$

Let us show that $D(\sigma, t)$ is increasing by σ. Indeed, consider the derivative

$$D'_\sigma(\sigma, t) = \frac{4 - c_2\pi^2}{t^2} - \frac{2c_1\pi}{t^3} + \delta'(\sigma)\underbrace{\frac{d_1\pi^2}{6}\frac{2t + 3d_2}{\sinh^2 \pi t}}_{>0},$$

here

$$\delta'(\sigma) = \begin{cases} -2\pi, & \text{for } 1/2 < \sigma \leqslant 3/4, \\ 2\pi, & \text{for } 3/4 < \sigma < 1. \end{cases}$$

Note that $D'_\sigma(\sigma_1, t) < D'_\sigma(\sigma_2, t)$ for $1/2 < \sigma_1 < 3/4$ and $3/4 < \sigma_2 < 1$, hence, it is enough to consider the interval $1/2 < \sigma < 3/4$.

For $1/2 < \sigma < 1$ we have

$$\begin{aligned}D'_\sigma(\sigma, t) &= \frac{4 - c_2\pi^2}{t^2} - \frac{2c_1\pi}{t^3} - \frac{d_1\pi^3}{3}\frac{2t + 3d_2}{\sinh^2 \pi t} \\ &= \frac{1}{t^3}\left(a_1 t - a_2 - \underbrace{\frac{a_3 t^4 + a_4 t^3}{\sinh^2 \pi t}}_{=E(t)>0}\right).\end{aligned}$$

Here $a_1 = 4 - c_2\pi^2 = 3.999...$, $a_2 = 2c_1\pi = 6.283...$, $a_3 = 2d_1\pi^3/3 = 20.672...$ and $a_4 = d_1\pi^3 = 31.009...$. Since the function $E(t)$ is decreasing for $t \geqslant t_0$, we have

$$D'_\sigma(\sigma,t) \geqslant \frac{1}{t^3}\left(a_1 t - a_2 - E(t_0)\right) > 0$$

for $t > (a_2+E(t_0))/a_1 = 1.577....$ Hence, $D'_\sigma(\sigma,t) \geqslant 0$ for $t \geqslant t_0$, and $D(\sigma,t) > D(1/2,t) = 0$ yielding us the statement of the lemma.

Lemma 2 *For $1/2 < \sigma < 1$, the derivative $L'_t(\sigma,t)$*

(i) *is positive* for $t \in (0,\theta_1]$,
(ii) *is negative* for $t \in [\theta_2,\infty)$.

Here $\theta_1 = 3.53$ and $\theta_2 = 3.83$.

Proof Let us calculate the first partial derivative

$$\begin{aligned} L'_t(\sigma,t) &= \underbrace{\Theta'_t(\sigma,t)}_{=N_1(\sigma,t)} + \underbrace{\frac{\pi}{3}\frac{t\sin 2\pi\sigma}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_2(\sigma,t)} \\ &+ \underbrace{\frac{\pi}{6}(2\sigma-1)\frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_3(\sigma,t)} + \underbrace{\frac{-\pi\cos\pi\sigma\sinh\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_4(\sigma,t)} \\ &+ \underbrace{\frac{2\pi}{9}\frac{\sinh\frac{2\pi t}{3}\left(\cos\frac{2\pi(\sigma-1)}{3} - \cos\frac{2\pi\sigma}{3}\right)}{\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}\right)}}_{=N_5(\sigma,t)} . \end{aligned} \tag{45}$$

Estimating $N_1(\sigma,t)$ we obtain (cf. (14)) for $1/2 < \sigma < 1$ and $t > 0$,

$$\begin{aligned} N_1(\sigma,t) = \Theta'_t(\sigma,t) &= \sum_{n=0}^{\infty}\frac{4t}{(2-2\sigma+n)^2+4t^2} - \sum_{n=0}^{\infty}\frac{4t}{(2\sigma+n)^2+4t^2} \\ &= \underbrace{16t(2\sigma-1)}_{>0}\sum_{n=0}^{\infty}\frac{n+1}{\underbrace{\left((2-2\sigma+n)^2+4t^2\right)}_{>0}\underbrace{\left((2\sigma+n)^2+4t^2\right)}_{>0}}. \end{aligned} \tag{46}$$

Estimating the numerator and the denominator in $N_5(\sigma,t)$ we have for $1/2 < \sigma < 1$ and $t > 0$,

$$\sinh\frac{2\pi t}{3}\left(\cos\frac{2\pi(\sigma-1)}{3} - \cos\frac{2\pi\sigma}{3}\right) = \frac{\sqrt{3}}{2}\underbrace{\sinh\frac{2\pi t}{3}}_{>0}\underbrace{\sin\frac{2\pi\sigma-\pi}{3}}_{>0}, \tag{47}$$

and

$$\begin{aligned} &\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cos\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}\right) \\ &\quad > \underbrace{\left(1-\cos\frac{2\pi(\sigma-1)}{3}\right)}_{>0}\underbrace{\left(1-\cos\frac{2\pi\sigma}{3}\right)}_{>0} > 0. \end{aligned} \tag{48}$$

Combining (46), (47) and (48) we get $N_1(\sigma,t)>0$ and $N_5(\sigma,t)>0$. Now let us show that $N_2(\sigma,t)+N_3(\sigma,t)+N_4(\sigma,t)>0$. It is sufficient to prove that

$$\underbrace{\frac{1}{3}t\sin 2\pi\sigma-\frac{2\sigma-1}{6}\sinh 2\pi t-\cos\pi\sigma\sinh\pi t}_{=N(\sigma,t)}>0.$$

The derivative is

$$\begin{aligned}N_t^{'}(\sigma,t)=&\frac{1}{3}\sin 2\pi\sigma-\frac{2\sigma-1}{3}\pi\cosh 2\pi t-\pi\cos\pi\sigma\cosh\pi t\\ =&-\frac{1}{3}(2\pi(2\sigma-1)\cosh^2\pi t+3\pi\cos\pi\sigma\cosh\pi t-(\sin 2\pi\sigma+(2\sigma-1)\pi)).\end{aligned}$$

The positive root of the last quadratic equation is

$$r(\sigma)=\frac{-3\pi\cos\pi\sigma+\sqrt{9\pi^2\cos^2\pi\sigma+8\pi(2\sigma-1)\sin 2\pi\sigma+8\pi^2(2\sigma-1)^2}}{4\pi(2\sigma-1)}.$$

For $1/2<\sigma<1$,

$$\frac{3+\sqrt{17}}{4}<r(\sigma)<\frac{3\pi}{4}.$$

For $0<t\leqslant 0.37$

$$1<\cosh 2\pi t\leqslant 1.755<\frac{3+\sqrt{17}}{4}.$$

Hence, $N_t^{'}(\sigma,t)>0$ and $N(\sigma,t)>N(\sigma,0)=0$, yielding us the statement of the lemma for $t\in(0,0.37]$.

Consider the first partial derivative in the interval $t\in[0.37,3.53]$,

$$\begin{aligned}L_t^{'}(\sigma,t)=&(2\sigma-1)\frac{2t}{\sigma^2+t^2}+\underbrace{\frac{\pi}{3}\frac{\cos\pi\sigma(2t\sin\pi\sigma-3\sinh\pi t)}{\cosh 2\pi t-\cos 2\pi\sigma}}_{=H_2(\sigma,t)}\\ &+\underbrace{\frac{\pi}{6}(2\sigma-1)\frac{-\sinh 2\pi t}{\cosh 2\pi t-\cos 2\pi\sigma}}_{=H_3(\sigma,t)}\\ &+\underbrace{\frac{2\pi}{9}\frac{\sinh\frac{2\pi t}{3}\left(\cos\frac{2\pi(\sigma-1)}{3}-\cos\frac{2\pi\sigma}{3}\right)}{\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi\sigma}{3}\right)}}_{=H_4(\sigma,t)}.\end{aligned}\qquad(49)$$

Estimating $H_2(\sigma,t)$ and $H_4(\sigma,t)$, we obtain that $H_2(\sigma,t)>0$ and $H_4(\sigma,t)>0$, for $1/2<\sigma<1$. Thus,

$$\begin{aligned} L_t^{'}(\sigma,t) >&(2\sigma-1)\frac{2t}{\sigma^2+t^2}+\frac{\pi}{6}(2\sigma-1)\frac{-\sinh 2\pi t}{\cosh 2\pi t-\cos 2\pi\sigma} \\ >&(2\sigma-1)\left(\frac{2t}{\sigma^2+t^2}-\frac{\pi}{6}\frac{\sinh 2\pi t}{\cosh 2\pi t-1}\right) \\ =&(2\sigma-1)\underbrace{\left(\frac{2t}{\sigma^2+t^2}-\frac{\pi}{6}\coth \pi t\right)}_{=M(\sigma,t)}. \end{aligned}$$

The function $M(\sigma,t)$ is decreasing by σ, hence

$$M(\sigma,t)\geqslant \frac{2t}{1+t^2}-\frac{\pi}{6}\coth\pi t\geqslant \underbrace{\frac{2t}{1+t^2}-\frac{\pi}{6}A_k}_{=M_k(t)}.$$

Here

$$A_k=\begin{cases}\coth 0.37\pi & \text{for } 0.37\leqslant t<2.77,\\ \coth 2.77\pi & \text{for } 2.77\leqslant t\leqslant 3.53.\end{cases}$$

Now

$$M_k'(t)=2\frac{1-t^2}{(1+t^2)^2}.$$

For $0.37\leqslant t<2.77$ the function $M_1(t)$ increases in the interval $(0.37,1)$ and decreases in the interval $(1,2.77)$. At endpoints the function is positive, $M_1(0.37)>0$ and $M_1(2.77)>0$.

For $2.77\leqslant t\leqslant 3.53$ the function $M_2(t)$ decreases. At the endpoint the function is positive, $M_2(3.53)>0$, yielding us the statement of the lemma for the interval $0.37\leqslant t\leqslant 3.53$.

Estimating functions $N_k(\sigma,t)$ in (45) for $t\geqslant\theta_2$ we obtain

$$N_1(\sigma,t)<16t(2\sigma-1)\underbrace{\sum_{n=0}^{\infty}\frac{n}{(n^2+4t^2)^2}}_{=h_3(t)}.$$

Using Euler-Maclaurin formula, we have

$$h_3(t)\sim\frac{1}{8t^2}-\frac{1}{192t^4}+\ldots,$$

hence

$$N_1(\sigma,t)<(2\sigma-1)\frac{2}{t}.$$

Next,

$$N_3(\sigma,t)<(2\sigma-1)\underbrace{\frac{\pi}{6}\frac{-\sinh 2\pi\theta_2}{\cosh 2\pi\theta_2+1}}_{=c_3},$$

$$N_4(\sigma,t) < -\pi\cos\pi\sigma\frac{\sinh\pi\theta_2}{\cosh 2\pi\theta_2-1} < (2\sigma-1)\underbrace{\frac{\sinh\pi\theta_2}{\cosh 2\pi\theta_2-1}\frac{\pi^2}{2}}_{=c_4},$$

and

$$\begin{aligned}N_5(\sigma,t) &< \frac{2\pi}{3\sqrt{3}}\sin\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)\frac{\sinh\frac{2\pi\theta_2}{3}}{\left(\cosh\frac{2\pi\theta_2}{3}-1\right)\left(\cosh\frac{2\pi\theta_2}{3}-\frac{1}{2}\right)}\\ &< (2\sigma-1)\underbrace{\frac{2\pi^2}{9\sqrt{3}}\frac{\sinh\frac{2\pi\theta_2}{3}}{\left(\cosh\frac{2\pi\theta_2}{3}-1\right)\left(\cosh\frac{2\pi\theta_2}{3}-\frac{1}{2}\right)}}_{=c_5}.\end{aligned}$$

Noticing that $N_2(\sigma,t)<0$, we obtain

$$L_t'(\sigma,t) < (2\sigma-1)\left(\frac{2}{t}+\underbrace{c_3+c_4+c_5}_{=-0.523...}\right) < 0$$

for $t \geqslant 3.83$, yielding us the statement of the lemma.

4 Auxiliary lemmas

Two following lemmas will be useful in the proof of Theorem 3.

Let us denote

$$\Psi_1(t) = L(1,t). \tag{50}$$

Lemma 3 *The function $\Psi_1(t)$*

(i) *is negative* for $t\in(0,t_1)\cup(t_2,\infty)$,
(ii) *is positive* for (t_1,t_2),
(iii) *has unique maximum point* at $t^*\in(t_1,t_2)$.

Here $t_1 = 2.233...$ and $t_2 = 5.967...$ are the roots of the function $\Psi_1(t)$.

Proof By (50) and (11) we obtain

$$\begin{aligned}\Psi_1(t) &= \Theta(1,t) - \left(\log(2\pi)+\frac{\log 2}{12}\right) - \frac{1}{12}\log(\cosh 2\pi t-1)\\ &+ \frac{1}{4}\log\frac{\cosh\pi t-1}{\cosh\pi t+1} + \frac{1}{3}\log\frac{\cosh\frac{2\pi t}{3}-1}{\cosh\frac{2\pi t}{3}+\frac{1}{2}}.\end{aligned} \tag{51}$$

Next, by (12),

$$\begin{aligned}\Theta(1,t) &= \log\left|\frac{\Gamma(2+2it)}{\Gamma(-2it)}\right|\\ &= \log\underbrace{\frac{|\Gamma(2it)|}{|\Gamma(-2it)|}}_{=1} + \log|(1+2it)(2it)|\\ &= \log|2it-4t^2| = \frac{1}{2}\log(4t^2+1)+\log 2t.\end{aligned} \tag{52}$$

Hence,

$$\Psi_1(t) = \frac{1}{2}\log(1+4t^2) + \log\frac{t}{\pi} - \frac{1}{6}\log 2 - \frac{1}{6}\log(\sinh \pi t)$$

$$+ \frac{1}{2}\log\frac{\sinh\frac{\pi t}{2}}{\cosh\frac{\pi t}{2}} + \frac{1}{3}\log\frac{4\sinh^2\frac{\pi t}{3}}{4\sinh^2\frac{\pi t}{3}+3} \tag{53}$$

$$= \frac{1}{2}\log(1+4t^2) + \log\frac{t}{\pi} + \log\frac{\sinh\frac{\pi t}{3}}{\cosh\frac{\pi t}{2}}.$$

Having that

$$\frac{\cosh\frac{\pi t}{2}}{2\sinh\frac{\pi t}{3}} = \frac{\left(2\sinh\frac{\pi t}{6}\right)^2+1}{2\left(2\sinh\frac{\pi t}{6}\right)} = \cosh\log\left(2\sinh\frac{\pi t}{6}\right),$$

we obtain

$$\Psi_1(t) = \frac{1}{2}\log(1+4t^2) + \log\frac{t}{2\pi} - \log\cosh\log\left(2\sinh\frac{\pi t}{6}\right). \tag{54}$$

Note that

$$\lim_{t\to 0+}\Psi_1(t) = -\infty, \qquad \lim_{t\to +\infty}\Psi_1(t) = -\infty. \tag{55}$$

Next, $\Psi_1(t) = 0$ iff

$$\underbrace{\cosh\log\left(2\sinh\frac{\pi t}{6}\right)}_{\geqslant 1} = \frac{t\sqrt{1+4t^2}}{2\pi}. \tag{56}$$

Hence, there are no zeros in the interval $(0, \theta_0)$, since the function $\Psi_1(t)$ is negative in the interval (cf. (56)). Here

$$\theta_0 = \sqrt{\sqrt{\pi^2+1/64}-1/8} = 1.738...$$

By (53),

$$\Psi_1(t) = \underbrace{\frac{1}{2}\log(1+4t^2) + \log\frac{t}{\pi} - \frac{\pi t}{6}}_{=P_1(t)}$$

$$+ \underbrace{\log\left(1 - \frac{1}{e^{\frac{2\pi t}{3}} - e^{\frac{\pi t}{3}} + 1}\right)}_{=P_2(t)}. \tag{57}$$

The function $P_1(t)$ is concave. Indeed, consider the second derivative,

$$P_1''(t) = -\frac{32t^4+4t^2+1}{t^2(1+4t^2)^2} < 0. \tag{58}$$

Next, consider the second derivative of $P_2(t)$,

$$P_2''(t) = \frac{-\pi^2 e^{\frac{2\pi t}{3}}}{9\left(e^{\frac{2\pi t}{3}} - e^{\frac{\pi t}{3}} + 1\right)^2 \left(e^{\frac{\pi t}{3}} - 1\right)^2} \times \underbrace{\left(4e^{\frac{2\pi t}{3}} - 7e^{\frac{\pi t}{3}} + 4\right)}_{>0} < 0. \tag{59}$$

Combining (55) and (56) we obtain that the function $\Psi_1(t)$ is concave. Thus, the concave function on an open set takes negative, then positive (e.g. $\Psi_1(3) > 0$), then again negative values (cf. (55)). Hence, it has unique positive maximum in the interval (t_1, t_2). Here t_1 and t_2 are the roots of the function $L_1(t)$. The values of the roots we obtain numerically with any sufficient accuracy.

Next, let us denote

$$\Psi_0(\sigma) = L(\sigma, 0). \tag{60}$$

Lemma 4 *For $1/2 < \sigma < 1$, the function $\Psi_0(\sigma)$ is negative.*

Proof By (7), (11) and (14) we have

$$\Psi_0(\sigma) = \underbrace{\log \frac{\Gamma(2\sigma)}{\Gamma(2-2\sigma)}}_{=H(\sigma)} + \underbrace{(1-2\sigma)\log 2\pi}_{<0} + Q(\sigma).$$

For $1/2 < \sigma < 1$, we have $0 < \Gamma(2\sigma) < 1$ and $1 < \Gamma(2-2\sigma) < \infty$, hence $0 < \Gamma(2\sigma)/\Gamma(2-2\sigma) < 1$ and $H(\sigma) < 0$.

The derivative of the function $Q(\sigma)$ (7) is

$$\begin{aligned} Q'(\sigma) &= -\frac{\pi}{6}(2\sigma - 1)\cot \pi\sigma - \frac{\pi}{2\sin \pi\sigma} \\ &\quad - \frac{4\pi}{3\sqrt{3}} \frac{\cos\left(\frac{\pi\sigma}{3} - \frac{\pi}{6}\right)}{\sin \pi\sigma} \\ &= \underbrace{\frac{-\pi}{6\sin \pi\sigma}((2\sigma - 1)\cos \pi\sigma + 3)}_{=Q_1(\sigma)} \\ &\quad + \underbrace{\frac{4\pi}{3\sqrt{3}} \frac{1}{1 - 2\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}}_{=Q_2(\sigma)}. \end{aligned} \tag{61}$$

Let us give an upper bounds of the functions $Q_1(\sigma)$ and $Q_2(\sigma)$. For $1/2 < \sigma < 1$, the function

$$Q_1(\sigma) = \underbrace{\frac{-\pi}{6\sin \pi\sigma}}_{<0} \underbrace{((2\sigma - 1)\cos \pi\sigma + 3)}_{>0} < 0.$$

For $1/2 < \sigma < 1$, the function

$$\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right) > \frac{1}{2},$$

hence $Q_2(\sigma) < 0$. With $Q_1(\sigma)$ and $Q_2(\sigma)$ both negative we obtain $\Psi_0'(\sigma) < 0$, with $\Psi_0(1/2) = 0$ yielding us the statement of the lemma.

5 Proof of the theorem for the modified Selberg zeta-function

Now we can prove the Theorem 3.

Proof Consider *max* value of the function $L(\sigma, t)$ in the rectangle $(\sigma, t) \in (1/2, 1) \times (0, t_1)$. By Lemma 2, the function $L(\sigma, t)$ has no stationary points in the interior of the rectangle, so it suffices to investigate the behaviour of the function on vertices of the rectangle. By Lemma 1, the function $L(\sigma, t)$ is convex by σ and the derivative by t is positive, hence we must consider the first zero of the function $\Psi_1(t)$ (cf. Lemma 3). Note that

$$\lim_{\sigma \to 1/2+} \Psi(\sigma, t) = 0. \tag{62}$$

Next let us consider *max* value of the function $L(\sigma, t)$ in the strip $(\sigma, t) \in (1/2, 1) \times (t_2, \infty)$. By Lemma 2, function $L(\sigma, t)$ has no stationary points in the interior of the strip, so it suffices to investigate the behaviour of the function on vertices. By Lemma 1, the function $L(\sigma, t)$ is convex by σ and the derivative by t is negative, hence we must consider the second zero of the function $\Psi_1(t)$ (cf. Lemma 3). By Lemma 4 and (62), it gives us $L(\sigma, t) < 0$. Consequently (cf. (10))

$$\log\left|\frac{W(s)}{W(1-s)}\right| < 0,$$

yielding us the statement of the theorem.

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