Nonlinear Analysis: Modelling and Control, Vol. 25, No. 2, 282–300
https://doi.org/10.15388/namc.2020.25.16518 ISSN: 1392-5113
eISSN: 2335-8963

Modeling the Dirichlet distribution using multiplicative functions

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Received: February 27, 2019 / **Revised:** August 27, 2019 / **Published online:** March 2, 2020

Abstract. For $q, m, n, d \in \mathbb{N}$ and some multiplicative function $f \geqslant 0$, we denote by $T_3(n)$ the sum of $f(d)$ over the ordered triples (q, m, d) with $qmd = n$. We prove that Cesaro mean of distribution functions defined by means of T_3 uniformly converges to the one-parameter Dirichlet distribution function. The parameter of the limit distribution depends on the values of f on primes. The remainder term is estimated as well.

Keywords: natural divisor, multiplicative function, Dirichlet distribution.

1 Introduction and result

Let a, b, c be positive constants and

$$E(u, v) := \bigl\{(s, t) \bigm| 0 \leqslant s \leqslant u,\ 0 \leqslant t \leqslant v,\ s + t \leqslant 1\bigr\}.$$

The two-dimensional Dirichlet distribution $\mathcal{D}(a, b, c)$ concentrated on the triangle $E(1, 1)$ is defined by the distribution function

$$D(u, v; a, b, c) = \frac{\Gamma(a + b + c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \iint\limits_{E(u,v)} \frac{\mathrm{d}t\,\mathrm{d}s}{s^{1-a}t^{1-b}(1 - t - s)^{1-c}},$$

where Γ denotes the Gamma function. The one-dimensional Dirichlet distribution is well-known Beta law $\mathcal{B}(a, b)$ with distribution function

$$B(u; a, b) := \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \int_0^u \frac{\mathrm{d}t}{t^{1-a}(1 - t)^{1-b}}, \quad u \in [0, 1].$$

A two-dimensional Dirichlet distribution usually is called as bivariate Beta distribution.

It is known various high efficiency algorithms for generating the Dirichlet random vectors (see, e.g., [6, 10, 11]). Usually, in this case, the efficiency is empirically estimated in terms of computer generation time.

In the probabilistic number theory, we have a slightly different problem. We need to construct a sequence of vectors, whose "average" distribution function tends to Dirichlet distribution. These "arithmetical" vectors are supposed to be related to the arithmetical functions. In this paper, we consider the construction of two-dimensional "arithmetical" vectors and the convergence of their distributions to some given Dirichlet distribution.

In the one-dimensional case, the first attempt to simulate $\mathcal{B}(1/2, 1/2)$, that is, the Arcsine law, by means of the divisor function was made in [9]. Manstavičius [12] noticed that not only the Arcsine law but also some other distributions can occur as a limits. Some ideas of these papers were extensively used for the simulation of some Beta related distributions by means of the divisor function with multiplicative weight (see [1, 2, 4, 5, 7, 8]). General result of this type for any Beta distribution $\mathcal{B}(a, b)$ was obtained in [3].

The number of ordered factorisations of $n \in \mathbb{N}$ into three factors is known as function $\tau_3(n) := \sum_{l_1 l_2 l_3 = n} 1$. For any multiplicative function $f : \mathbb{N} \to \mathbb{R}$, we define

$$T_3(n) := \sum_{d|n} f(d)\tau\left(\frac{n}{d}\right),$$

where $\tau(n) := \sum_{d|n} 1$ is the classical divisor function. If $f(n) \equiv 1$, then $T_3(n) = \tau_3(n)$.

Let $(X_n; Y_n)$ be the random vector, which takes values $(\ln d_1/\ln n; \ln d_2/\ln n)$ when $(d_1; d_2)$ run trough all ordered pairs of divisors of n with uniform probability $1/T_3(n)$. Its distribution function is

$$F_n(u, v) := \frac{1}{T_3(n)} \sum_{\substack{qm|n,\\ q \leqslant n^u,\, m \leqslant n^v}} f\left(\frac{n}{qm}\right).$$

This sequence of distributions does not converge pointwise on $[0, 1] \times [0, 1]$ (see [13]). Therefore, following [9], the corresponding Cesaro mean

$$S_x(u, v) := \frac{1}{x} \sum_{n \leqslant x} F_n(u, v)$$

can be considered. When $f \equiv 1$, Nyandwi and Smati [13] proved that $S_x(u, v)$ tends to the Dirichlet distribution function $D(u, v; 1/3, 1/3, 1/3)$. In this paper, we generalize the result of [13] showing that some class of the one-parameter Dirichlet distributions can be simulated by $S_x(u, v)$. The parameter of the limit distribution depends on the values of multiplicative function f on primes. More precisely, we assume that the values $f(p)$ satisfy some regularity conditions so that the multiplicative function $1/T_3$ belongs to the class $\mathcal{G}$:

Definition 1. We say that multiplicative function $\varphi : \mathbb{N} \to [0; \infty)$ belongs to the class $\mathcal{G}(\varkappa, \delta)$ for some constants $\varkappa, \delta \geqslant 0$ if $\varphi(p^k) \leqslant C$ for all primes p and $k \in \mathbb{N}$ and the function

$$L(s) := \sum_p \frac{\varphi(p) - \varkappa}{p^s}, \quad s = \sigma + \mathrm{i}\tau \in \mathbb{C}, \ \ \sigma > 1,$$

for some $0 < c_0 \leqslant 1/2$, has an analytic continuation $P(s)$ into the region

$$\sigma \geqslant \sigma(\tau) := 1 - \frac{c_0}{\ln(|\tau| + 3)},$$

where $P(s)$ is holomorphic, and $|P(s)| \leqslant \delta \log(|\tau| + 1) + c_1$ with some $c_0 \geqslant 0$.

We assume that c and C with or without subscripts denote constants.
The aim of this paper is to prove the following:

Theorem 1. *Let f be a nonnegative multiplicative function such that $1/T_3 \in \mathcal{G}(\alpha, \delta)$, $0 < \alpha < 1/2$ and $0 \leqslant \delta < 1/2$. Then for all $u, v \in [0, 1]$,*

$$S_x(u, v) = D(u, v; \alpha, \alpha, 1 - 2\alpha) + O\big(\rho_x(u + v)\big).$$

Here

$$\rho_x(z) := \begin{cases} \ln^{-\alpha} x & \text{if } z \leqslant 1, \\ \ln^{-\alpha} x + \ln^{2\alpha - 1} x & \text{if } z > 1. \end{cases}$$

Unless otherwise indicated, here and in what follows, we assume that $x \to \infty$, the implicit constants in the $\ll$ or $O(\cdot)$ symbols depend at most on the parameters and constants involved in the definitions of the corresponding classes $\mathcal{G}(\cdot)$.

2 Preliminaries

For $\varkappa > 0$ and any multiplicative function θ, set

$$A(\varkappa, \theta) := \frac{1}{\Gamma(\varkappa)} \prod_p \left(1 - \frac{1}{p}\right)^{\varkappa} \sum_{k=0}^{\infty} \frac{\theta(p^k)}{p^k}.$$

Here and in what follows, we assume that p is prime. A slight modification to the proof of Lemma 3.1 in [2] yields

Lemma 1. *Let φ and g be the nonnegative multiplicative functions such that*

$$0 \leqslant \varphi\big(p^j\big) g\big(p^k\big) \leqslant C_1 \tag{1}$$

for some $C_1 > 0$ and all integers j, k with $0 \leqslant j \leqslant k$. Assume furthermore that $\varphi \cdot g \in \mathcal{G}(\varkappa, \delta)$, $\varkappa > 0$ and $0 \leqslant \delta < 1$. Then, uniformly for all $x \geqslant 1$ and $d \in \mathbb{N}$,

$$\sum_{n \leqslant x} \varphi(n) g(nd) = \frac{x}{\ln^{1-\varkappa}(\mathrm{e}x)} \left(A(\varkappa, \varphi \cdot g) \cdot \tilde{h}(d | \varphi, g) + O\left(\frac{\hat{h}(d | \varphi, g)}{\ln(\mathrm{e}x)} \right) \right),$$

where the multiplicative functions $\tilde{h}$ *and* $\hat{h}$ *are defined by*

$$\begin{aligned} \tilde{h}(p^k|\varphi,g) &:= \left(\sum_{j=0}^{\infty}\frac{\varphi(p^j)g(p^j)}{p^j}\right)^{-1}\sum_{j=0}^{\infty}\frac{\varphi(p^j)g(p^{k+j})}{p^j},\\ \hat{h}(p^k|\varphi,g) &:= \left(1+\frac{c_2}{p^{\sigma_0}}\right)\sum_{j=0}^{\infty}\frac{\varphi(p^j)g(p^{k+j})}{p^{j\sigma_0}}. \end{aligned} \tag{2}$$

Here $\sigma_0=\sigma(0)$*, and* $c_2\geqslant 0$ *is a constant depending on* c_0*,* $\varkappa$ *and* C_1*.*

Remark. If (1) holds, then

$$\tilde{h}(p^k|\varphi,g)=g(p^k)+O(p^{-1}),\qquad \hat{h}(p^k|\varphi,g)=g(p^k)+O(p^{-\sigma_0})$$

for any $k\in\mathbb{N}$. Hence $\tilde{h},\hat{h}\in\mathcal{G}(\varkappa,\delta)$, provided $g\in\mathcal{G}(\varkappa,\delta)$. In the sequel, we will constantly use this property.

Lemma 2. (*See* [3].) *Let* $g\in\mathcal{G}(\varkappa,\delta)$ *with some* $\varkappa\geqslant 0$ *and* $0\leqslant\delta<1$*. Then, uniformly for all* $m\in\mathbb{N}$ *and* $x\geqslant 1$*,*

$$\sum_{n\leqslant x} g(nm)\ll x\cdot\hat{h}(m|1,g)\nu_x(\varkappa),$$

where

$$\nu_x(\varkappa):=\begin{cases}\mathrm{e}^{-c_3\sqrt{\ln x}} & \text{if } \varkappa=0,\\ \ln^{\varkappa-1}(\mathrm{e}x) & \text{if } \varkappa>0,\ c_3>0.\end{cases}$$

For $0\leqslant u\leqslant w\leqslant 1$, $x\geqslant 1$, $b\in\mathbb{R}$, we set

$$\Theta_x(u,w,b):=\sum_{x^u<m\leqslant x^w}\frac{a_m}{m\ln^b(\frac{\mathrm{e}x}{m})},\qquad a_m\geqslant 0.$$

This sum may be evaluated in terms of the integral

$$I(u,w;a,b,\eta):=\int_u^w\frac{\mathrm{d}v}{(\eta+v)^a(\eta+1-v)^b},$$

provided some information about the behaviour of the sum

$$M(v):=\sum_{m\leqslant v}a_m$$

is given. For $x>1$, set $\eta_x:=\ln^{-1}x$ and

$$r_x(u,w;a,b):=\frac{\eta_x}{(\eta_x+u)^a(\eta_x+1-u)^b}+\frac{\eta_x}{(\eta_x+w)^a(\eta_x+1-w)^b}.$$

The following consequence of Lemmas 3 and 4 in [3] will be applied to evaluate the sum $\Theta_x(u,w,b)$.

Lemma 3. *Assume that* $x \geqslant \mathrm{e}$ *and*

$$\left| M(v) - \frac{Av}{\ln^a(\mathrm{e}v)} \right| \leqslant \frac{Bv}{\ln^{a+1}(\mathrm{e}v)}$$

for some $A, B \geqslant 0$ *and all* $1 \leqslant v \leqslant x$. *If* $a \neq 0$ *and* $b \neq 1$, *then*

$$\begin{aligned} &\ln^{a+b} x \left| \Theta_x(u, w, b) - \frac{A}{\ln^{a+b-1} x} I(u, w; a, b, \eta_x) \right| \\ &\quad \ll (A + B)\bigl(1 + r_x(u, w; a, b-1) \ln x\bigr). \end{aligned}$$

The implicit constant in $\ll$ *symbol depends at most on* a *and* b.

We will need some estimates of the integrals

$$J_1(\varepsilon, \eta, u, v, a, b, c) := \int_\varepsilon^u \int_\varepsilon^v \frac{\mathrm{d}z\, \mathrm{d}s}{(\eta + s)^a (\eta + z)^b (\eta + 1 - s - z)^c},$$

$$J_2(\varepsilon, \eta, u, a, b, c) := \int_\varepsilon^u \int_\varepsilon^{1-s-\varepsilon} \frac{\mathrm{d}z\, \mathrm{d}s}{(\eta + s)^a (\eta + z)^b (\eta + 1 - s - z)^c}.$$

Lemma 4. *If* $a, b, c \in (0, 1)$, *then*

$$J_1(0, 0, u, v, a, b, c) \ll \min\bigl(u^{1-a}, v^{1-b}\bigr), \tag{3}$$

$$J_2(0, 0, u, a, b, c) \ll u^{1-a} \tag{4}$$

uniformly for $u, v \in [0, 1]$, $u + v \leqslant 1$. *The implied constant in* $\ll$ *depending on* a, b, c *only.*

Proof. Let us begin with the proof of (4). By the definition, for $0 \leqslant u < 1$,

$$\begin{aligned} J_2(0, 0, u, a, b, c) &= \int_0^u \frac{\mathrm{d}s}{s^a (1 - s)^{b+c-1}} \int_0^1 \frac{\mathrm{d}t}{t^b\, (1 - t)^c} \\ &= B(u; 1 - a, 2 - b - c)\, \frac{\Gamma(1 - a)\Gamma(1 - b)\Gamma(1 - c)}{\Gamma(3 - a - b - c)}. \end{aligned} \tag{5}$$

This implies (4) since $b + c < 2$. Further, observe that

$$J_1(0, 0, u, v, a, b, c) \leqslant J_1(0, 0, u, 1 - u, a, b, c) \leqslant J_2(0, 0, u, a, b, c)$$

and

$$J_1(0, 0, u, v, a, b, c) = J_1(0, 0, v, u, b, a, c) \leqslant J_2(0, 0, v, b, a, c).$$

Therefore (3) follows from (4). This completes the proof of Lemma 4. □

Lemma 5. *Let* $0 \leqslant \varepsilon \leqslant 1/4$, $0 \leqslant \eta \leqslant 1$ *and* $a, b > 0$, $a, b, c \neq 1$, $c \neq 2$. *Then*

$$J_1(\varepsilon, \eta, u, v, a, b, c) \ll 1 + (\varepsilon+\eta)^{\gamma_1} + \eta^{2-c}\big((\varepsilon+\eta)^{-a} + (\varepsilon+\eta)^{-b}\big)$$

uniformly for $u, v \in [\varepsilon, 1]$, $u + v \leqslant 1$. *Here*

$$\gamma_1 := \min(1-a,\, 1-b,\, 2-a-b,\, 2-a-c,\, 2-b-c),$$

the implied constant in $\ll$ *depending at most on a, b, c.*

Proof. For brevity, we set $J(u,v) := J_1(\varepsilon, \eta, u, v, a, b, c)$. Let $\varepsilon \leqslant u \leqslant 1/2$. Then

$$J(u,v) \leqslant J(u, 1-u) = J\left(u, \frac{1}{4}\right) + \Delta J, \tag{6}$$

where

$$\Delta J := \int_{\varepsilon}^{u} \frac{\mathrm{d}s}{(\eta+s)^a} \int_{1/4}^{1-u} \frac{\mathrm{d}z}{(\eta+z)^b(\eta+1-s-z)^c}.$$

It is clear that

$$\begin{aligned} J\left(u, \frac{1}{4}\right) &\leqslant 4^c \int_{\varepsilon}^{1/2} \frac{\mathrm{d}s}{(\eta+s)^a} \int_{\varepsilon}^{1/4} \frac{\mathrm{d}z}{(\eta+z)^b} \\ &\ll \big(1 + (\varepsilon+\eta)^{1-a}\big)\big(1 + (\varepsilon+\eta)^{1-b}\big). \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} \Delta J &\leqslant \frac{4^b}{1-c} \int_{\varepsilon}^{u} \left(\left(\eta + \frac{3}{4} - s\right)^{1-c} - (\eta+u-s)^{1-c}\right) \frac{\mathrm{d}s}{(\eta+s)^a} \\ &\ll 1 + (\varepsilon+\eta)^{1-a} + I, \end{aligned} \tag{8}$$

where

$$I = \int_{\varepsilon}^{u} \frac{\mathrm{d}s}{(\eta+s)^a(\eta+u-s)^{c-1}} = (\eta+u)^{2-a-c} \int_{\varepsilon/(\eta+u)}^{u/(\eta+u)} \left(\frac{\eta}{\eta+u} + t\right)^{-a} \frac{\mathrm{d}t}{(1-t)^{c-1}}$$

with $t = s/(\eta+u)$. Separately estimating the latter integral over $t \leqslant 1/2$ and $t > 1/2$, we obtain

$$I \ll \left(1 + \left(\frac{\varepsilon+\eta}{\eta+u}\right)^{1-a} + \left(\frac{\eta}{\eta+u}\right)^{2-c}\right)(\eta+u)^{2-a-c}.$$

From this, (8), (7) and (6) we deduce

$$J(u,v) \ll 1 + (\varepsilon+\eta)^{\beta} + \eta^{2-c}(\varepsilon+\eta)^{-a} \tag{9}$$

uniformly for $\varepsilon \leqslant u \leqslant 1/2$, $u + v \leqslant 1$. Here

$$\beta = \min(1 - a,\ 1 - b,\ 2 - a - b,\ 2 - a - c).$$

If $1/2 < u \leqslant 1$, then $\varepsilon \leqslant v \leqslant 1 - u < 1/2$, and the proof of Lemma 5 follows from (9) since $J_1(\varepsilon, \eta, u, v, a, b, c) = J_1(\varepsilon, \eta, v, u, b, a, c)$. □

Lemma 6. *Let* $0 \leqslant \varepsilon \leqslant 1/4$, $0 \leqslant \eta \leqslant 1$ *and all numbers* $a, b, c \neq 1$. *Then*

$$J_2(\varepsilon, \eta, u, a, b, c) \ll 1 + (\varepsilon + \eta)^{\gamma_2}$$

uniformly for $u \in [\varepsilon, 1 - 2\varepsilon]$. *Here* $\gamma_2 := \min(\gamma_1,\ 1 - c)$, *the implied constant in* $\ll$ *depending at most on a, b, c.*

Proof. Consider the inner integral of $J_2(\varepsilon, \eta, u, a, b, c)$

$$I(s) := \int\limits_{\varepsilon}^{1-s-\varepsilon} \frac{\mathrm{d}z}{(\eta + z)^b (\eta + 1 - s - z)^c}$$

when $1 - s \geqslant 2\varepsilon$. Set

$$t_1 = \frac{\varepsilon}{\eta + 1 - s}, \qquad t_2 = 1 - \frac{\eta + \varepsilon}{\eta + 1 - s}, \qquad t_3 = \min\left(\frac{1}{2}, t_2\right).$$

Changing integration variable z by $t = z/(\eta + 1 - s)$, we obtain

$$I(s) \ll (\eta + 1 - s)^{1-b-c} \left(\int\limits_{t_1}^{t_3} \left(\frac{\eta}{\eta + 1 - s} + t \right)^{-b} \mathrm{d}t + \int\limits_{t_3}^{t_2} \frac{\mathrm{d}t}{(1 - t)^c} \right).$$

Hence

$$I(s) \ll \frac{(\varepsilon + \eta)^{1-b}}{(\eta + 1 - s)^c} + \frac{(\varepsilon + \eta)^{1-c}}{(\eta + 1 - s)^b} + (\eta + 1 - s)^{1-b-c}. \tag{10}$$

Thus

$$J_2(\varepsilon, \eta, u, a, b, c) \leqslant \int\limits_{\varepsilon}^{1-2\varepsilon} \frac{I(s)}{(\eta + s)^a}\, \mathrm{d}s \ll (\eta + \varepsilon)^{\gamma_2} + 1.$$

□

Lemma 7. *Let* $0 \leqslant \eta \leqslant 1$ *and* $a, b, c \in (0, 1)$. *Then*

$$J_1(0, 0, u, v, a, b, c) - J_1(0, \eta, u, v, a, b, c) \ll \eta^{1-a} + \eta^{1-b} \tag{11}$$

and

$$J_2(0, 0, u, a, b, c) - J_2(0, \eta, u, a, b, c) \ll \eta^{1-a} + \eta^{1-b} + \eta^{1-c} \tag{12}$$

uniformly for $u, v \in [0, 1]$, $u + v \leqslant 1$. *The implied constant in* $\ll$ *depending at most on a, b, c.*

Proof. We may assume that $\eta > 0$. Let $0 < \varepsilon \leqslant 1/4$. For short, set

$$\begin{aligned}\Delta_1(\varepsilon, u, v) &:= J_1(\varepsilon, 0, u, v, a, b, c) - J_1(\varepsilon, \eta, u, v, a, b, c),\\ \Delta_2(\varepsilon, u) &:= J_2(\varepsilon, 0, u, a, b, c) - J_2(\varepsilon, \eta, u, a, b, c).\end{aligned}$$

Firstly, we prove (11). If $\min(u, v) \leqslant \varepsilon$, then by Lemma 4

$$\Delta_1(0, u, v) \leqslant J_1(0, 0, u, v, a, b, c) \ll \varepsilon^{1-a} + \varepsilon^{1-b}. \tag{13}$$

Now assume that $u, v \in (\varepsilon,\ 1]$. We have

$$\Delta_1(0, u, v) - \Delta_1(\varepsilon, u, v) = \Delta_1(0, \varepsilon, v) + \Delta_1(0, u, \varepsilon) - \Delta_1(0, \varepsilon, \varepsilon).$$

This and (13) yield

$$\Delta_1(0, u, v) \ll \Delta_1(\varepsilon, u, v) + \varepsilon^{1-a} + \varepsilon^{1-b}. \tag{14}$$

According to the Lagrange mean value theorem,

$$\begin{aligned}\Delta_1(\varepsilon, u, v) = \eta a J_1(\varepsilon, t, u, v, a+1, b, c) + \eta b J_1(\varepsilon, t, u, v, a, b+1, c)\\ + \eta c J_1(\varepsilon, t, u, v, a, b, c+1)\end{aligned} \tag{15}$$

with some $t \in (0, \eta)$. We can estimate the integrals on the right-hand side using Lemma 5. It gives

$$J_1(\varepsilon, t, u, v, a+1, b, c) \leqslant J_1(\varepsilon, 0, u, v, a+1, b, c) \ll \varepsilon^{-a}.$$

Similarly, the next two integrals are $\ll \varepsilon^{-b}$ and $\ll \varepsilon^{-a} + \varepsilon^{-b}$. Thus

$$\Delta_1(\varepsilon, u, v) \ll \eta\bigl(\varepsilon^{-a} + \varepsilon^{-b}\bigr).$$

This inequality together with (14) and (13) imply (11) by choosing $\varepsilon = \eta/4$.

Estimate (12) can be proved in much the same way. So, if $0 \leqslant u \leqslant \varepsilon$, then by Lemma 4

$$\Delta_2(0, u) \leqslant J_2(0, 0, u, a, b, c) \ll \varepsilon^{1-a}. \tag{16}$$

For $\varepsilon < u \leqslant 1 - 2\varepsilon$, we have

$$\Delta_2(0, u) - \Delta_2(\varepsilon, u) \leqslant \Delta_2(0, \varepsilon) + \Delta_1(0, u, \varepsilon) + 2\int_{\varepsilon}^{u} \frac{I_\varepsilon(s)\,\mathrm{d}s}{s^a}, \tag{17}$$

where

$$I_\varepsilon(s) := \int_{1-s-\varepsilon}^{1-s} \frac{\mathrm{d}z}{z^b(1-s-z)^c}$$

and $1 - s \geqslant 2\varepsilon$. Routine calculation yields

$$I_\varepsilon(s) = (1-s)^{1-b-c} \int_{1-\varepsilon/(1-s)}^{1} \frac{\mathrm{d}t}{t^b(1-t)^c} \ll (1-s)^{-b}\varepsilon^{1-c}.$$

From this, (13), (16) and (17) it follows

$$\Delta_2(0,u) \ll \Delta_2(\varepsilon,u) + \varepsilon^{1-a} + \varepsilon^{1-b} + \varepsilon^{1-c}. \tag{18}$$

For $\Delta_2(\varepsilon,u)$, we employ the Lagrange mean value theorem again and obtain the expression similar to that in (15) with integrals J_2 instead of J_1. Then we estimate integrals J_2 by means of Lemma 6 and find that

$$\Delta_2(\varepsilon,u) \ll \eta\left(\varepsilon^{-a} + \varepsilon^{-b} + \varepsilon^{-c}\right). \tag{19}$$

When $1-2\varepsilon < u < 1$, we have

$$\Delta_2(0,u) - \Delta_2(0,\,1-2\varepsilon) \leqslant 2\big(J_2(0,0,u,a,b,c) - J_2(0,0,1-2\varepsilon,a,b,c)\big).$$

Hence, in view of (5),

$$\Delta_2(0,u) \ll \Delta_2(0,1-2\varepsilon) + \int\limits_{1-2\varepsilon}^{u} \frac{\mathrm{d}s}{(1-s)^{b+c-1}}.$$

Set $\varepsilon = \eta/4$. Then the latter estimate together with (19), (18) and (16) show that (12) holds uniformly for $u \in [0,1]$. □

Next lemmas deal with multiplicative functions $\theta \in \mathcal{G}(1-a,\delta)$ and $\theta_1(q) = \tilde{h}(q|1,\theta)$, $\theta_2(q) = \hat{h}(q|1,\theta)$ defined by (2). Note that $\theta_1, \theta_2 \in \mathcal{G}(1-a,\delta)$ as well.

Lemma 8. *Assume that $\theta \in \mathcal{G}(1-a,\delta)$ for some $0<a<1$. Then for $q \in \mathbb{N}$, $x \geqslant \mathrm{e}$, $\eta_x \leqslant w \leqslant 1$, $0 < t \leqslant x^{1-w}$, $b \neq 1$, we have*

$$\begin{aligned} Z_x(q,t,w,b;\theta) &:= \sum_{m\leqslant x^w} \frac{\theta(qm)}{m(\ln(\frac{\mathrm{e}x}{mt}))^b} \\ &= A(1-a,\theta)\theta_1(q)\left(\frac{1}{\ln x}\right)^{a+b-1} \int\limits_0^w \frac{\mathrm{d}s}{(\eta_x+s)^a(\eta_x+1-\frac{\ln t}{\ln x}-s)^b} \\ &\quad + O\left(\theta_2(q)\left(\left(\ln\frac{x}{t}\right)^{-b} + \frac{(1+w\ln x)^{-a}}{(1+\ln\frac{x}{t}-w\ln x)^{b-1}\ln\frac{x}{t}}\right)\right). \end{aligned}$$

Moreover,

$$Z_x(q,t,w,b;\theta) \ll \theta_2(q)\left(\left(\ln\frac{x}{t}\right)^{1-a-b} + \frac{(1+w\ln x)^{1-a}}{(1+\ln\frac{x}{t}-w\ln x)^{b-1}\ln\frac{x}{t}}\right). \tag{20}$$

Proof. Using notations of Lemma 3 and taking $a_m = \theta(qm)$, we can write

$$Z_x(q,t,w,b;\theta) = \Theta_{x/t}(0,z,b) + \theta(q)\left(\ln\frac{\mathrm{e}x}{t}\right)^{-b}, \tag{21}$$

where $z = w \ln x / \ln(x/t)$. By Lemma 1, taking $\varphi \equiv 1$, $g = \theta$, $d = q$, we have

$$\sum_{m \leqslant v} \theta(qm) = \frac{v}{\ln^a(\mathrm{e}v)} \left(A(1-a,\theta)\theta_1(q) + O\left(\frac{\theta_2(q)}{\ln(\mathrm{e}v)} \right) \right). \tag{22}$$

Therefore we may evaluate $\Theta_{x/t}(0, z, b)$ by means of Lemma 3. Then (21) becomes

$$\begin{aligned} Z_x(q,t,w,b;\theta) &= \frac{A(1-a,\theta)\theta_1(q)}{(\ln \frac{x}{t})^{a+b-1}} I(0, z; a, b, \eta_{x/t}) \\ &\quad + O\left(\theta_2(q) \left(\ln \frac{x}{t} \right)^{-b} \left(1 + \left(\ln \frac{x}{t} \right)^{1-a} r_{x/t}(0, z; a, b-1) \right) \right) \end{aligned}$$

since $\theta_1(q) \leqslant \theta_2(q)$. Changing integration variable $s = v \ln(x/t)/\ln x$, we have

$$\begin{aligned} I(0, z; a, b, \eta_{x/t}) &= \int_0^w \frac{\ln x \, \mathrm{d}s}{\ln \frac{x}{t} (\eta_{x/t} + s \frac{\ln x}{\ln \frac{x}{t}})^a (\eta_{x/t} + 1 - s \frac{\ln x}{\ln \frac{x}{t}})^b} \\ &= \left(\frac{\ln \frac{x}{t}}{\ln x} \right)^{a+b-1} \int_0^w \frac{\mathrm{d}s}{(\eta_x + s)^a (\eta_x + 1 - \frac{\ln t}{\ln x} - s)^b}. \end{aligned}$$

A simple calculation shows that

$$\left(\ln \frac{x}{t} \right)^{1-a-b} r_{x/t}(0, z; a, b-1) \ll \left(\ln \frac{x}{t} \right)^{-b} + \frac{(1 + w \ln x)^{-a}}{(1 + \ln \frac{x}{t} - w \ln x)^{b-1} \ln \frac{x}{t}}.$$

It remains to prove estimate (20). By (22) we have

$$\sum_{m \leqslant v} \theta(qm) \ll \frac{\theta_2(q)\, v}{\ln^a(\mathrm{e}v)}. \tag{23}$$

Therefore (20) follows from (21) and Lemma 3 by taking $A = 0$ and choosing $a - 1$ instead of a. Lemma is proved. □

Lemma 9. *Assume that* $\theta \in \mathcal{G}(1-a, \delta)$ *for some* $0 < a < 1$. *If* $b \neq 1$, *then uniformly for* $0 \leqslant u, v \leqslant 1$, $u + v \leqslant 1$,

$$\begin{aligned} T_x(u,v,b;\theta) &:= \sum_{q \leqslant x^u} \frac{1}{q} Z_x(q,q,v,b;\theta) \\ &= \frac{A(1-a,\theta)A(1-a,\theta_1)}{\ln^{2a+b-2} x} J_1(0, \eta_x, u, v, a, a, b) \\ &\quad + O\left(\frac{1}{\ln^a x} + \frac{1}{\ln^{a+b-1} x} \right). \end{aligned}$$

Moreover,

$$T_x(u,v,b;\theta) \ll \ln^{2-2a-b} x + \ln^{-a} x. \tag{24}$$

Proof. Note that (20) implies

$$Z_x(1,1,v,b;\theta) \leqslant Z_x(1,1,1,b;\theta_2) \ll \ln^{-a} x + \ln^{1-a-b} x. \tag{25}$$

Consider two cases.

(i) Let $\min(u,v) \leqslant \eta_x$. Then we apply (23) and (25) to obtain

$$T_x(u,v,b;\theta) \ll Z_x(1,1,1,b;\theta_2) \ll \ln^{-a} x + \ln^{1-a-b} x. \tag{26}$$

(ii) Let $\min(u,v) > \eta_x$. Then $u,v \in (\eta_x,\, 1-\eta_x)$ since $u+v \leqslant 1$. To evaluate $Z_x(q,q,v,b;\theta)$, we will employ Lemma 8. Note that for $q \leqslant x^u$ and $0 \leqslant s \leqslant v$,

$$\eta_x + 1 - \frac{\ln q}{\ln x} - s = \frac{1}{\ln x}\cdot \ln\frac{\mathrm{e}x^{1-s}}{q},$$
$$\left(1+\ln\frac{x}{q} - v\ln x\right)^{b-1} \ln\frac{x}{q} \geqslant \left(\ln\frac{\mathrm{e}x^{1-v}}{q}\right)^{b}.$$

Therefore by Lemma 8

$$T_x(u,v,b;\theta) = \frac{A(1-a,\theta)}{\ln^{a-1} x}\int_0^v \frac{Z_{x^{1-s}}(1,1,\frac{u}{1-s},b;\theta_1)}{(\eta_x+s)^a}\,\mathrm{d}s + O\big(Z_x(1,1,u,b;\theta_2) + (1+v\ln x)^{-a} Z_{x^{1-v}}(1,1,1,b;\theta_2)\big). \tag{27}$$

In view of (25), the remainder term in (27) is $\ll \ln^{-a} x + \ln^{1-a-b} x$. We deal with the main term using Lemma 8 again:

$$Z_{x^{1-s}}\left(1,1,\frac{u}{1-s},b;\theta_1\right) = \frac{A(1-a,\theta_1)}{(\ln x^{1-s})^{a+b-1}}\int_0^{u/(1-s)} \frac{(\eta_{x^{1-s}}+t)^{-a}}{(\eta_{x^{1-s}}+1-t)^b}\,\mathrm{d}t + O\left(\big((1-s)\ln x\big)^{-b} + \frac{(1+(1-s-u)\ln x)^{1-b}}{(1+u\ln x)^a(1-s)\ln x}\right) \tag{28}$$

with $0 \leqslant s \leqslant v < 1-\eta_x$. When $u \leqslant (1-s)/2$, the remainder term in (28) is $\ll ((1-s)\ln x)^{-b}$. Otherwise, this remainder is $\ll ((1-s)\ln x)^{-b} + ((1-s)\ln x)^{-a-1}$. Therefore, changing the integration variable t by $z = t(1-s)$, we transform (28) as follows:

$$Z_{x^{1-s}}\left(1,1,\frac{u}{1-s},b;\theta_1\right) = \frac{A(1-a,\theta_1)}{\ln^{a+b-1} x}\int_0^u \frac{\mathrm{d}z}{(\eta_x+z)^a(\eta_x+1-s-z)^b} + O\big(\big((1-s)\ln x\big)^{-b} + \big((1-s)\ln x\big)^{-a-1}\big). \tag{29}$$

It remains to estimate the integral

$$I(a,b) := \int_0^v \frac{\mathrm{d}s}{(\eta_x + s)^a (1-s)^b}$$

with $0 < a < 1$, $b \neq 1$ and $\eta_x < v < 1 - \eta_x$. We have

$$I(a,b) \ll \int_0^{1/2} \frac{\mathrm{d}s}{(\eta_x + s)^a} + \int_{1/2}^{1-\eta_x} \frac{\mathrm{d}s}{(1-s)^b} \ll 1 + \ln^{b-1} x.$$

Analogously, $I(a, a+1) \ll \ln^a x$. Hence the first assertion of lemma follows from (29), (27) together with (26).

From what we have already established

$$T_x(u,v,b;\theta) \ll \frac{J_1(0,\eta_x,u,v,a,a,b)}{\ln^{2a+b-2} x} + \frac{1}{\ln^a x} + \frac{1}{\ln^{a+b-1} x}.$$

Therefore (24) follows from Lemma 5. □

Lemma 10. *Assume that* $\theta \in \mathcal{G}(1-a,\delta)$ *for some* $0 < a < 1$*. If* $b \neq 1$*, then uniformly for* $0 \leqslant u \leqslant 1 - \eta_x$*,*

$$\begin{aligned} T_x^*(u,b;\theta) &:= \sum_{q \leqslant x^u} \frac{1}{q} Z_{x/q}(q,1,1,b;\theta) \\ &= \frac{A(1-a,\theta)A(1-a,\theta_1)}{\ln^{2a+b-2} x} J_2(0,\eta_x,u,a,a,b) \\ &\quad + O\biggl(\frac{1}{\ln^{a+b-1} x} + \frac{1}{\ln^{2a-1} x}\biggr). \end{aligned}$$

Proof. By Lemma 8

$$\begin{aligned} T_x^*(u,b;\theta) = A(1-a,\theta) &\sum_{q \leqslant x^u} \frac{\theta_1(q)}{q(\ln \frac{x}{q})^{a+b-1}} \int_0^1 \frac{\mathrm{d}s}{(\eta_{\frac{x}{q}} + s)^a (\eta_{\frac{x}{q}} + 1 - s)^b} \\ &+ O\biggl(\sum_{q \leqslant x^u} \frac{\theta_2(q)}{q} \biggl(\biggl(\ln \frac{x}{q}\biggr)^{-b} + \frac{(1 + \ln \frac{x}{q})^{-a}}{\ln \frac{x}{q}}\biggr)\biggr) \\ =:&\ A(1-a,\theta)P_1 + O(R_1 + R_2). \end{aligned} \tag{30}$$

Making use of Lemma 8, we derive that

$$R_1 \ll Z_x(1,1,1-\eta_x,b;\theta_2) \ll \frac{1}{\ln^{a+b-1} x} + \frac{1}{\ln^a x}$$

and

$$R_2 \ll Z_x(1,1,1-\eta_x,1+a;\theta_2) \ll \frac{1}{\ln^a x}.$$

Therefore

$$R_1 + R_2 \ll \frac{1}{\ln^{a+b-1} x} + \frac{1}{\ln^a x}. \tag{31}$$

Consider the main term of (30). Setting $\omega(q) = 1 - \ln q/\ln x$, $z = s/\omega(q)$, we have

$$P_1 = \frac{1}{\ln^{a+b-1} x} \sum_{q \leqslant x^u} \theta_1(q) \frac{F(q)}{q}, \tag{32}$$

where

$$F(t) := \int_0^{\omega(t)} \frac{\mathrm{d}z}{(\eta_x + z)^a (\eta_x + \omega(t) - z)^b}.$$

Partial summation gives

$$\begin{aligned} \sum_{q \leqslant x^u} \theta_1(q) \frac{F(q)}{q} &= \frac{F(x^u)}{x^u} \sum_{q \leqslant x^u} \theta_1(q) \\ &\quad + \int_{1-}^{x^u} \sum_{q \leqslant s} \theta_1(q) \frac{F(s)}{s^2}\,\mathrm{d}s - \int_{1-}^{x^u} \sum_{q \leqslant s} \theta_1(q) \frac{F'(s)}{s}\,\mathrm{d}s \\ &=: H_1 + H_2 + H_3. \end{aligned} \tag{33}$$

Lemma 2 and (10) yield

$$H_1 \ll \frac{1}{(1+u\ln x)^a} \int_0^{1-u} \frac{\mathrm{d}t}{(\eta_x + t)^a (\eta_x + 1 - u - t)^b} \ll 1 + \frac{1}{\ln^{1-b} x}.$$

Evaluating the derivative in H_3, we obtain

$$\begin{aligned} H_3 &= \ln^b x \int_{1-}^{x^u} \sum_{q \leqslant s} \theta_1(q) \frac{\eta_x\,\mathrm{d}s}{s^2(\eta_x + \omega(s))^a} \\ &\quad - \int_{1-}^{x^u} \frac{b}{s^2} \sum_{q \leqslant s} \theta_1(q) \int_0^{\omega(s)} \frac{\eta_x\,\mathrm{d}z\,\mathrm{d}s}{(\eta_x + z)^a (\eta_x + \omega(s) - z)^{b+1}}. \end{aligned}$$

In view of Lemma 2,

$$H_3 \ll \ln^{b-a} x \int_0^u \frac{\mathrm{d}t}{(\eta_x + t)^a (\eta_x + 1 - t)^a} + \frac{J_2(0,\eta_x,u,a,a,b+1)}{\ln^a x}.$$

The first integral of the last relation is $\ll 1$, and the second one we estimate using Lemma 6. This gives

$$H_3 \ll \ln^{b-a} x + \ln^{-a} x.$$

It remains to evaluate the second term in (33). Using Lemma 1, we get

$$\begin{aligned} H_2 &= A(1-a,\theta_1) \int_1^{x^u} \frac{1}{s \ln^a(\mathrm{e}s)} \int_0^{\omega(s)} \frac{\mathrm{d}z}{(\eta_x + z)^a(\eta_x + \omega(s) - z)^b} \, \mathrm{d}s \\ &\quad + O\Biggl(\int_1^{x^{1-v}} \frac{1}{s \ln^{a+1}(\mathrm{e}s)} \int_0^{\omega(s)} \frac{\mathrm{d}z}{(\eta_x + z)^a(\eta_x + \omega(s) - z)^b} \, \mathrm{d}s \Biggr) \\ &= \frac{A(1-a,\theta_1)}{\ln^{a-1} x} J_2(0,\eta_x,u,a,a,b) + O\biggl(\frac{J_2(0,\eta_x,u,a+1,a,b)}{\ln^a x} \biggr). \end{aligned}$$

Application of Lemma 6 gives

$$J_2(0,\eta_x,u,a+1,a,b) \ll \ln^a x + \ln^{a+b-1} x.$$

Therefore

$$H_2 = \frac{A(1-a,\theta_1)}{\ln^{a-1} x} J_2(0,\eta_x,u,a,a,b) + O\bigl(1 + \ln^{b-1} x\bigr).$$

Combining estimates of H_1, H_2, H_3 with (33) and (32), we get

$$P_1 = \frac{A(1-a,\theta_1)}{\ln^{2a+b-2} x} J_2(0,\eta_x,u,a,a,b) + O\biggl(\frac{1}{\ln^{a+b-1} x} + \frac{1}{\ln^{2a-1} x} \biggr).$$

This estimate together with (30) and (31) complete the proof of Lemma 10. □

3 Proof of Theorem 1

We have

$$\begin{aligned} &S_x(u,v) \\ &\quad = \frac{1}{x} \sum_{d \leqslant x} f(d) \sum_{\substack{m \leqslant n^u,\, q \leqslant n^v \\ n := qmd \leqslant x}} \frac{1}{T_3(qmd)} \\ &\quad = \frac{1}{x} \sum_{d \leqslant x} f(d) \sum_{\substack{m \leqslant x^u,\, q \leqslant x^v \\ qm \leqslant x/d}} \frac{1}{T_3(qmd)} - \frac{1}{x} \sum_{d \leqslant x} f(d) \sum_{\substack{n^u < m \leqslant x^u,\, q \leqslant n^v \\ n := qmd \leqslant x}} \frac{1}{T_3(qmd)} \\ &\qquad - \frac{1}{x} \sum_{d \leqslant x} f(d) \sum_{\substack{m \leqslant n^u,\, n^v < q \leqslant x^v \\ n := qmd \leqslant x}} \frac{1}{T_3(qmd)} - \frac{1}{x} \sum_{d \leqslant x} f(d) \sum_{\substack{n^u < m \leqslant x^u,\, n^v < q \leqslant x^v \\ n := qmd \leqslant x}} \frac{1}{T_3(qmd)} \\ &\quad =: H(u,v) - R_1(u,v) - R_2(u,v) - R_3(u,v). \end{aligned} \tag{34}$$

We set
$$R(u,v) := \frac{1}{x}\sum_{d\leqslant x} f(d) \sum_{m\leqslant x^u} \sum_{q\leqslant \min(x^v,\, x^{1-u}/d)} \frac{1}{T_3(qmd)}.$$
Then
$$R_1(u,v) \leqslant \frac{1}{x}\sum_{d\leqslant x} f(d) \sum_{q\leqslant x^v} \sum_{(qmd)^u<m\leqslant x^u} \frac{1}{T_3(qmd)}$$
$$\leqslant \frac{1}{x}\sum_{d\leqslant x} f(d) \sum_{m\leqslant x^u} \sum_{\substack{q\leqslant x^v\\ q<m^{1/u-1}/d}} \frac{1}{T_3(qmd)} \leqslant R(u,v).$$
Moreover,
$$R_2(u,v) = R_1(v,u) \leqslant R(v,u), \qquad R_3(u,v) \leqslant \min\bigl\{R(u,v), R(v,u)\bigr\}. \tag{35}$$
Since $1/T_3 \in \mathcal{G}(\alpha,\delta)$, by Lemma 2 we get
$$R(u,v) \ll \frac{1}{x}\sum_{d\leqslant x^{1-u}} f(d) \sum_{m\leqslant x^u} \sum_{q\leqslant x^{1-u}/d} \frac{1}{T_3(qmd)}$$
$$\ll \frac{1}{x^u}\sum_{d\leqslant x^{1-u}} \frac{f(d)}{d(\ln\frac{\mathrm{e}x^{1-u}}{d})^{1-\alpha}} \sum_{m\leqslant x^u} f_1(md),$$
where the multiplicative function f_1 is defined by
$$f_1(p^k) := \hat{h}\biggl(p^k|1, \frac{1}{T_3}\biggr) = \frac{1}{T_3(p^k)}\biggl(1 + O\biggl(\frac{1}{p^{\sigma_0}}\biggr)\biggr).$$
Having in mind that $f_1 \in \mathcal{G}(\alpha,\delta)$, by Lemma 2 we obtain
$$R(u,v) \ll \frac{1}{(1+u\ln x)^{1-\alpha}} \sum_{d\leqslant x^{1-u}} \frac{f(d)f_2(d)}{d(\ln\frac{\mathrm{e}x^{1-u}}{d})^{1-\alpha}},$$
where the multiplicative function f_2 is defined by $f_2(p^k) := \hat{h}(p^k|1, f_1)$. Since $f\cdot f_2 \in \mathcal{G}(1-2\alpha,\, 2\delta)$, using (20), for $0 \leqslant u \leqslant 1-\eta_x$, we get
$$R(u,v) \ll \frac{((1-u)\ln x)^{-\alpha}}{(1+u\ln x)^{1-\alpha}} \ll \ln^{-\alpha} x.$$
Note that if $1-\eta_x < u \leqslant 1$, then
$$R(u,v) \ll \ln^{\alpha-1} x.$$
These estimates and (35) imply
$$R_1(u,v) + R_2(u,v) + R_3(u,v) \ll \ln^{-\alpha} x \tag{36}$$

uniformly for $u, v \in [0,1]$. Changing order of summation, we have

$$H(u,v) = \frac{1}{x}\sum_{d\leqslant x} f(d) \sum_{\substack{qm\leqslant x/d,\\ m\leqslant x^u,\, q\leqslant x^v}} \frac{1}{T_3(qmd)} = \frac{1}{x}\sum_{m\leqslant x^u}\sum_{q\leqslant x^v}\sum_{d\leqslant x/(qm)} \frac{f(d)}{T_3(qmd)}. \tag{37}$$

Since $f/T_3 \in G(1-2\alpha,\, 2\delta)$, applying Lemma 1, for $qm \leqslant x$, we obtain

$$\sum_{d\leqslant \frac{x}{qm}} \frac{f(d)}{T_3(qmd)} = \frac{x}{qm(\ln(\frac{ex}{qm}))^{2\alpha}} \times \biggl(g_1(qm)\cdot A\biggl(1-2\alpha, \frac{f}{T_3}\biggr) + O\biggl(\frac{h_1(qm)}{\ln(\frac{ex}{qm})}\biggr)\biggr), \tag{38}$$

where

$$g_1(\cdot) := \tilde{h}\biggl(\cdot\,|f, \frac{1}{T_3}\biggr), \qquad h_1(\cdot) := \hat{h}\biggl(\cdot\,|f, \frac{1}{T_3}\biggr).$$

Easy to check that $g_1, h_1 \in \mathcal{G}(\alpha, \delta)$.

Let us split the unit square $K = [0,1]\times[0,1]$ into parts $K = K_1 \cup K_2$, $K_1 := \{(u,v)\in K \mid u+v\leqslant 1\}$ and $K_2 := \{(u,v)\in K \mid u+v > 1\}$.

(i) Let $(u,v)\in K_1$. Then, in view of (38) and (37), we obtain

$$\begin{aligned} H(u,v) &= A\biggl(1-2\alpha, \frac{f}{T_3}\biggr)\sum_{m\leqslant x^u}\frac{1}{m}\sum_{q\leqslant x^v}\frac{g_1(qm)}{q\,\ln^{2\alpha}(\frac{ex}{qm})} \\ &\quad + O\biggl(\sum_{m\leqslant x^u}\frac{1}{m}\sum_{q\leqslant x^v}\frac{h_1(qm)}{q\,\ln^{2\alpha+1}(\frac{ex}{qm})}\biggr) \\ &= A\biggl(1-2\alpha, \frac{f}{T_3}\biggr)T_x(u,v,2\alpha; g_1) + O\bigl(T_x(u,v,2\alpha+1; h_1)\bigr). \end{aligned} \tag{39}$$

By Lemma 9

$$T_x(u,v,2\alpha+1; h_1) \ll \ln^{\alpha-1} x \tag{40}$$

and

$$\begin{aligned} T_x(u,v,2\alpha; g_1) &= A(\alpha, g_1)Ag(\alpha, g_2)J_1(0, \eta_x, u, v, 1-\alpha, 1-\alpha, 2\alpha) \\ &\quad + O\bigl(\ln^{-\alpha} x\bigr), \end{aligned}$$

where $g_2(\cdot) := \tilde{h}(\cdot|1, g_1)$. From this, (40) and (39) it follows

$$H(u,v) = A_3 J_1(0, \eta_x, u, v, 1-\alpha, 1-\alpha, 2\alpha) + O\bigl(\ln^{-\alpha} x\bigr) \tag{41}$$

uniformly for $(u,v)\in K_1$. Here

$$A_3 := A\biggl(1-2\alpha, \frac{f}{T_3}\biggr)A(\alpha, g_1)A(\alpha, g_2).$$

We have

$$A_3 = \frac{1}{\Gamma(1-2\alpha)\Gamma^2(\alpha)} \prod_p \left(1 - \frac{1}{p}\right) \sum_{k=0}^{\infty} \frac{1}{p^k} \sum_{j=0}^{\infty} \frac{1}{p^j} \sum_{i=0}^{\infty} \frac{f(p^i)}{p^i T_3(p^{i+j+k})}.$$

Changing order of summation in the last triple sum, we get

$$\prod_p \left(1 - \frac{1}{p}\right) \sum_{k=0}^{\infty} \frac{1}{p^k} \sum_{s=0}^{\infty} \frac{1}{p^s T_3(p^{s+k})} \sum_{i=0}^{s} f(p^i) = 1$$

since

$$T_3(p^t) = \sum_{i=0}^{t} f\big(p^i\big)(t - i + 1).$$

Thus

$$A_3 = \frac{1}{\Gamma(1-2\alpha)\Gamma^2(\alpha)}, \tag{42}$$

and, in view of (11), relation (41) becomes

$$H(u,v) = \frac{J_1(0,0,u,v,1-\alpha,1-\alpha,2\alpha)}{\Gamma(1-2\alpha)\Gamma^2(\alpha)} + O\big(\ln^{-\alpha} x\big). \tag{43}$$

(ii) Let $(u,v) \in K_2$. If $u \leqslant \eta_x$, then (38) and Lemma 8 yield

$$H(u,v) \ll \sum_{q \leqslant \mathrm{e}} \frac{1}{q} Z_x(q,1,1,2\alpha,h_1) \ll \ln^{-\alpha} x \tag{44}$$

since $h_1 \in \mathcal{G}(\alpha,\delta)$. It is clear that $H(u,v) = H(v,u)$. Therefore (44) holds uniformly for $\{(u,v) \mid \min(u,v) \leqslant \eta_x,\ (u,v) \in K\}$. Now assume that $u > \eta_x$ and $v > \eta_x$. For any $t \in [0,1]$, define

$$D(t) := \frac{1}{x} \sum_{q \leqslant x^{1-t}} \sum_{m \leqslant x/q} \sum_{d \leqslant x/(qm)} \frac{f(d)}{T_3(qmd)} - \frac{1}{x} \sum_{q \leqslant x^{1-t}} \sum_{m \leqslant x^t} \sum_{d \leqslant x/(qm)} \frac{f(d)}{T_3(qmd)}.$$

From (34) and (36) it follows

$$H(1,1) = S_x(1,1) + O\big(\ln^{-\alpha} x\big).$$

Hence

$$H(u,v) = H(1,1) - D(u) - D(v) = 1 - D(u) - D(v) + O\big(\ln^{-\alpha} x\big). \tag{45}$$

Since $f/T_3 \in G(1-2\alpha, 2\delta)$, by Lemma 1

$$D(v) = A\left(1-2\alpha, \frac{f}{T_3}\right)\big(T_x^*(1-v,2\alpha;g_1) - T_x(1-v,v,2\alpha;g_1)\big) + O\big(T_x^*(1-v,2\alpha+1;h_1) + T_x(1-v,v,2\alpha+1;h_1)\big),$$

here T_x and T_x^* are defined in Lemmas 9 and 10. These lemmas together with Lemmas 6 and 7 yield

$$\begin{aligned} D(v) = A_3\bigl(&J_2(0,0,1-v,1-\alpha,1-\alpha,2\alpha) \\ &- J_1(0,0,1-v,v,1-\alpha,1-\alpha,2\alpha)\bigr) \\ &+ O\bigl(\ln^{-\alpha} x + \ln^{2\alpha-1} x\bigr). \end{aligned}$$

It follows from (5) that

$$J_2(0,0,1,1-\alpha,1-\alpha,2\alpha) = \Gamma(1-2\alpha)\Gamma^2(\alpha).$$

Combining the last two relations with (45) and having in mind (44) and (42), we obtain

$$\begin{aligned} H(u,v) = {}& \frac{1}{\Gamma(1-2\alpha)\Gamma^2(\alpha)} \iint\limits_{E(u,v)} \frac{\mathrm{d}t\,\mathrm{d}s}{t^{1-\alpha}s^{1-\alpha}(1-t-s)^{2\alpha}} \\ &+ O\bigl(\ln^{-\alpha} x + \ln^{2\alpha-1} x\bigr) \end{aligned}$$

uniformly for $(u,v) \in K_2$.

The proof of Theorem 1 follows now from this estimate, (43), (36) and (34).

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