

# On sheaves on midsymmetrical quantaloids

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## 1. Introduction

In an earlier paper [2] we studied presheaves and sheaves on an arbitrary quantaloid (category enriched in the category of complete sup-lattices subject to certain laws). In [3] we began a study of presheaves on quantaloids from a special subclass of quantaloids satisfying "Midsymmetry". In this note we discuss sheaves which specialize in that "midsymmetrical" case.

## 2. Midsymmetrical quantaloids

DEFINITION 2.1. A *quantaloid* is a locally small category  $Q$  such that:

- (i) for all  $u, v$  objects in  $Q$ , the hom-set  $Q(u, v)$  is a complete lattice,
- (ii) composition of morphisms of  $Q$  (in this paper denoted by  $\&$ ) preserves arbitrary joins in both variables:  $p\&\bigvee_i q_i = \bigvee_i p\&q_i$  and  $(\bigvee_i p_i)\&q = \bigvee_i p_i\&q$  for all morphisms  $p, q$  of  $Q$  and for all families  $(p_i), (q_i)$  of morphisms of  $Q$  (forming respective composable pairs).

A quantaloid  $Q$  will be called midsymmetrical whenever it satisfies

**Midsymmetry:**  $p\&(q\&r)\&p' = p\&(r\&q)\&p'$  for all  $p \in Q(u, v), q, r \in Q(v, v), p' \in Q(v, v')$ .

We focus on midsymmetrical quantaloids and the traditional requirement on the existence of their units is not imposed. Examples of the one-object quantaloids (which are called *quantales*) include frames (and thus complete Boolean algebras) and various ideal lattices of rings or  $C^*$ -algebras. Many other quantales and quantaloids can be found in [5]. Our basic example is the following

EXAMPLE 2.2. Let  $Q$  be a right-sided idempotent quantale (belonging to a class of quantales considered in [4]), i.e., an one-object quantaloid such that, for any  $p, q \in Q$ ,  $p\&p \leq p$  and  $p\&p = p$ . The routine check shows that it is midsymmetrical.

From now  $Q$  will be an arbitrary midsymmetrical quantaloid having a small set of objects. Let  $Q_0$  denote this set and  $Q_1$  the set of morphisms of  $Q$ .

### 3. Sheaves on a (midsymmetrical) quantaloid

The notion of a sheaf and a few facts (with omitted proofs) are taken from [2]. We will present the axioms of a sheaf (in Proposition 3.4) and the concept of the compatibility of sub- $Q$ -sets (in Corollary 3.6) in a more close form to those of C.J. Mulvey and M. Nawaz. For the concepts not defined here see [2] (or [3]).

DEFINITION 3.1. Let  $(X, A, \overset{\circ}{\rhd})$  be a separated presheaf on a quantaloid  $Q$ .

- (i) We say that, for a singleton  $\mathcal{S} = (S, S^\#)$  of the underlying  $Q$ -set  $(X, A)$ , there exist enough restrictable triplets  $(s_x, x, s_x^\#) \in Q_1 \times X \times Q_1$  if the following conditions hold:

$$s_x = \bigvee \{s_{x'} \& a_{x',x} | x' \in X, (s_{x'}, x', s_{x'}^\#) \text{ restrictable}\} \quad (1)$$

and

$$s_x^\# = \bigvee \{a_{x,x'} \& s_{x'}^\# | x' \in X, (s_{x'}, x', s_{x'}^\#) \text{ restrictable}\} \quad (2)$$

for all  $x \in X$ . (Note that  $(s_x, x, s_x^\#)$  is restrictable if and only if  $s_x = s_x \& s_x^\# \& s_x$  and  $s_x^\# = s_x^\# \& s_x \& s_x^\#$ .)

- (ii) A sub- $Q$ -set  $(J, J A) \subseteq (X, A)$  with  $J \subseteq X_u$  (for some  $u \in Q_0$ ) is said to be *compatible* if the pair  $\mathcal{E} = (E, E^\#)$  with  $E = (e_x)_{x \in J}$ ,  $E^\# = (e_x^\#)_{x \in J}$ , and  $e_x = e_x^\# = a_x$  (representing a "diagonal" element of  $A$ ) for  $x \in J$  constitutes a singleton of  $(J, J A) \subseteq (X, A)$ . The singleton  $\mathcal{E}$  itself is called an *extent* of  $(J, J A) \subseteq (X, A)$ .

Observe that the bottom extension  ${}^X \mathcal{E} = ({}^X E, {}^X E^\#)$  of the extent  $\mathcal{E} = (E, E^\#)$  of a compatible sub- $Q$ -set  $(J, J A) \subseteq (X, A)$  to a singleton of  $(X, A)$  is obtained by the formulas obtained by:

$${}^X e_x = \bigvee_{x' \in J} a_{x',x} \text{ and } {}^X e_x^\# = \bigvee_{x' \in J} a_{x,x'} \quad (3)$$

for all  $x' \in X$ .

DEFINITION 3.2. We say that a separated presheaf  $(X, A, \overset{\circ}{\rhd})$  on  $Q$  is a *sheaf* on  $Q$  if it satisfies the following

*Sheaf Conditions:*

- (i) for every singleton  $\mathcal{S}$  of the underlying  $Q$ -set  $(X, A)$ , there exist enough restrictable triplets  $(s_x, x, s_x^\#)$ ;  
 (ii) for every compatible sub- $Q$ -set  $(J, J A) \subseteq (X, A)$ , the bottom extension of its extent  $\mathcal{E}$  to a singleton of  $(X, A)$  (given by (3)) is of the form

$${}^X \mathcal{E} = \mathcal{A}_y \quad (4)$$

for some (unique) element  $y \in X$  (where  $\mathcal{A}_y = ((a_{y,x})_{x \in X}, (a_{x,y})^{x \in X})$ ).

The next fact is taken from [2].

**PROPOSITION 3.3.** Let  $(X, A, \dashv)$  be a separated presheaf and let  $(J, {}_J A) \subseteq (X, A)$  be a sub- $Q$ -set with  $J \subseteq X_u$  (for some  $u \in Q_0$ ). Then the following two conditions are equivalent:

- (i)  $(J, {}_J A) \subseteq (X, A)$  is compatible;
- (ii)  $a_x \& a_{x'} = a_{x,x'} \leq a_x \wedge a_{x'}$  for all  $x, x' \in J$ .

The following proposition shows that we can weaken the axiom (ii) of Definition 3.2.

**PROPOSITION 3.4.** Let  $(X, A, \dashv)$  be a separated presheaf and  $(J, {}_J A) \subseteq (X, A)$  be a compatible sub- $Q$ -set. Then the following assertions are equivalent:

- (i) the bottom extension of the extent of  $(J, {}_J A)$  is of the form

$${}^X \mathcal{E} = \mathcal{A}_y$$

for some element  $y \in X$ , that is, there exists a unique (by Separation) element  $y \in X$  for which

$$a_{y,x} = \bigvee_{x' \in J} a_{x',x} \quad \text{and} \quad a_{x,y} = \bigvee_{x' \in J} a_{x,x'} \quad (5)$$

for all  $x \in X$ ;

- (ii) the extent of  $(J, {}_J A)$  is of the form

$$\mathcal{E} = {}_J \mathcal{A}_y$$

for some unique element  $y \in X$  for which

$$a_y = \bigvee_{x \in J} a_x, \quad (6)$$

where  ${}_J \mathcal{A}_y = ((a_{y,x})_{x \in J}, (a_{x,y})^{x \in J})$ , that is, there exists a unique element  $y \in X$  satisfying (6) and the following relations

$$a_{y,x} = a_x = a_{x,y} \quad (7)$$

hold for all  $x \in J$ .

The implication from (i) to (ii) is trivial (taking  $x = y$  in (5), one has (6)). To prove that (ii) implies (i), assume (ii). Then, given  $x \in X$ , on the one hand, we have that

$$\bigvee_{x' \in J} a_{x',x} = \bigvee_{x' \in J} a_{x'} \& a_{x',x} = \bigvee_{x' \in J} a_{y,x'} \& a_{x',x} \leq a_{y,x},$$

while, on the other,

$$a_{y,x} = a_y \& a_{y,x} = \bigvee_{x' \in J} a_{x'} \& a_{y,x} = \bigvee_{x' \in J} a_{x',y} \& a_{y,x} \leq \bigvee_{x' \in J} a_{x',x},$$

whence the first equality of (5). The second part follows similarly.

**PROPOSITION 3.5.** Let  $(X, A, \overset{\circ}{\rightarrow})$  be a separated presheaf and let  $(J, {}_J A) \subseteq (X, A)$  be a sub- $Q$ -set with  $J \subseteq X_u$  (for some  $u \in Q_0$ ) such that the inequality

$$a_x \& a_{x'} \leq a_x \wedge a_{x'} \quad (8)$$

holds for all  $x, x' \in J$ . Then the following conditions are equivalent:

- (i)  $a_x \& a_{x'} \overset{\circ}{\rightarrow} x \overset{\circ}{\leftarrow} a_{x'} = a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x$  for all  $x, x' \in J$ ,
- (ii)  $a_x \& a_{x'} = a_{x,x'}$  for all  $x, x' \in J$ .

First of all, we note that the assumption (8) ensures the restrictability of the triplets  $(a_x \& a_{x'}, x, a_{x'})$  and  $(a_x, x', a_x)$  for all  $x, x' \in J$  (and thus legitimates the expression in (i)). To prove the implication (i) $\Rightarrow$ (ii), assume (i). Then, given  $x, x' \in J$ , we have the following chain of implications:

$$\begin{aligned} a_{a_x \& a_{x'} \overset{\circ}{\rightarrow} x \overset{\circ}{\leftarrow} a_{x'}, a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x} &= a_{a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x} \\ \Rightarrow a_x \& a_{x'} \& a_{x,x'} \& a_x &= a_x \& a_{x'} \& a_x \\ \Rightarrow a_x \& a_{x'} \& a_{x,x'} \& a_x \& a_{x'} &= a_x \& a_{x'} \& a_x \& a_{x'} \\ \Rightarrow a_x \& a_x \& a_{x,x'} \& a_{x'} \& a_{x'} &= a_x \& a_x \& a_{x'} \& a_{x'} \text{ (by Midsymmetry)} \\ \Rightarrow a_{x,x'} &= a_x \& a_{x'} \text{ (by Strictness),} \end{aligned}$$

whence (ii). To prove that (ii) implies (i), assume (ii). Then, given  $x, x' \in J$ , we have that  $a_x \& a_{x'} \& a_x \& a_{x'} = a_x \& a_{x'} \& a_x$ ,  $a_x \& a_{x'} \& a_x = a_x \& a_{x'} \& a_{x,x'} \& a_x$ , and that  $a_x \& a_{x'} \& a_x \& a_{x'} = a_x \& a_{x',x} \& a_{x'}$ , which may be written

$$a_{a_x \& a_{x'} \overset{\circ}{\rightarrow} x \overset{\circ}{\leftarrow} a_{x'}} = a_{a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x} = a_{a_x \& a_{x'} \overset{\circ}{\rightarrow} x \overset{\circ}{\leftarrow} a_{x'}, a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x} = a_{a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x, a_x \& a_{x'} \overset{\circ}{\rightarrow} x \overset{\circ}{\leftarrow} a_{x'}},$$

whence (i) (by Separation).

In view of Proposition 3.3, we have

**COROLLARY 3.6.** In the setting of the preceding proposition, the following two conditions are equivalent:

- (i)  $(J, {}_J A) \subseteq (X, A)$  is compatible;
- (ii)  $a_x \& a_{x'} \leq a_x \wedge a_{x'}$  and  $a_x \& a_{x'} \overset{\circ}{\rightarrow} x \overset{\circ}{\leftarrow} a_{x'} = a_x \overset{\circ}{\rightarrow} x' \overset{\circ}{\leftarrow} a_x$  for all  $x, x' \in J$ .

Now consider the case of a right-sided idempotent quantale in order to compare our concept of sheaf with that of C.J. Mulvey and M. Nawaz.

DEFINITION 3.7. (Definition 27 [4] and Definition 28 [4]). Let  $Q$  be a right-sided idempotent quantale and let  $(X, E, \downarrow, \uparrow)$  be a presheaf in the sense of C.J. Mulvey and M. Nawaz.

(i) A subset  $J$  of  $X$  is said to be compatible if

$$Ex \downarrow x' = x \uparrow Ex'$$

for all  $x, x' \in J$ .

(ii) A quadruple  $(X, E, \downarrow, \uparrow)$  is called a sheaf on  $Q$  if, for any compatible subset  $J \subseteq X$ , there exists a "join" of  $J$ , i.e., a unique element  $y \in X$  such that

$$Ey = \bigvee_{x \in J} Ex$$

and  $Ex \downarrow y = x$  for all  $x \in J$ .

PROPOSITION 3.8. (Proposition 7.6 [2]). Let  $(X, A)$  be a separated quantal set on a right-sided idempotent quantale  $Q$  such that its every singleton  $\mathcal{S} = (S, S^\#)$  satisfies the condition that  $s_x^\# = a_x \& s_x$  for all  $x \in X$  (observing that every singleton in the sense of C.J. Mulvey and M. Nawaz is so). Let  $(X, A, \uparrow^\#)$  be a presheaf on  $Q$  and let  $(X, E, \downarrow, \uparrow)$  be the presheaf associated to  $(X, A, \uparrow^\#)$  in the setting of Proposition 6.5 [2]. If  $(X, E, \downarrow, \uparrow)$  is a sheaf in the sense of Definition 3.7, then the underlying presheaf  $(X, A, \uparrow^\#)$  is a sheaf in our sense.

## References

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## Apie pluoštus virš midsimetriškų kvantaloidų

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Nagrinėjami priešpluoščiai ir pluoštai virš midsimetriškų kvantaloidų. Nustatytos sąlygos, kuriomis priešpluoštis tampa pluoštu.