

On weak convergence of an approximation of a fractional Brownian motion

Kęstutis KUBILIUS (MII, VGTU)

e-mail: kubilius@ktl.mii.lt

In this paper we consider a problem of approximation of a fractional Brownian motion (fBm). Several schemes of such approximations are considered in [1,2,3,5,6]. In this note we extend Sottinen's result [5].

A fBm with the Hurst index $0 < H < 1$ is a centered Gaussian process $X = \{X_t, t \geq 0\}$ with $X_0 = 0$ and with the covariance

$$\text{Cov}(X_t, X_s) = \frac{1}{2} \text{Var}(X_1)(t^{2H} + s^{2H} - |t - s|^{2H}),$$

for all $t, s \geq 0$. If $\text{Var}(X_1) = 1$, we write $X = B^H$. If $H > 1/2$, then we have the following kernel representation of B^H with respect to the standard Brownian motion

$$B_t^H = \int_0^t K_H(t, s) dW_s$$

with a deterministic kernel

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2} du,$$

where c_H is the normalizing constant

$$c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}},$$

$\beta(\cdot, \cdot)$ is the beta function.

Let $\pi^n = \{t_k^n: 0 \leq k \leq n\}$ be a sequence of partitions of the interval $[0, 1]$ such that $t_k^n = k/n$. Denote $\rho^n(t) = [nt]/n$. Define

$$A_t^n = \sum_{k=1}^{[nt]} K_H(\rho^n(t), t_k^n) \frac{\xi_k^n}{\sqrt{n}}, \quad M_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k^n,$$

where $[nt]$ denotes the integer part of nt , $\{\xi_k^n\}$ is an i.i.d. random variables with $\mathbf{E}\xi_1^n = 0$ and $\mathbf{D}\xi_1^n = 1$.

From the equality

$$K_H(t, s) = c_H(t - s)^\alpha \int_0^1 \left[\left(\frac{t}{s} - 1 \right) v + 1 \right]^\alpha v^{\alpha-1} dv, \quad s < t,$$

we see that $K_H(t, s)$ is continuous in $\{0 < s < t\}$, $K_H(\cdot, s)$ is increasing and $K_H(t, \cdot)$ is decreasing, where $\alpha = H - 1/2$.

Lemma 1. (cf. [5]) *The finite-dimensional distributions of (M^n, A^n) converges to those of (W, B^H) .*

Proof. Let $\lambda_k, \mu_k \in \mathbb{R}$ for $1 \leq k \leq m$ and $t_1, \dots, t_m \in (0, 1]$ be arbitrary. We want to show that

$$X^n := \sum_{i=1}^m (\lambda_i M^n(t_i) + \mu_i A^n(t_i))$$

converges to a normal distribution with variance

$$\mathbf{E} \left\{ \sum_{i=1}^m (\lambda_i W(t_i) + \mu_i B^H(t_i)) \right\}^2.$$

Note that

$$\begin{aligned} \mathbf{D}X^n &= \mathbf{E} \left\{ \sum_{i=1}^m \sum_{k=1}^{[nt_i]} \left(\lambda_i + \mu_i K_H(\rho^n(t_i), t_k^n) \right) \frac{\xi_k^n}{\sqrt{n}} \right\}^2 \\ &= \frac{1}{n} \sum_{i,j=1}^m \sum_{k=1}^{[n(t_i \wedge t_j)]} \left(\lambda_i + \mu_i K_H(\rho^n(t_i), t_k^n) \right) \left(\lambda_j + \mu_j K_H(\rho^n(t_j), t_k^n) \right) \\ &= \sum_{i,j=1}^m \lambda_i \lambda_j \frac{[n(t_i \wedge t_j)]}{n} + \frac{1}{n} \sum_{i,j=1}^m \mu_i \lambda_j \sum_{k=1}^{[n(t_i \wedge t_j)]} K_H(\rho^n(t_i), t_k^n) \\ &\quad + \frac{1}{n} \sum_{i,j=1}^m \mu_j \lambda_i \sum_{k=1}^{[n(t_i \wedge t_j)]} K_H(\rho^n(t_j), t_k^n) \\ &\quad + \frac{1}{n} \sum_{i,j=1}^m \mu_i \mu_j \sum_{k=1}^{[n(t_i \wedge t_j)]} K_H(\rho^n(t_i), t_k^n) K_H(\rho^n(t_j), t_k^n). \end{aligned} \tag{1}$$

We have

$$\begin{aligned} |K_H(t, s) - K_H(\rho^n(t), s)| &= c_H s^{-\alpha} \int_{\rho^n(t)}^t u^\alpha (u - s)^{\alpha-1} du \\ &\leq \frac{c_H}{\alpha} \left(\frac{t}{s} \right)^\alpha (t - \rho^n(t))^\alpha \leq \left(\frac{t}{s} \right)^\alpha \frac{c_H}{\alpha n^\alpha}. \end{aligned}$$

Let $s \in (t_{i-1}^n, t_i^n]$. Then

$$\begin{aligned} & |K_H(\rho^n(t), t_i^n) - K_H(\rho^n(t), s)| \\ & \leq c_H |(\rho^n(t) - t_i^n)^\alpha - (\rho^n(t) - s)^\alpha| \int_0^1 \left[\left(\frac{\rho^n(t)}{t_i^n} - 1 \right) v + 1 \right]^\alpha v^{\alpha-1} dv \\ & \quad + c_H (\rho^n(t) - s)^\alpha \int_0^1 \left| \left[\left(\frac{\rho^n(t)}{t_i^n} - 1 \right) v + 1 \right]^\alpha \right. \\ & \quad \left. - \left[\left(\frac{\rho^n(t)}{s} - 1 \right) v + 1 \right]^\alpha \right| v^{\alpha-1} dv := I_1 + I_2. \end{aligned}$$

Further

$$\begin{aligned} I_1 & \leq c_H (t_i^n - s)^\alpha \int_0^1 \left[\left(\frac{t}{t_i^n} - 1 \right) v + 1 \right]^\alpha v^{\alpha-1} dv \leq \frac{c_H}{\alpha n^\alpha} \left[\left(\frac{1}{t_i^n} \right)^\alpha + 1 \right], \\ I_2 & \leq c_H \int_0^1 \left| \frac{\rho^n(t)}{t_i^n} - \frac{\rho^n(t)}{s} \right|^\alpha v^{2\alpha-1} dv \leq \frac{c_H}{2\alpha} \left(\frac{\rho^n(t)(t_i^n - s)}{st_i^n} \right)^\alpha \leq \frac{c_H}{\alpha n^\alpha} \left(\frac{t}{s} \right)^\alpha. \end{aligned}$$

Assume that $s > \rho^n(\varepsilon)$. Then

$$\left| K_H(t, s) - \sum_{i=[n\varepsilon]+1}^{[nt]} K_H(\rho^n(t), t_i^n) \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s) \right| \leq \frac{c_H}{\alpha n^\alpha} \left\{ 3 \left(\frac{1}{\rho^n(\varepsilon)} \right)^\alpha + 1 \right\}. \quad (2)$$

Note that

$$|K_H(t, s)| \leq c_H \alpha^{-1} s^{-\alpha} := \widehat{c}_H s^{-\alpha}. \quad (3)$$

Let $\varepsilon \leq t_1$. From (2) and (3) we get

$$\begin{aligned} & \left| \frac{1}{n} \sum_{k=1}^{[n(t_i \wedge t_j)]} K_H(\rho^n(t_i), t_k^n) - \int_0^{t_i \wedge t_j} K_H(t_i, s) ds \right| \\ & \leq \left| \frac{1}{n} \sum_{k=1}^{[n\varepsilon]} K_H(\rho^n(t_i), t_k^n) \right| + \left| \int_0^{\rho^n(\varepsilon)} K_H(t_i, s) ds \right| \\ & \quad + \sum_{k=[n\varepsilon]+1}^{[n(t_i \wedge t_j)]} \left| \frac{1}{n} K_H(\rho^n(t_i), t_k^n) - \int_{t_{k-1}^n}^{t_k^n} K_H(t_i, s) ds \right| \\ & \quad + \left| \int_{[n(t_i \wedge t_j)]/n}^{t_i \wedge t_j} K_H(t_i, s) ds \right| \\ & \leq 2\widehat{c}_H \frac{\varepsilon^{1-\alpha}}{1-\alpha} + \frac{c_H}{\alpha n^\alpha} \left\{ 3 \left(\frac{1}{\rho^n(\varepsilon)} \right)^\alpha + 1 \right\} + \widehat{c}_H (1-\alpha)^{-1} \frac{1}{n^{1-\alpha}}. \end{aligned}$$

Thus the term on the left-hand side of the above inequality vanishes as $n \rightarrow \infty$.

Similarly one can prove that the second and fourth terms in (1) converges to the corresponding integrals. Thus $\sigma_n^2 := \mathbf{D}X^n \rightarrow \mathbf{D} \sum_{i=1}^m (\lambda_i W(t_i) + \mu_i B^H(t_i))$ as $n \rightarrow \infty$.

Let us write X^n as a sum in k . Denote

$$c_{k,i} = \begin{cases} 1, & k \leq [nt_i], \\ 0, & k > [nt_i]. \end{cases}$$

Note that the function $K_H(t, s) = 0$ if $s > t$. Then

$$X^n = \sum_{k=1}^{[nt_m]} \left\{ \sum_{i=1}^m \left(c_{k,i} \lambda_i + \mu_i K_H(\rho^n(t_i), t_k^n) \right) \right\} \frac{\xi_k^n}{\sqrt{n}} = \sum_{k=1}^{[nt_m]} X_k^n.$$

Now we verify the Lindeberg's condition

$$\lim_{n \rightarrow \infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{[nt_m]} \mathbf{E}(X_k^n)^2 \mathbf{1}_{\{|X_k^n| > \varepsilon \sigma_n\}} = 0,$$

where $\varepsilon > 0$. Using the properties of the function $K_H(\cdot, \cdot)$ we get

$$\begin{aligned} (X_k^n)^2 &\leq \frac{(\xi_k^n)^2}{n} \left\{ \sum_{i=1}^m \lambda_i + K_H(\rho^n(t_m), t_k^n) \sum_{i=1}^m \mu_i \right\}^2 \\ &\leq 2 \frac{(\xi_k^n)^2}{n} \left\{ \left(\sum_{i=1}^m \lambda_i \right)^2 + \left(\sum_{i=1}^m \mu_i \right)^2 K_H^2(\rho^n(t_m), t_k^n) \right\} \\ &\leq 2A \frac{(\xi_k^n)^2}{n} (K_H^2(\rho^n(t_m), t_1^n) + 1) = 2A(\xi_k^n)^2 \delta_n, \end{aligned} \tag{4}$$

where

$$A := \max \left\{ \left(\sum_{i=1}^m \lambda_i \right)^2, \left(\sum_{i=1}^m \mu_i \right)^2 \right\}, \quad \delta_n := \frac{1}{n} (K_H^2(\rho^n(t_m), t_1^n) + 1).$$

We obtain

$$\{|X_k^n| > \varepsilon \sigma_n\} \subseteq \{2A(\xi_k^n)^2 \delta_n > \varepsilon^2 \sigma_n^2\} =: A^n(\xi_k^n). \tag{5}$$

Using inequality (4) and the inclusion (5) we obtain

$$\mathbf{E}(X_k^n)^2 \mathbf{1}_{\{|X_k^n| > \varepsilon \sigma_n\}} \leq \sigma_{n,k}^2 \mathbf{E}(\xi_k^n)^2 \mathbf{1}_{A^n(\xi_k^n)} = \sigma_{n,i}^2 \mathbf{E} \xi^2 \mathbf{1}_{A^n},$$

where $\xi := \xi_1^1$, $A^n := A^n(\xi_1^1)$, and $\sigma_{n,k}^2 = \mathbf{D}X_k^n$. Thus

$$\frac{1}{\sigma_n^2} \sum_{k=1}^{[nt_m]} \mathbf{E}(X_k^n)^2 \mathbf{1}_{\{|X_k^n| > \varepsilon \sigma_n\}} \leq \frac{\sigma_{n,1}^2 + \dots + \sigma_{n,[nt_m]}^2}{\sigma_n^2} \mathbf{E} \xi^2 \mathbf{1}_{A^n} = \mathbf{E} \xi^2 \mathbf{1}_{A^n}.$$

Since $\delta_n \rightarrow \infty$ as $n \rightarrow \infty$, then $\mathbf{E}\xi^2 \mathbf{1}_{A^n}$ tends to zero. Thus we obtain the convergence of corresponding finite-dimensional distributions. \square

For weak convergence of (M^n, A^n) to (W, B^H) it remains to prove the tightness of (M^n, A^n) .

First we prove the following lemma.

Lemma 2. *Assume that $\{\xi_k^n\}$ are standard normal random variables. For every $n \geq 1$, $\beta \geq 1$, and $0 < s < t$ there is a constant $C = C(H, \beta)$ such that*

$$\mathbf{E}|A^n(t) - A^n(s)|^\beta \leq C \left(\frac{[nt]}{n} - \frac{[ns]}{n} \right)^{\beta(H-1/2)}. \quad (6)$$

Proof. Note that $A^n(t) - A^n(s)$ is a centered Gaussian random variable for every $n \geq 1$. Thus it is enough to prove that for some constant C_1

$$\mathbf{E}|A^n(t) - A^n(s)|^2 \leq C_1 \left(\frac{[nt]}{n} - \frac{[ns]}{n} \right)^{2H-1}. \quad (7)$$

By simple calculations we get

$$\begin{aligned} \mathbf{E}|A^n(t) - A^n(s)|^2 &= \frac{1}{n} \sum_{k=[ns]+1}^{[nt]} K_H^2(\rho^n(t), t_k^n) \\ &\quad + \frac{1}{n} \sum_{k=1}^{[ns]} \left\{ K_H(\rho^n(t), t_k^n) - K_H(\rho^n(s), t_k^n) \right\}^2 \\ &\leq c_H^2 \frac{t^{2H-1}}{n} \sum_{k=[ns]+1}^{[nt]} (t_k^n)^{-2\alpha} \left\{ \int_{t_k^n}^{\rho^n(t)} (u - t_k^n)^{\alpha-1} du \right\}^2 \\ &\quad + c_H^2 \frac{t^{2H-1}}{n} \sum_{k=1}^{[ns]} (t_k^n)^{-2\alpha} \left\{ \int_{\rho^n(s)}^{\rho^n(t)} (u - t_k^n)^{\alpha-1} du \right\}^2 \\ &\leq \frac{2c_H^2}{\alpha^2} \frac{1}{1-2\alpha} (\rho^n(t) - \rho^n(s))^{2\alpha}. \end{aligned}$$

Lemma 3. *Under conditions of Lemma 2 the sequence $\{(M^n, A^n)\}$ is tight.*

Proof. We shall use the tightness criterium formulated in Theorem 6.4.1 [4]. From

$$\mathbf{E}|M_\delta^n - M_0^n|^2 + \mathbf{E}|A_\delta^n - A_0^n|^2 \leq \frac{[n\delta]}{n} + C \left(\frac{[n\delta]}{n} \right)^{2\alpha}$$

we get

$$\lim_{\delta \downarrow 0} \limsup_n \mathbf{P}(|(M_\delta^n, A_\delta^n) - (M_0^n, A_0^n)| > \varepsilon) = 0$$

for all $\varepsilon > 0$. It is evident that for every $\lambda > 0$, $s < r < t$

$$\begin{aligned} & \mathbf{P}(|(M_r^n, A_r^n) - (M_s^n, A_s^n)| \geq \lambda, |(M_t^n, A_t^n) - (M_r^n, A_r^n)| \geq \lambda) \\ & \leq \mathbf{P}(|M_r^n - M_s^n||M_t^n - M_r^n| + |M_r^n - M_s^n||A_t^n - A_r^n| \\ & \quad + |A_r^n - A_s^n||M_t^n - M_r^n| + |A_r^n - A_s^n||A_t^n - A_r^n| \geq \lambda^2). \end{aligned}$$

Let $\beta > (2\alpha)^{-1}$. By inequality (6) and the fact that the increments of the process M^n are independent and Gaussian we get

$$\begin{aligned} I & := \mathbf{E}|M_r^n - M_s^n|^2 \mathbf{E}|M_t^n - M_r^n|^2 + c \mathbf{E}^{\beta/2} |M_r^n - M_s^n|^2 \mathbf{E}^{1/2} |A_t^n - A_r^n|^{2\beta} \\ & \quad + c \mathbf{E}^{1/2} |A_r^n - A_s^n|^{2\beta} \mathbf{E}^{\beta/2} |M_t^n - M_r^n|^2 + \mathbf{E}^{1/2} |A_r^n - A_s^n|^{2\beta} \mathbf{E}^{1/2} |A_t^n - A_r^n|^{2\beta} \\ & \leq \tilde{C} \left(\frac{[nt]}{n} - \frac{[ns]}{n} \right)^{2\alpha\beta}, \end{aligned}$$

where $c = c(\beta)$, \tilde{C} , and \tilde{C} are constants. If now $t - s \geq 1/n$ we have

$$I \leq \tilde{C} |2(t - s)|^{2\alpha\beta}. \quad (8)$$

If on the other hand $t - s < 1/n$ then either s and r or r and t lie in the same subinterval $[i/n, (i + 1)/n)$ for some i . Thus $I = 0$. Therefore (8) holds for all $s < r < t$. Thus for some constant C we obtain

$$\begin{aligned} & \mathbf{P}(|(M_r^n, A_r^n) - (M_s^n, A_s^n)| \geq \lambda, |(M_t^n, A_t^n) - (M_r^n, A_r^n)| \geq \lambda) \\ & \leq C(\lambda^{-4} \vee \lambda^{-2\beta})(t - s)^{2\alpha\beta}. \end{aligned}$$

References

- [1] J.-M. Bardet, Lang, G., Oppenheim, G., Philippe, A., Taqqu, M.S., Generators of long-range dependent processes: A survey, in: *Theory and Applications of Long-Range Dependence*, P. Doukhan, G. Oppenheim, M.S. Taqqu (Eds), Birkhauser, Boston (2003).
- [2] I. Descrusefond, A.S. Üstünel, Stochastic analysis of the fractional Brownian motion, *Potential Analysis*, **10**, 177–214 (1999).
- [3] J. Jacod, A.N. Shiriyayev, *Limit Theorems for Stochastic Processes*, Springer–Verlag, Berlin Heidelberg New York London Paris Tokyo (1987).
- [4] T. Sottinen, Fractional Brownian motion, random walks and binary market models, *Finance Stochast.*, **5**, 343–355 (2001).
- [5] T. Szabados, Strong approximation of fractional Brownian motion by moving averages of simple random walks, *Stochastic Processes Appl.*, **92**, 31–60 (2001).

Trupmeninio Brauno judesio aproksimacijos silpno konvergavimo klausimu

K. Kubilius

Naudojantis trupmeninio Brauno judesio išraiška $B_t^H = \int_0^t K_H(t, s) dW_s$, čia W – standartinis Brauno judesys, $K_H(t, s)$ – neatsitiktinis branduolys, sukonstruota pakankamai paprasta aproksimacija, kuri silpnai konverguoja į trupmeninį Brauno judesį.