Liet. matem. rink., T. 43, spec. nr., 449–453

Prefixed tableaux for multi-modal logic of knowledge with inclusions

Jūratė SAKALAUSKAITĖ (MII)
e-mail: jurates@ktl.mii.lt

1. Introduction

The logic denoted by $S5$ and its multi-modal version denoted by $S5_n$ are widely recognised as logics of knowledge (see, e.g., [4]).

In this paper we consider the certain extensions of the logic $S5_n$. $S5_n$ contains an indexed set of n independent unary modal connectives that allow us to represent the information possessed by a group of agents. We consider a family of the extensions of $S5_n$, where these connectives are interrelated, i.e., an accessibility relation can be included into another one. We denote a set of such inclusions by **I** and we denote the corresponding extension of $S5_n$ by $S5_n^{\mathbf{I}}$.

In this paper, we present a prefixed analytic tableau proof system for $S5_n^{\mathbf{I}}$ which is a generalization of the prefixed tableau proof system described by Massacci for $S5$ [5]. We use prefixes reasoning about accessibility. But prefixes are more complex than in the mono-modal case. We prove soundness and completeness of the presented system.

Some prefixed tableau systems for $S5_n^{\mathbf{I}}$ are presented in [1] and [7]. Our system is closer to that described in [7] but we do not use the side conditions on inference rules reasoning about accessibility.

Our paper is organized as follows. In Section 2 we give syntax and semantics of the logic $S5_n^{\mathbf{I}}$. In Section 3 we present a tableau calculus for $S5_n^{\mathbf{I}}$. In Section 4 we sketch the proof of soundness and completeness of this calculus.

2. Syntax and semantics of $S5_n^{\mathbf{I}}$

Given the set $\mathcal{P} = \{p, \ldots\}$ of propositional variables, the set of formulas is inductively defined as follows:

1) any $p \in \mathcal{P}$ is a formula and is called an atomic formula;

2) if A, B are formulas then so are $\neg A$, $A \wedge B$, $\Box_1 A, \ldots, \Box_n A$.

A *frame* is a structure $\mathcal{F} =< W, R_1, \ldots, R_n >$ such that W is a non-empty set, whose elements are called *worlds*, and R_i is a binary relation on W, $i \in \{1, \ldots, n\}$. We write wR_iw' iff $(w, w') \in R_i$, and we say that w' is accessible from w by the relation R_i. A *model* is a structure $\mathcal{M} =< \mathcal{F}, V >$ such that $\mathcal{F}$ is a frame and V is a valuation V: $\mathcal{P} \to 2^W$. The satisfiability relation $\models$ is defined inductively in the usual way:

$\mathcal{M}, w \models p$	iff	$w \in V(p)$ for every $p \in \mathcal{P}$;
$\mathcal{M}, w \models \neg A$	iff	$\mathcal{M}, w \not\models A$;
$\mathcal{M}, w \models A \wedge B$	iff	$\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$;
$\mathcal{M}, w \models \Box_i A$	iff	for every v such that wR_iv, we have $\mathcal{M}, v \models A$.

A formula A is said to be *true* in the model (written $\mathcal{M} \models A$) iff for every world $w \in W$ $\mathcal{M}, w \models A$. A formula A is said to be *true* in the frame $\mathcal{F}$ (written $\mathcal{F} \models A$) iff A is true in every model based on $\mathcal{F}$.

Let $\mathbf{I}$ be a subset of the set $\{R_{i_1} \subseteq R_{i_2} | 1 \leqslant i_1, i_2 \leqslant n\}$ of inclusions such that: 1) $R_i \subseteq R_i \in \mathbf{I}$ for each $1 \leqslant i \leqslant n$; 2) if $R_i \subseteq R_j \in \mathbf{I}$ and $R_j \subseteq R_k \in \mathbf{I}$, then $R_i \subseteq R_k \in \mathbf{I}$.

A *frame of logic* $S5_n^{\mathbf{I}}$ is a frame $< W, R_1, \ldots, R_n >$ such that: 1) for every $1 \leqslant i \leqslant n$ R_i is an equivalence relation; 2) $R_1, \ldots, R_n$ satisfy the property $\mathbf{I}$. A formula A is said to be $S5_n^{\mathbf{I}}$*-satisfiable* iff there exists a model $\mathcal{M}$ based on some frame of the logic $S5_n^{\mathbf{I}}$ and $w \in W$ such that $\mathcal{M}, w \models A$. A formula A is said to be $S5_n^{\mathbf{I}}$*-valid* iff for every frame $\mathcal{F}$ of the logic $S5_n^{\mathbf{I}}$, we have $\mathcal{F} \models A$.

3. Tableau calculus for $S5_n^{\mathbf{I}}$

By $\mathcal{I}$, $\mathcal{J}$ below we denote subsets of $\{1, \ldots, n\}$. A *prefix* is an alternating sequence of integers and subsets of $\{1, \ldots, n\}$. (This is a generalization of modal case (e.g., [2])).

DEFINITION 3.2. The set of prefixes Σ is the least set such that $1 \in \Sigma$ and if $\sigma \in \Sigma$, $\mathcal{I} \subseteq \{1, \ldots, n\}$ and m is an integer, then $\sigma\mathcal{I}m \in \Sigma$.

Intuitively a prefix $\sigma\{i_1, \ldots, i_k\}m$ names a world accessible by relations $R_{i_1}, \ldots, R_{i_k}$ from a world named by σ.

Tableaus use *prefixed formulas*, i.e., pairs σ: A where σ is a prefix and A is a formula. A *tableau* $\mathcal{T}$ is a binary tree each node of which is labelled with a prefixed formula. A *branch* $\mathcal{B}$ is a path from the root downwards in such a tree. Nodes are added and labelled by applying the tableaux rules presented below. The rules are applied downwards as defined for prefixed tableaux by Fitting [2]. A prefix σ is *present* in a branch if there is a prefixed formula σ: A for some formula A in that branch. A prefix $\sigma\mathcal{I}n$ is *new* to a branch if there is no prefix of a kind $\sigma\mathcal{J}n$ present in that branch.

The tableau calculus for $S5_n^{\mathbf{I}}$ consists of the following tableaux rules.

1. Propositional connective tableaux rules:

$$\frac{\sigma: A \wedge B}{\begin{array}{c}\sigma: A\\ \sigma: B\end{array}} (\wedge) \quad \frac{\sigma: \neg(A \wedge B)}{\sigma: \neg A | \sigma: \neg B} (\vee) \quad \frac{\sigma: \neg\neg A}{\sigma: A} (\neg)$$

2. Modal tableaux rules. Below we fix $i \in \{1, \ldots, n\}$.

2.1. π_i-rule.

Let $R_i \subseteq R_{i_1}, \ldots, R_i \subseteq R_{i_k}$ be all inclusions from **I** where R_i is on the left side of an inclusion.

$$\frac{\sigma\colon \neg\Box_i A}{\sigma\{i, i_1, \ldots, i_k\}n\colon \neg A}\ (\pi_i), \text{ where } \sigma\{i, i_1, \ldots, i_k\}n \text{ is new to the current branch.}$$

2.2. ν_i-rules:

$$\frac{\sigma\colon \Box_i A}{\sigma\mathcal{I}n\colon A}\,(K_i) \qquad \frac{\sigma\colon \Box_i A}{\sigma\colon A}\,(T_i) \qquad \frac{\sigma\colon \Box_i A}{\sigma\mathcal{I}n\colon \Box_i A}\,(4_i) \qquad \frac{\sigma\mathcal{I}n\colon \Box_i A}{\sigma\colon \Box_i A}\,(4^r{}_i)$$

Here $\sigma, \sigma\mathcal{I}n$ must be present in the branch and $i \in \mathcal{I}$.

A branch $\mathcal{B}$ is *closed* if it contains some prefixed formula $\sigma\colon A$ and also $\sigma\colon \neg A$. A tableau is *closed* if every branch of it is closed, otherwise it is open. A *tableau proof* for a formula A is a closed tableau starting from $1\colon \neg A$.

4. Soundness and completeness

Theorem 4.1 (soundness). *If a formula A has a tableau proof, then A is a $S5_n^{\mathbf{I}}$-valid formula.*

The proof of the theorem is similar to that presented in [6].

The rest of the section is devoted to the sketch of the proof of the following

Theorem 4.2 (completeness). *If a formula A is a $S5_n^{\mathbf{I}}$-valid formula, then A has a tableau proof.*

The completeness proof follows the ideas of Fitting [2]: a systematic tableau construction is presented that must produce a tableau proof if one exists and, if one does not, it will give us the information necessary to construct a counter-model. The counter-model is built from saturated open branches (see definitions below).

For calculus for $S5_n^{\mathbf{I}}$ we get a systematic tableau construction by adapting one presented by Gore in [3], page 374: instead of using π-, ν- rules of the mono-modal case use π_i-, ν_i- rules $(i \in \{1, \ldots, n\})$, respectively, of the calculus for $S5_n^{\mathbf{I}}$.

In order to prove the model existence theorem (see Theorem 4.9 below) which is the basis for the proof of the completeness theorem we need some preliminary definitions and lemmas.

Let $\mathcal{B}$ be a branch. We introduce the binary relations $\rhd_i$, $i \in \{1, \ldots, n\}$ over prefixes present in $\mathcal{B}$. $\rhd_i$ is defined as follows: 1) $\sigma \rhd_i \sigma$; 2) $\sigma \rhd_i \sigma\mathcal{I}_1 m_1 \ldots \mathcal{I}_k m_k$, $k \geqslant 1$, $i \in \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_k$; 3)$\sigma\mathcal{I}_1 m_1 \ldots \mathcal{I}_k m_k \rhd_i \sigma$, $k \geqslant 1$, $i \in \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_k$; 4)$\sigma\mathcal{I}_1 m_1 \ldots \mathcal{I}_k m_k \rhd_i \sigma\mathcal{J}_1 n_1 \ldots \mathcal{J}_l n_l$, $k \geqslant 1$, $l \geqslant 1$, $i \in \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_k$, $i \in \mathcal{J}_1 \cap \ldots \cap \mathcal{J}_l$. Here σ is a non-empty sequence.

It can be verified

Lemma 4.3. 1) $\rhd_i$ *is an equivalence relation. 2) If* $R_i \subseteq R_j \in \mathbf{I}$, *then* $\rhd_i \subseteq \rhd_j$.

Let $\mathcal{B}$ be a branch of a tableau.

DEFINITION 4.4. A prefixed formula $\sigma: A$ is *reduced* by the rule (ρ) in $\mathcal{B}$

- if (ρ) has the form $\dfrac{\sigma: A}{\sigma_1: A_1}$, then $\sigma_1: A_1$ is in $\mathcal{B}$;
- if (ρ) has the form $\dfrac{\sigma: A}{\begin{array}{c}\sigma_1: A_1\\ \sigma_2: A_2\end{array}}$, then $\sigma_1: A_1$ and $\sigma_2: A_2$ are in $\mathcal{B}$;
- if (ρ) has the form $\dfrac{\sigma: A}{\sigma_1: A_1 | \sigma_2: A_2}$, then at least one of $\sigma_1: A_1$ and $\sigma_2: A_2$ is in $\mathcal{B}$.

The formula $\sigma: A$ is *fully reduced* in $\mathcal{B}$ if it is reduced by all applicable rules. A prefix σ is *fully reduced* if all prefixed formula $\sigma: A$ are fully reduced.

DEFINITION 4.5. A branch $\mathcal{B}$ is *saturated* if all prefixes in $\mathcal{B}$ are fully reduced.

Similarly as in [6] we can prove

Lemma 4.6. *Let $\mathcal{B}$ be a branch of a tableau. If $\mathcal{B}$ is saturated, then the following holds: if $\sigma: \Box_i A$ is in $\mathcal{B}$, σ^* is present in $\mathcal{B}$, and $\sigma \rhd_i \sigma^*$, then $\sigma^*: A$ is also in $\mathcal{B}$.*

Now we define a mapping between names and worlds and introduce the notion of satisfiable branch.

DEFINITION 4.7. Let $\mathcal{B}$ be a branch. Let $< W, R_1, \ldots, R_n, V >$ be a model, an *interpretation* $\imath()$ in the model is a mapping from the set of prefixes that occur in $\mathcal{B}$ to worlds W such that for all σ and $\sigma\mathcal{I}n$ present in $\mathcal{B}$ one has $\imath(\sigma)R_j\imath(\sigma\mathcal{I}n)$ for each $j \in \mathcal{I}$.

DEFINITION 4.8. A tableau branch $\mathcal{B}$ is satisfiable iff there is a model $< W, R_1, \ldots, R_n, V >$ and an interpretation $\imath()$ in this model such that for every prefixed formula $\sigma: A$ in $\mathcal{B}$, $\imath(\sigma) \models A$.

Theorem 4.9 (model existence). *If $\mathcal{B}$ is a saturated and an open branch, then there exists a model where $\mathcal{B}$ is satisfiable.*

Proof. Construct the model as follows:

$$W = \{\sigma | \sigma \text{ is present in } \mathcal{B}\};$$
$$\sigma R_i \sigma^* \text{ iff } \sigma \rhd_i \sigma^*, \quad 1 \leqslant i \leqslant n;$$
$$V(p) = \{\sigma | \sigma: p \in \mathcal{B}\}.$$

By Lemma 4.2 $< W, R_1, \dots, R_n >$ is a frame we are looking for. We must show that $\mathcal{B}$ is satisfiable in the model $< W, R_1, \dots, R_n, V >$ with interpretation $\imath()$ (here $\imath()$ is the identity function): for every A we must prove $\sigma \colon A \in \mathcal{B}$ then $\sigma \models A$ by induction on the construction of A.

We consider only modal connectives. Let $\sigma \colon \neg\Box_i A \in \mathcal{B}$. Then we have $\sigma\mathcal{I}n \colon \neg A \in \mathcal{B}$ for some $\sigma\mathcal{I}n$, where $i \in \mathcal{I}$ (since $\sigma \colon \neg\Box_i A$ must be reduced by the rule (π_i)). By inductive hypothesis $\sigma\mathcal{I}n \models \neg A$ and $\sigma R_i \sigma\mathcal{I}n$ by construction. Therefore $\sigma \models \neg\Box_i A$. Let $\sigma \colon \Box_i A \in \mathcal{B}$. Then by Lemma 4.5, for every σ^* present in $\mathcal{B}$ such that $\sigma \rhd_\imath \sigma^*$, one has $\sigma^* \colon A \in \mathcal{B}$. By inductive hypothesis $\sigma^* \models A$ and $\sigma R_i \sigma^*$ by construction. Therefore $\sigma \models \Box_i A$.

The proof of completeness Theorem 4.2 is now standard (see page 410 in [2]).

References

[1] M. Baldoni, Normal multimodal logics with interaction axioms, in: D. Basin *et al.* (Eds.), *Labelled Deduction*, **17**, Applied Logic Series, Kluwer Academic Publishers (2000), pp. 33–53.

[2] M. Fitting, *Proof Methods for Modal and Intuitionistic Logics*, **169**, Synthese Library, D. Reidel Publishing Company, Dordrecht (1983).

[3] R. Goré, Tableau methods for modal and temporal logics, in: D'Agostino *et al.* (Eds.), *Handbook of Tableau Methods*, Kluwer Academic Publishers (1999), pp. 297–396.

[4] J. Halpern, Y. Moses, A guide to completeness and complexity for modal logics of knowledge and belief, *Artificial Intelligence*, **54**, 319–379 (1992).

[5] F. Massacci, Strongly analytic tableaux for normal modal logics, in: *Proc. 12th Conference on Automated Deduction*, LNAI, **814** (1994), pp. 123–137.

[6] F. Massacci, Single step tableaux for modal logics: methodology, computations, algorithms, *Journal of Automated Reasoning*, **24**, 319–364 (2000).

[7] M. Stone, *First-Order Multi-Modal Deduction*, Technical Report RUCCS TR 55, Rutgers University Center for Cognitive Science (1999).

Semantinės lentelės su prefiksais žinojimo logikos praplėtimui

J. Sakalauskaitė

Darbe pateikiamas semantinių lentelių su prefiksais skaičiavimas daugelio modalumų žinojimo logikos tam tikram praplėtimui. Įrodomas šio skaičiavimo neprieštaringumas bei pilnumas.