Liet. matem. rink., **44**, *spec. nr.*, 2004, 525–529

Decision procedure for a combination of logics $KD4$ and PDL

Aida PLIUŠKEVIČIENĖ (MII)
e-mail: aida@ktl.mii.lt

1. Introduction

Combinations of modal logics are used as a formal theory that can be helpful for the specification, development, modelling and even the execution of rational agents (see, e.g., [6]). The best known logical theories of rational agents are BDI logics (for Belief, Desire and Intention) [5] and $KARO$ logics (for Knowledge, Abilities, Results and Opportunities) [2]. In BDI logics each rational agent is viewed as having three mental attitudes: belief (the main component of BDI logics), desire and intention. The BDI logics are fusion of various propositional temporal logics and propositional multi-modal logics expressing properties of the mental attitudes. The $KARO$ logics focus on the dynamic of mental states: how actions can change agents knowledge (believes), desires, and so on. The $KARO$ logics are combination of propositional dynamic logic (PDL) [1] and logics of knowledge [2]. In [6] it is described a rich logic $LORA$ (Logic of Rational Agents) based on BDI logic and dynamic logic. Tableau-based decision procedures for BDI logics are presented in [5]. Sequent-based decision procedure for BDI logics is presented in [3]. Sequent-based decision procedures for PDL are presented in [4].

In this paper a propositional dynamic logic of belief ($PDLB$) is considered. The aim of this paper is to present a deduction-based decision procedure for a fragment of $PDLB$. A $PDLB$ is a fusion of a deterministic propositional dynamic logic and logic $KD4$ (which describes properties of asymmetric belief operator [6]) containing variables for actions and propositional variables. Belief operator is the main one describing behaviour of rational agents [6]. The object of consideration in the presented fragment of $PDLB$ is R-sequents (Section 2). Because of a possibility to reduce a R-sequent to a set of R-sequents having some normal form (reduced primary R-sequents, Section 2) a traditional non-invertible, in general, rules for belief modality [3] can be inserted into the disjunctive invertible separation rules (Section 3).

Here a procedural approach of decidable logical calculi is used and we assume that the notions of a decidable calculus and the deduction-based decision procedure are identical. The presented decision procedure is based on sequent-like calculus DB with loop-type axioms (analogously as in [3]).

2. Language of $FTLB$

$PDLB$ is a fusion of two logics, namely, modal $KD4$ and deterministic PDL. $PDLB$ contains two-sorts of variables: actions and propositional ones.

A language of $PDLB$ contains: a denumerable set of propositional variables; a denumerable set of actions variables $\alpha_1, \alpha_2, \ldots$; action modalities $[\alpha_k]$; belief modality $\mathcal{B}$; logical symbols: $\supset, \wedge, \vee, \neg$; action operators: $\circ$ ("composition"), $\cup$ ("non-deterministic choice"), $*$ ("non-deterministic iteration" or "star"), ? ("test"). An arbitrary propositional variable is an atomic formula. Formulas and actions of $PDLB$ are defined inductively as follows: an atomic formula is formula; any action variable is an action; if α and β are actions then $(\alpha \circ \beta)$, $(\alpha \cup \beta)$, α^* are actions; if A, B are formulas, α is an action then $A \supset B$, $A \wedge B$, $A \vee B$, $\neg(A)$, $[\alpha]A$, $\mathcal{B}A$ are formulas; a formula A is a logical one if A contains only logical and propositional variables; if P is a logical formula then $P?$ is an action.

Thus, in formulas we consider two types of modalities, namely, belief modality $\mathcal{B}$, and action modalities $[\alpha_i]$.

DEFINITION 1 (R-sequent, induction-free R-sequent). *A sequent S is an R-sequent if S satisfies the following regularity condition: if a formula σA (where $\sigma \in \{\mathcal{B}, [\alpha^*]\}$) occurs negatively in S then A does not contain positive occurrences of the belief modality, and action variables. An R-sequent S is an induction-free one, if S does not contain positive occurrences of the operator $*$ ("star"). Otherwise, an R-sequent S is a non-induction-free one.*

Since we consider asymmetric belief modality, belief accessibility relation is only distributive, serial and transitive, but asymmetric. Therefore the formulas $\mathcal{B}(P \supset Q) \wedge \mathcal{B}P \supset \mathcal{B}Q$, $\mathcal{B}P \supset \mathcal{B}\mathcal{B}P$, $\mathcal{B}P \supset \neg\mathcal{B}\neg P$ (expressing, correspondingly, the distributive, transitive, and serial properties of the accessibility relation) are valid in $KD4$. But formula $P \supset \mathcal{B}\neg\mathcal{B}\neg P$ (expressing symmetric property of the accessibility relation) is invalid in $KD4$.

3. Some auxiliary tools of the decision algorithm

In this section, we present the main axiomatic tools of the decision algorithm for R-sequents: separation and reduction rules, and marked contraction rules. First, let us introduce some canonical forms of R-sequents.

An R-sequent S is *a primary* R-sequent, if $S = \Sigma_1, \mathcal{B}\Gamma_1, [\alpha_k]\Pi_1, [\alpha^*]\Omega_1 \to \Sigma_2, \mathcal{B}\Gamma_2, [\beta_l]\Pi_2, [\beta^*]\Omega_2$, where for every i ($i \in \{1, 2\}$) Σ_i is empty or consists of logical formulas; $\mathcal{B}\Gamma_i$ is empty or consists of formulas of the shape $\mathcal{B}A$; $[\alpha_k]\Pi_1$ ($[\beta_l]\Pi_2$)is empty or consists of formulas of the shape $[\alpha_k]C$ ($[\beta_l]D$, respectively); $[\alpha^*]\Omega_1$ ($[\beta^*]\Omega_2$)is empty or consists of formulas of the shape $[\alpha^*]M$ ($[\beta^*]N$, respectively). An R-sequent S is a *reduced primary* R-sequent if S is a primary one and does not contain $[\alpha^*]\Omega_1$, $[\beta^*]\Omega_2$.

Let us define reduction rules by means of which each R-sequent can be reduced to a set of primary and reduced primary R-sequents.

Reduction rules consist of the traditional invertible rules for logical symbols and the following rules for actions:

$$\frac{\Gamma \to \Delta, [\alpha][\beta]A}{\Gamma \to \Delta, [\alpha \circ \beta]A}(\to \circ) \qquad \frac{[\alpha][\beta]A, \Gamma \to \Delta}{[\alpha \circ \beta]A, \Gamma \to \Delta}(\circ \to)$$

$$\frac{\Gamma \to \Delta, [\alpha]A;\ \Gamma \to \Delta, [\beta]A}{\Gamma \to \Delta, [\alpha \cup \beta]A}(\to \cup) \qquad \frac{[\alpha]A, [\beta]A, \Gamma \to \Delta}{[\alpha \cup \beta]A, \Gamma \to \Delta}(\cup \to)$$

$$\frac{\Gamma, P \to \Delta, B}{\Gamma \to \Delta, [P?]B}(\to ?) \qquad \frac{\Gamma \to \Delta, P;\ B, \Gamma \to \Delta}{[P?]B, \Gamma \to \Delta}(? \to)$$

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, [\alpha][\alpha^*]A}{\Gamma \to \Delta, [\alpha^*]A}(\to *) \qquad \frac{A, [\alpha][\alpha^*]A, \Gamma_1 \to \Delta_1}{[\alpha^*]A, \Gamma_1 \to \Delta_1}(* \to),$$

where $\Gamma_1 \to \Delta_1$ contains positive occurrences of $*$ ("star") and action constants,

$$\frac{A, \Pi \to \Theta}{[\alpha^*]A, \Pi \to \Theta}(*_0 \to),$$

where $\Pi \to \Theta$ does not contain positive occurrences either $*$ ("star") or action constants.

From the shape of the primary R-sequent it is easy to see that bottom-up applying logical rules, and action rules, except the rules for of the "star" operator, each R-sequent can be reduced to a set of primary R-sequents. As follows from the shape of reduced primary R-sequents bottom-up applying reduction rules each primary R-sequent can be reduced to a set of reduced primary R-sequents.

To define separation rules let us introduce a belief-type mark. The mark is of the shape σ^* (where $\sigma \in \{\mathcal{B}, [\alpha_k]\}$), and is defined as follows: if $B = \mathcal{B}A$, then each occurrence of belief and action modalities in A is marked and $\sigma^{**} = \sigma^*$. The mark is meant to restrict applications of separation rules for belief modality and to exclude loops for induction-free R-sequents.

The separation rules (SR_i), $i \in \{1, 2\}$ is of the following shape, where the conclusion of the rules is a reduced primary R-sequent, such that the sequent $\Sigma_1 \to \Sigma_2$ is not derivable in a propositional logic.

$$\frac{\Pi^{\circ}_{1,i} \to \Pi_{2,j}}{\Sigma_1, \mathcal{B}\Gamma_1, [\alpha_k]\Pi_1 \to \Sigma_2, \mathcal{B}\Gamma_2, [\beta_l]\Pi_2}(SR_1),$$

where $[\alpha_k]\Pi_1 = [\alpha_1]\Pi_{1,1}, \ldots, [\alpha_n]\Pi_{1,n}, (n \geqslant 0)$, i.e., $[\alpha_k]\Pi_1$ may be the empty word; $[\beta_l]\Pi_2 = [\beta_1]\Pi_{2,1}, \ldots, [\beta_m]\Pi_{2,m}$ $(m \geqslant 1)$, and $[\alpha_i]\Pi_{1,i}$, $0 \leqslant i \leqslant n$, $([\beta_j]\Pi_{2,j}$, $1 \leqslant j \leqslant m)$ consists of formulas of the shape $[\alpha_i]B_{i,l}$ (of the shape $[\beta_j]M_{j,p}$, respectively); $\Pi^{\circ}_{1,i} = \varnothing$, if $\alpha_i \neq \beta_j$ and $\Pi^{\circ}_{1,i} = \Pi_{1,i}$ in opposite case.

$$\frac{\mathcal{B}^*\Gamma_1, \Gamma_1 \to A_i}{\Sigma_1, \mathcal{B}\Gamma_1, [\alpha_k]\Pi_1 \to \Sigma_2, \mathcal{B}\Gamma_2, [\beta_l]\Pi_2}(SR_2),$$

where $\mathcal{B}\Gamma_2 = \mathcal{B}A_1, \ldots, \mathcal{B}A_i, \ldots, \mathcal{B}A_k$, $(k \geqslant 0)$ and $A_i = \varnothing$, if $k = 0$.

Marked contraction rule is defined using an equality $\sigma^* A$, $\sigma A = \sigma^* A$ (where $\sigma \in \{\mathcal{B}, [\alpha_k]\}$). During the reduction to primary and reduced primary R-sequents the marked contraction rule and the ordinary contraction rule (using an equality $A, A = A$ which follows from the set-type notion of a sequent) will be used implicitly.

4. Decision procedure for *R*-sequents

A decision procedure for induction-free R-sequents is realized by means of calculus $IFDB$ for the induction-free propositional dynamic logic of belief. The calculus $IFDB$ consists of the separation rules (SR_l) ($l \in \{1, 2\}$), the reduction rules, except the rule $(\to *)$, and logical axiom $\Gamma, A \to \Delta, A$.

Using induction on the height of derivation, we can show that all the reduction rules of the calculus $IFTBA$ are invertible. The separation rules (SR_i) are not simply invertible, but they are disjunctively invertible.

Using induction on the height of derivation, we can prove the following

LEMMA 1. *Let S be a conclusion of the rules (SR_i) ($i \in \{1, 2\}$) and $IFDB \vdash S$, then either there exist i, j such that $IFDB \vdash \Pi^{\circ}_{1,i} \to \Pi_{2,j}$, or there exists i such that $IFDB \vdash \mathcal{B}^*\Gamma_1, \Gamma_1 \to A_i$.*

An R-sequent S^* is *b-final* if S^* is not an axiom and contains only propositional variables and/or marked modalities.

The decision procedure for an induction-free sequent S is realized by constructing ordered derivations in the calculus $IFTBA$.

DEFINITION 2 (ordered derivation, successful and unsuccessful derivation). *An ordered derivation D for induction-free R-sequents consists of several horizontal levels. Each level consists of bottom-up applications of reduction rules. At each level, where a set consisting of only reduced-primary R-sequents is received, all possible bottom-up applications of the separation rules (SR_i), $i \in \{1, 2\}$ to every reduced-primary R-sequent are realized. Each bottom-up application of the separation rules (SR_i) ($i \in \{1, 2\}$) provides a possibility to construct a different (in general) ordered derivation D_k ($k \geqslant 1$).*

An ordered derivation D_k is a successful one, if each leaf of D_k is a logical axiom. An ordered derivation D_k is a unsuccessful one if in D_k there exists a branch having such a leaf that either a sequent in this leaf contains only atomic formulas and is not an axiom, or a sequent in this leaf is an induction-free R-sequent S^* such that S^* is a b-final R-sequent.

Using the shape of the calculus $IFDB$ and invertibility of the rules of $IFDB$ we can get

THEOREM 1. *Let S be an induction-free R-sequent. Then either one can automatically construct a successful ordered derivation D_k of the R-sequent S in $IFDB$, i.e., $IFDB \vdash S$, or all possible ordered derivations D_k are unsuccessful, i.e., $IFDB \nvdash S$. The process of construction of the ordered derivation D_k of the R-sequent S in $IFDB$ always terminates.*

A decision algorithm for an arbitrary R-sequent is realized by means of a calculus DB containing non-logical (loop-type) axiom.

Let D be a derivation in some calculus and (i) be a branch in D. The R-sequent $S^* = \Gamma \rightarrow \Delta$ from the branch (i) is a *saturated* R-sequent if, in the branch (i) above S^*, there exists an R-sequent of the shape $S^{**} = \Gamma, \Pi \rightarrow \Delta, \Theta$, in a special case, $S^* = S^{**}$.

A saturated R-sequent S^* is *a-saturated* if $S^* = \Gamma \rightarrow \Delta, [\alpha^*]A$. These sequents will be used as non-logical axiom.

A calculus DB is obtained from the calculus $IFDB$ by adding: (1) a non-logical axiom of the shape $\Gamma \rightarrow \Delta, [\alpha^*]A$ and (2) the reduction rule $(\rightarrow *)$. Disjunctive invertibility of separation rules in DB is provable using infinitary rule for the "star" operator (instead of the rule $(\rightarrow *)$ and non-logical axiom) and proving that this infinitary rule is admissible in the calculus DB for R-sequents.

We can present the decision procedure for an arbitrary R-sequent in the same way as in the case of induction-free R-sequents. Namely, we construct ordered derivations in the same manner, as described above. But there is a new substantial point: along with the logical axiom there is a *non-logical axiom*. If there *exists* an ordered derivation D_k of R-sequent S such that in *each* leaf of D there is either a logical axiom, or a non-logical axiom, then $DB \vdash S$. If in *all* the possible ordered derivations D_k of an R-sequent S there *exists* a branch having an induction-free R-sequent S^+ such that $IFDB \nvdash S^+$, then $DB \nvdash S$.

THEOREM 2. *Let S be a non-induction-free R-sequent. Then one can automatically construct a successful or unsuccessful ordered derivation D of the R-sequent S in DB. This process always terminates.*

Proof. The automatic way of construction of an ordered derivation D and correctness (i.e., preservation of derivability) follow from invertibility of the rules of DB; the termination follows from finiteness of the generated subformulas in D.

References

1. D. Harel, D. Kozen, J. Tiuryn, *Dynamic Logic*, MIT Press (2000).
2. J.-J.C. Meyer, W. van der Hoek, B. van Linder, A logical approach to the dynamics of commitments, *Artificial Intelligence*, **113**(1–2), 1–40 (1999).
3. N.Nide, S.Takata, Deduction systems for BDI logics using sequent calculus, *Proc. of AAMAS'02*, 928–935 (2002).
4. V.R. Pratt, A near-optimal method for reasoning about action, *Journal of Computer and System Sciences*, **20**, 231–254 (1980).
5. A.S. Rao, M.P.Georgeff, Decision procedures for BDI logics, *Journal of Logic and Computation*, **8**(3), 292–343 (1998).
6. M. Wooldridge. *Reasoning about Rational Agents*, The MIT Press (2000).

REZIUMĖ

A. Pliuškevičienė. Išsprendžiamoji procedūra $KD4$ ir PDL logikų apjungimui

Pasiūlyta dedukcija pagrįsta išprendžiamoji procedūra modalinės logikos $KD4$ ir propozicinės dinaminės logikos PDL apjungimui. Pasiūlyta išprendžiamoji procedūra yra korektiška ir pilna.