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# Decision procedures for quantified fragments of reflexive common knowledge logic

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## 1. Introduction

Logics of knowledge, especially the common knowledge logics, have a lot of applications in computer science and artificial intelligence (see, e.g., [2], [3], [4]). On the other hand, common knowledge operator satisfies induction-like postulates and for this reason is interesting from a logical point of view. A decision procedure for propositional irreflexive common knowledge logic (based on multi-modal logics $K_n$) can be get relying on sequent-like calculus with analytic cut presented in [1]. Propositional logics for knowledge-based logics are often insufficient for more complex real world situations. First-order extensions of these logics are necessary whenever an application domain is infinite or a cardinality of application domain is not known in advance. In [6] it is presented decidability of some fragments of first-order one-sorted irreflexive common knowledge logics. In [2] it is proved general decidability results for some fragments of first-order one-sorted agent-based logics.

In this paper decidable fragments of first-order two-sorted logic of reflexive common knowledge ($FRCL$) are considered. A language of $FRCL$ is based on first-order two-sorted extension of common knowledge logic [4], containing individual knowledge operators, reflexive "common knowledge" operator and "everyone knows" operator. A reflexive common knowledge is based on reflexive and transitive closure of individual knowledge. The irreflexive knowledge (see, e.g., [3]) is based only on transitive closure of individual knowledge. Individual knowledge operators satisfy modal postulates of first-order two-sorted multi-modal logic $K_n$. A language of $FRCL$ contains two sorts of variables and constants, namely, variables and constants for agents, variables and constants for other individuals.

## 2. Language of *FRCL* and calculi

$FRCL$ is a first-order version of two-sorted multi-modal logic of reflexive common knowledge denoted as $K_n(\mathcal{C})$.

A language of $FRCL$ contains: a denumerable set of predicate symbols; a denumerable set of agent constants $a_1, a_2, \ldots$; a denumerable set of constants for other individuals $c_1, c_2, \ldots$; a denumerable set of agent variables $x^a, y^a, z^a, x_1^a, y_1^a, \ldots$; a denumerable set of variables for other individuals $x, y, z, x_1, y_1, \ldots$; logical symbols: $\supset, \wedge, \vee, \neg, \forall, \exists$; knowledge operators: individual knowledge operators

$[t_k^a]$ (where $k \in \{1, \dots, m\}$ and $m \geq 1$, $t_k^a$ is an agent term); "everyone knows" operator $\mathcal{E}$ and "common knowledge" operator $\mathcal{C}$. A term is a constant or a variable. An agent term is an agent constant or an agent variable. Formulas are constructed in a traditional way. A formula (sequent) is a logical one if it contains only logical symbols and atomic formulas.

The formula $[t_i^a](A)$ means: "agent $t_i$ knows that $A$". The knowledge operators $[t_i^a]$ $(1 \leqslant i \leqslant n)$ satisfy axioms of the basic multi-modal logic $K_n$ (as in [1]). The formula $\mathcal{E}(A)$ means: "everybody agent $i \in \{1, \dots, n\}$ knows $A$", i.e., $\mathcal{E}(A) \equiv \bigwedge_{i=1}^n [t_i^a](A)$. The formula $\mathcal{C}(A)$ means: "$A$ is common knowledge of all agents" (therefore we use only so-called *public* common knowledge operator). We consider so-called reflexive common knowledge operator [4], which satisfies the following axioms: $\mathcal{C}(A) \supset (A \wedge \mathcal{E}(\mathcal{C}(A)))$ (common knowledge axiom) and $A \wedge \mathcal{C}(A \supset \mathcal{E}(A)) \supset \mathcal{C}(A)$ (induction axiom). In the case of irreflexive common knowledge operator [3] instead of these axioms there are the following common knowledge axiom $\mathcal{C}(A) \supset \mathcal{E}(A \wedge \mathcal{C}(A))$ and the following induction rule: $A \supset \mathcal{E}(A \wedge B)$ implies $A \supset \mathcal{C}(B)$. A formal semantics of formulas with the knowledge operators $[t_i^a]$, $\mathcal{E}$ and $\mathcal{C}$ can be found in [4].

A sequent $S$ is a *miniscoped* sequent if all negative (positive) occurrences of $\forall$ ($\exists$, correspondingly) in $S$ occur only in formulas of the shape $Q\bar{x}A(\bar{x})$ and $Qx^a[x^a]B$, where $Q \in \{\forall, \exists\}$, $\bar{x} = x_1, \dots, x_n$, $n \geq 0$ , $Q\bar{x}A(\bar{x})$ is a decidable logical formula (a logical formula (sequent) is decidable if it belongs to a decidable class of classical first-order logic).

A sequent $S$ is an *RC-sequent*, if $S$ satisfies the following conditions: (a) the sequent $S$ is a miniscoped one (miniscoped condition); (b) if any formula of the shape $\mathcal{C}(A)$ occur negatively in $S$ then $A$ does not contain positive occurrences of operator $\sigma$ (where $\sigma \in \{[t_i^a], \mathcal{C}, \mathcal{E}\}$)(regularity condition); (c) the sequent $S$ contains at most one positive occurrence of a formula $\sigma(A)$ where $\sigma \in \{[t^a], \mathcal{E}, \mathcal{C}\}$ and $\sigma(A)$ is not a subformula of another formula, but $A$ can contain occurrences of formulas of the shape $\sigma B$ (Horn-type condition). An $RC$-sequent is an induction-free one if $S$ does not contain positive occurrences of the induction-type operator $\mathcal{C}$.

Let us introduce some canonical forms of $RC$-sequents.

An $RC$-sequent $S$ is a *primary* $RC$-sequent, if $S = \Sigma_1,\ \forall\mathcal{K}_i\Gamma,\ \mathcal{C}\Theta \rightarrow \Sigma_2,\ \exists\mathcal{K}_j A,\ \mathcal{C}(B)$, where for every $k$ ($k \in \{1, 2\}$), $\Sigma_k$ is empty or consists of decidable logical formulas; $\forall\mathcal{K}_i\Gamma$ is empty or consists of formulas of the shape $\forall x_i^a[x_i^a]M$ or $[a_i]M$ $(1 \leq i \leq m)$; $\mathcal{C}\Theta$ is empty or consists of formulas of the shape $\mathcal{C}(A)$; $\exists\mathcal{K}_j A$ is empty or is a formula of the shape $\exists x_j^a[x_j^a]A$ or $[a_j]A$ $(j \in \{1, \dots, n\})$; $\mathcal{C}(B)$ is empty or is a formula of the shape $\mathcal{C}(B)$. An $RC$-sequent $S$ is a *reduced primary*, if $S$ is a primary one not containing $\mathcal{C}\Theta$ and $\mathcal{C}(B)$.

*Log* is a calculus in which logical sequents are decidable.

As in [1] let us introduce a calculus $K_nC_\omega$ containing infinitary rule for the common knowledge operator. This rule defines the semantics of the reflexive common knowledge operator. The calculus $K_nC_\omega$ is convenient to prove disjunctive invertibility of separation rules (see below). The calculus $K_nC_\omega$ is defined by the following postulates:

Logical axiom: $\Sigma_1 \rightarrow \Sigma_2$, where $Log \vdash \Sigma_1 \rightarrow \Sigma_2$.

Logical rules consist of traditional invertible rules for logical symbols.

Rules for knowledge:

$$\frac{A, \mathcal{E}(\mathcal{C}(A)), \Gamma_1 \to \Delta_1}{\mathcal{C}(A), \Gamma_1 \to \Delta_1}\,(\mathcal{C} \to) \qquad \frac{A, \Pi \to \Theta}{\mathcal{C}(A), \Pi \to \Theta}\,(\mathcal{C}_0 \to),$$

where $\Gamma_1 \to \Delta_1$ contains a positive occurrence of knowledge operators; $\Pi \to \Theta$ does not contain positive occurrences of knowledge operators;

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, \mathcal{E}(A); \ldots; \Gamma \to \Delta; \mathcal{E}^k(A); \ldots}{\Gamma \to \Delta, \mathcal{C}(A)}\,(\to \mathcal{C}_\omega),$$

where $k \in \omega = \{0, 1, \ldots\}$; $\mathcal{E}^0(A) = A$, $\mathcal{E}^k(A) = \mathcal{E}(\mathcal{E}^{k-1}(A))$, $k \geqslant 1$;

$$\frac{\Gamma \to \Delta, \wedge_{i=1}^m [a_i]A}{\Gamma \to \Delta, \mathcal{E}(A)}\,(\to \mathcal{E}) \qquad \frac{\wedge_{i=1}^m [a_i]A, \Gamma \to \Delta}{\mathcal{E}(A), \Gamma \to \Delta}\,(\mathcal{E} \to).$$

Separation rules:

$$\frac{S_l}{\Sigma_1, \forall \mathcal{K}_i \Gamma \to \Sigma_2, \exists \mathcal{K}_j A}\,(SR_l),$$

where $l \in \{1, 2\}$; the conclusion of these rules is a reduced primary $RC$-sequent such that $Log \nvdash \Sigma_1 \to \Sigma_2$.

Let $\exists \mathcal{K}_j A = \exists x_j^a [x_j^a] A$ and $\forall \mathcal{K}_i \Gamma = \forall \mathcal{K}_i \Gamma_0, [a_1]\Gamma_1, \ldots, [a_n]\Gamma_n$, $(n \geq 0)$ where $\forall \mathcal{K}_i \Gamma_0$ is empty or consists of formulas of the shape $\forall x_i^a [x_i^a] M$; $[a_k]\Gamma_k$ $(1 \leq k \leq n)$ is empty or consists of formulas of the shape $[a_k]N$. Then $S_1 = \Gamma_0, \Gamma_k \to A$, $k \in \{0, \ldots, n\}$.

Let $\exists \mathcal{K}_j A = [a_j]A$ and $\forall \mathcal{K}_i \Gamma$ has the same shape as in the previous case. Then $S_2 = \Gamma_0, \Gamma_k^\circ \to A$, where $\Gamma_k^\circ = \Gamma_k$ if $k = j$, and $\Gamma_k^\circ = \varnothing$ in opposite case.

A calculus $K_n C$ is obtained from $K_n C_\omega$ by dropping the rule $(\to \mathcal{C}_\omega)$.

A calculus $K_n^* C$ is obtained from $K_n C$ by adding the following rule:

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, \mathcal{E}(\mathcal{C}(A))}{\Gamma \to \Delta, \mathcal{C}(A)}\,(\to C^+);$$

Now we define the basic calculus $K_n^+ C$. First, let us introduce some auxiliary notions. Formulas $A$ and $A^*$ are called *parametrically identical* ones (in symbols $A \approx A^*$) if either $A = A^*$, or $A$ and $A^*$ are congruent, or differ only by the corresponding occurrences of eigen-variables of the rules $(\to \forall)$, $(\exists \to)$. $RC$-sequents $S = A_1, \ldots, A_n \to A_{n+1}, \ldots, A_{n+m}$ and $S^* = A_1^*, \ldots, A_n^* \to A_{n+1}^*, \ldots, A_{n+m}^*$ are *parametrically identical* (in symbols $S \approx S^*$), if $\forall k$ $(1 \leqslant k \leqslant n + m)$ formulas $A_k$ and $A_k^*$ are parametrically identical ones. An $RC$-sequent $S = \Gamma \to \Delta$ *subsumes* an $RC$-sequent $S^* = \Pi,\ \Gamma^* \to \Delta^*,\ \Theta$ (in symbols $S \succeq S^*$), if $\Gamma \to \Delta \approx \Gamma^* \to \Delta^*$. In this case the $RC$-sequent $S^*$ is subsumed by $S$ (in a special case, $S = S^*$ or $S \approx S^*$).

In derivations in the calculus $K_n^+ C$ along with logical axioms non-logical axioms are used. These non-logical axioms are defined in following way. Let $(i)$ be a branch from a derivation and an $RC$-sequent $S^* = \Gamma^*, \Pi \to \Delta^*, \Theta$ belongs to the branch $(i)$. Let in the branch $(i)$ (below than $S^*$) there exists $RC$-sequent $S = \Gamma \to \Delta$ such that

$S \succeq S^*$. Then the $RC$-sequent $S$ is a *saturated* one. A saturated $RC$-sequent $S$ is a *non-logical axiom* (*loop axiom*) if $S$ has the following shape: $\Gamma \to \Delta, \mathcal{C}(A)$.

A calculus $K_n^+C$ is obtained from $K_n^*C$ by adding the non-logical axiom.

All rules of the calculi $K_nC_\omega$ and $K_n^*C$, except the separation rules $(SR_i)$ $(i \in \{1,2\})$, are invertible.

LEMMA 1 (disjunctive invertibility of $(SR_i)$). *Let $S$ be a reduced primary $RC$-sequent, and $S_i$, $(i \in \{1,2\})$ be a premise of $(SR_i)$. Then if $K_nC_\omega \vdash S$ then (1) either $Log \vdash \Sigma_1 \to \Sigma_2$, or (2) there exists such $k$ that $K_nC_\omega \vdash S_1$, or $K_nC_\omega \vdash S_2$.*

Bottom-up applying logical rules (except the rules $(\to \exists)$ $(\forall \to)$) and rules $(\to \mathcal{E})$, $(\mathcal{E} \to)$ of the calculus $K_n^*C$ any $RC$-sequent $S$ can be reduced to a set of primary $RC$-sequents. A reduction of $RC$-sequent $S$ to a set of reduced primary $RC$-sequents is carried out bottom-up applying (in all possible ways) rules of $K_n^*C$. Using the invertibility of these rules we get that if $K_n^*C \vdash S$ then $K_n^*C \vdash S_j$, where $j \in \{1, \ldots, n\}$ is primary (reduced primary) $RC$-sequent.

To prove that the separation rules $(SR_i)$, $(i \in \{1,2\})$ are disjunctive invertible in $K_n^+C$ let us introduce an invariant calculus $INK_nC$ which is a connecting link between the calculi $K_nC_\omega$ and $K_n^+C$. A calculus $INK_nC$ is obtained from the calculus $K_n^*C$ by adding the following rule:

$$\frac{\Gamma \to \Delta, I; \; I \to \mathcal{E}(I); \; I \to A}{\Gamma \to \Delta, \mathcal{C}(A)} \; (\to \mathcal{C}^*),$$

where a formula $I$ is called an invariant formula and is constructed automatically using the shape of non-logical axioms in a derivation in the calculus $K_n^+C$.

Analogously as in [5] we can prove that $K_n^+C \vdash S \iff INK_nC \vdash S \iff K_nC_\omega \vdash S$, where $S$ is an $RC$-sequent. Thus, the separation rules $(SR_i)$, $(i \in \{1,2\})$ are also disjunctive invertible in $K_n^+C$.

## 3. Decision procedure for $RC$-sequents

First, we present a decision procedure for induction-free $RC$-sequents. Decision procedure for induction-free $RC$-sequents is realized by constructing so-called ordered derivation in the calculus $K_nC$.

*An ordered derivation $D$* for induction-free $RC$-sequents is a derivation consisting of several horizontal levels. Each level consists of bottom-up applications of rules of the calculus $K_nC$. In each level, when a set consisting of only reduced primary $RC$-sequents is received all possible bottom-up applications of separation rules $(SR_i)$, $i \in \{1,2\}$ to every reduced primary $RC$-sequent are carried out. An ordered derivation $D$ is *successful* one, if each leaf of $D$ is a logical axiom. In opposite case $D$ is *unsuccessful*.

Each bottom-up application of the separation rules $(SR_i)$ $(i \in \{1,2\})$ supplies a possibility to construct a different (in general) ordered derivation.

From the invertibility of the rules of $K_nC$ and from the shape of these rules we get that one can automatically construct a successful or unsuccessful ordered derivation of an $RC$-sequent $S$ in $K_nC$. The process of construction of such derivation $D$ always terminates.

A decision procedure for non-induction-free $RC$-sequents is realized constructing an ordered derivation $D$ in the calculus $K_n^+C$ analogously as in the case of induction-free $RC$-sequent. If each leaf of an ordered derivation $D$ of $RC$-sequent $S$ is either a logical axiom, or a non-logical axiom then $K_n^+C \vdash S$. In this case $D$ is a *successful* ordered derivation. In opposite case $D$ is *unsuccessful.*

THEOREM 1. *Let $S$ be an $RC$-sequent. Then one can automatically construct a successful or unsuccessful ordered derivation of the $RC$-sequent $S$ in $K_n^+C$. This process always terminates.*

*Proof.* Automatic way of construction of an ordered derivation $D$ and correctness (i.e., preservation of derivability) follows from invertibility of the rules; termination follows from finiteness of a set of generated subformulas in $D$ (congruent subformulas are merged).

Depending on decision procedures for different fragments of first-order logic we can get decision procedures for different fragments of $FRCL$.

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REZIUMĖ

***R. Pliuškevičius. Refleksyviosios bendro žinojimo logikos išsprendžiami kvantoriniai fragmentai***

Pasiūlytos išsprendžiamosios procedūros refleksyviosios bendro žinojimo logikos kvantoriniams fragmentams. Išsprendžiamosios procedūros yra grindžiamos sekvenciniais skaičiavimais.