Liet. matem. rink., **44**, *spec. nr.*, 2004, 812–816

Hölderian functional central limit theorem for linear processes

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Abstract. Let $(X_t)_{t\geqslant 1}$ be a linear process defined by $X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}$ where $(\psi_i, i \geqslant 0)$ is a sequence of real numbers and $(\epsilon_i, i \in Z)$ is a sequence of i.i.d. random variables with null expectation and variance 1. This paper provides Hölderian FCLT for $(X_t)_{t\geqslant 1}$ with wide class of filters. Filters with $\psi(i) = l(i)/i$ for a slowly varying function $l(i)$ are allowed. The weak convergence of polygonal line process build from sums of $(X_t)_{t\geqslant 1}$ to the standard Brownian motion W in the Hölder space (H_α), $0 < \alpha < 1/2 - 1/\tau$ holds provided the proper noise behavior is satisfied: $E|\epsilon_1|^\tau < \infty$, $\tau > 2$.

Keywords: near convergence, linear process, Hölder space.

1. Introduction

Let

$$X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}, \quad t \in Z, \tag{1.1}$$

be a one-sided linear process, where $(\epsilon_i, i \in Z)$ are i.i.d. innovations. For the sake of simplicity lets assume throughout the paper that $E\epsilon_i = 0$ and $E\epsilon_i^2 = 1$. By Kolmogorov's Three Series Theorem X_t exists almost surely if and only if the sequence $(\psi_i, i = 0, 1, \ldots)$ satisfies $\sum_{i=0}^{\infty} \psi_i^2 < \infty$. The process $(X_t)_{t\in Z}$ under these assumptions is well defined in the sense that stationarity holds.

Set $S_0 = 0$ and

$$S_k = X_1 + \cdots + X_k, \quad k = 1, 2 \ldots,$$

and consider partial sum process

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0, 1].$$

The asymptotic behavior of this partial sum process (sometimes called Donsker line) have been extensively studied in the literature. Some representatives of linear process invariance principle in usual continuous functions space C[0,1] would be Davydov (1970), Phillips and Solo (1992) and more recently Wu, Min (2003). Let us recall Wu, Min's result: suppose that the filter of (1.1) satisfies the following condition:

$$\sum_{i=0}^{\infty} (\Psi_{n+i} - \Psi_i)^2 = \mathrm{o}(B_n^2), \quad B_n \to \infty, \tag{1.2}$$

where $\Psi_n = \sum_{i=0}^{n} \psi_i$ and $B_n^2 = \sum_{i=0}^{n-1} \Psi_i^2$. If also $\sigma_n := (E|S_n|^2)^{1/2}$ and $E|\epsilon_i|^{2+\delta} < \infty$, for some $\delta > 0$, then the classical Donsker–Prohorov invariance principle holds:

$$\sigma_n^{-1}\xi_n \to^D W$$

in $C[0, 1]$, where $W = (W_t\colon t \in [0, 1])$ is a standard Wiener process and $\to^D$ denotes convergence in distribution.

This paper exploits linear processes and invariance principle in the spaces with stronger topology.

Let us recall the general definition of Hölder spaces. For $0 < \alpha \leqslant 1$, let $H_\alpha^0[0, 1]$ be the set of real valued continuous functions $x\colon [0, 1] \to R$ such that $\lim_{\delta\to 0} \omega_\alpha(x, \delta) = 0$, where

$$\omega_\alpha(x, \delta) = \sup_{\substack{t,s\in[0,1] \\ 0<|t-s|<\delta}} \frac{|x(t) - x(s)|}{|t - s|^\alpha}.$$

The set $H_\alpha^0[0, 1]$ is a separable Banach space when endowed with the norm

$$||x||_\alpha = |x(0)| + \omega_\alpha(x, 1).$$

The first Hölderian Functional Central Limit Theorem is due to Lamperti ([6]) and it deals with i.i.d. random variables with finite moment of order > 2. Sometimes Hölderian FCLT is called Lamperti's invariance principle. An important advantage of Lamperti's invariance principle is that it provides more continuous functionals of the ξ_n paths (see Hamadouche [3]).

Now suppose that linear process filter satisfies

$$\sum_{i=0}^{\infty} |\psi_i| < \infty.$$

Under this condition Lamperti's invariance principle follows from Theorem 6 given in Račkauskas, Suquet [7] and the fact that this case allows Beveridge–Nelson decomposition ([4]).

Indeed let us consider Orlitcz spaces:

$$\Theta_p = \Big(\Phi\colon R^+ \to R^+,\ \Phi(0) = 0,\ \Phi \neq 0,\ convex,\ x^{-p}\Phi(x) \uparrow\Big)$$

for $p > 1$. And Luxembourg norms defined for $\Phi \in \Theta_p$ by

$$||X||_\Phi := \inf\left(t > 0\colon\ E\Phi\left(\frac{|X|}{t}\right) \leqslant 1\right).$$

Since

$$\left\|\sum_{j=0}^{\infty} \psi_j \sum_{k=1}^{n} \epsilon_{j-k}\right\|_\Phi \leqslant \sum_{j=0}^{\infty} \left\|\psi_j \sum_{k=1}^{n} \epsilon_{j-k}\right\|_\Phi \leqslant \sum_{j=0}^{\infty} |\psi_j| \cdot \left\|\sum_{k=1}^{n} \epsilon_{j-k}\right\|_\Phi$$

$$= \sum_{j=0}^{\infty} |\psi_j| \cdot \left\| \sum_{k=1}^{n} \epsilon_k \right\|_{\Phi}.$$

It follows that

$$P\big(||S_n||_{\Phi} > m^{1/2}u\big) \leqslant P\big(||\epsilon_1 + \ldots + \epsilon_n||_{\Phi} > C^{-1}m^{1/2}u\big) \tag{1.3}$$

with $C = \sum_{j=0}^{\infty} |\psi_j|$.

Now from (1.3) it follows that $(X_t)_{t\in Z}$ and $(\epsilon_t)_{t\in Z}$ satisfies the same conditions required in [7] Theorem 6.

The essence of this paper that it mainly deals with the Linear processes with diverging filter:

$$\sum_{i=0}^{\infty} |\psi_i| = \infty.$$

Of course the rate of divergence is not higher then slowly varying function since we must get Wiener process as a limit of Hölderian FCLT.

2. Results

Throughout this section it will be assumed that the filter of (1.1) satisfies condition (1.2). For the sake of simplicity lets assume that Donsker line admits the following form:

$$\chi_n(t) = \frac{1}{\sigma_n}\left(\sum_{i=1}^{j} X_i + (nt-j)X_{j+1}\right) \quad \frac{j}{n} \leqslant t \leqslant \frac{j+1}{n}, \quad j = 0, 1, \ldots, n-1.$$

Next suppose that the following restriction of linear process filter is satisfied:

$$\limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \frac{\sum_{i=0}^{k-1} \psi_i(1-\frac{i}{k})}{\sum_{i=0}^{n-1} \psi_i(1-\frac{i}{n})} < \infty \tag{2.1}$$

The main result of this paper is the following

THEOREM 2.1. *Let* $(X_i)_{i\geqslant 1}$ *be a strictly stationary sequence from* (1.1). *Assume that* $\sigma_n \geqslant \sqrt{n}$ *for each* $n \geqslant 1$ *and*

$$E|\epsilon_1|^{\tau} < \infty \tag{2.2}$$

for some $\tau > 2$.

If moreover (1.2) *and* (2.1) *holds, then*

$$\chi_n \to^{D} W \tag{2.3}$$

in the space $H_{\alpha}^{0}[0,1]$, *with* $0 < \alpha < \frac{1}{2} - \frac{1}{\tau}$.

Proof. The convergence of the finite dimensional distributions follows by [8] (Theorem 1), since condition (1.2) is satisfied and $\tau > 2$.

To prove tightness it is enough to show:

$$E\left|\chi_n(t) - \chi_n(s)\right|^{\tau} \leqslant C|t-s|^{1+\delta}, \quad \delta > 0.$$

Indeed using this condition and Markov inequality it follows that (Kerkyachrian, Roynette [5]) sufficient condition for the tightness in $H_\alpha^0[0,1]$ is satisfied for all $\alpha < \frac{\delta}{\tau}$.

First, if $\frac{j}{n} \leqslant t \leqslant s \leqslant \frac{j+1}{n}$, then $|\chi_n(t) - \chi_n(s)| = (\frac{n}{\sigma_n})|t-s||X_{j+1}|$. Since $(X_i)_{i\geqslant 1}$ is strictly stationary, $\sigma_n \geqslant \sqrt{n}$ and $n|t-s| \leqslant 1$ it follows that

$$E\left|\chi_n(t) - \chi_n(s)\right|^{\tau} \leqslant |t-s|^{\tau}\left(\frac{n}{\sigma_n}\right)^{\tau} E|X_1|^{\tau} \leqslant |t-s|^{\frac{\tau}{2}} E|X_1|^{\tau}.$$

Now if for some j and k, $\frac{j-1}{n} \leqslant s \leqslant \frac{j}{n} \leqslant \frac{j+k}{n} \leqslant t \leqslant \frac{j+k+1}{n}$, then we have

$$3^{1-\tau} E\left|\chi_n(t) - \chi_n(s)\right|^{\tau} \leqslant E\left|\chi_n(s) - \chi_n\left(\frac{j}{n}\right)\right|^{\tau} + E\left|\chi_n\left(\frac{j+k}{n}\right) - \chi_n\left(\frac{j}{n}\right)\right|^{\tau} + E\left|\chi_n(t) - \chi_n\left(\frac{j+k}{n}\right)\right|^{\tau}.$$

Since two components of this sum could be treated as in preceding display, thus it left to estimate the middle term:

$$E\left|\chi_n\left(\frac{j+k}{n}\right) - \chi_n\left(\frac{j}{n}\right)\right|^{\tau} = E\left|\frac{1}{\sigma_n}(X_1 + X_2 + \ldots + X_k)\right|^{\tau}.$$

To this end we use Wu, Min's [8] Lemma 6, which states that under condition (1.2) there exists constant C, independent of n, such that for all $n \in N$,

$$\left(E|S_n|^{\tau}\right)^{\frac{1}{\tau}} \leqslant C\sigma_n.$$

Also note that $\sigma_n = \sqrt{n}l^*(n)$ and $l^*(n) \sim \frac{|\sum_{i=0}^{n-1}\Psi_i|}{n}$ is slowly varying function.

By (2.1) it follows that

$$\limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \left(\frac{n}{k}\right)^{1/2} \frac{\sigma_k}{\sigma_n} = \limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \left(\frac{n}{k}\right)^{1/2} \frac{\sqrt{k}l^*(k)}{\sqrt{n}l^*(n)}$$
$$= \limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \frac{n}{k} \frac{|\sum_{i=0}^{k-1}\Psi_i|}{|\sum_{i=0}^{n-1}\Psi_i|} < \infty.$$

Thus

$$E\left|\frac{S_k}{\sigma_n}\right|^{\tau} \leqslant \frac{1}{\sigma_n^{\tau}} E(S_k)^{\tau} \leqslant C^{\tau}\left(\frac{\sigma_k}{\sigma_n}\right)^{\tau} < C^*\left(\frac{k}{n}\right)^{\tau/2} \leqslant C^*|t-s|^{\tau/2}$$

since $|t-s| \geqslant \frac{k}{n}$.

COROLLARY 2.1. *Let $(\epsilon_i)_{i\in Z}$ be Gaussian white noise (ϵ_i: i.i.d. $\sim N(0,1)$). $X_t := \sum_{i=1}^{\infty} \frac{1}{i}\epsilon_{t-i}$, $t \geqslant 1$. Then*

$$\chi_n \to^D W$$

in the space $H_\alpha^0[0,1]$, with $0 < \alpha < 1/2$.

Proof. For Gaussian case $EX_i^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, thus $X_i \sim N(0, \pi^2/6)$ and moment condition (2.2) is satisfied for all $\tau > 2$. Now $l^*(n) = \sum_{i=1}^{n} \frac{1}{i}(1 - \frac{i}{n}) \sim \ln n$. Thus

$$\sigma_n \sim \sqrt{n} \ln n \geqslant \sqrt{n}.$$

and

$$\limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \frac{l^*(k)}{l^*(n)} \leqslant 1.$$

So that all conditions required in Theorem 2.1 are satisfied.

Remark 2.1. It is interesting to note that a process considered in Corollary is associated and its covariance structure is $Cov(X_1, X_i) \sim \frac{\ln i}{i}$ with $\sum_{i=1}^{\infty} Cov(X_1, X_i) = \infty$. But even so if normalization rate is $\sigma_n = \sqrt{n}\ln n$ we still get invariance principle in $H_\alpha^0[0,1]$, with $0 < \alpha < 1/2$.

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REZIUMĖ

M. Juodis. Hiolderinė FCRT tiesiniams procesams

Nagrinėjami tiesiniai procesai su plačia klase filtrų. Įrodoma FCRT Hiolderio erdvėje su standartiniais momentiniais reikalavimais.