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On an extension of Girard algebras

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1. Introduction

In this note we present a generalization of the canonical extension of Girard algebras recently introduced and studied by U. Höhle and S. Weber [1]–[4]. Our results are submitted without proofs. We are going to detail it in a subsequent paper.

2. Girard algebras, MV-algebras and Boolean algebras

A lattice-theoretic definition of a Girard algebra, of a MV-algebra and of a Boolean algebra are the following:

DEFINITION 2.1. *A Girard algebra G is a bounded lattice (with the join $\vee$, with the meet $\wedge$, with the least element $0 = \bigwedge G$ and with the geatest element $1 = \bigvee G$) together with commutative semigroup operation $\circ$ with 1 as unit and 0 as zero such that*

1. $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$ *(distributivity between $\circ$ and $\vee$),*
2. *There exists a further binary operation $\setminus$: $G \times G \to G$ defined by*

$$a \setminus b = \bigvee \{c \in G \mid a \circ c \leqslant b\} \ \text{(existence of residuals)},$$

which satisfies the following axiom:

$$(a \setminus 0) \setminus 0 = a \ \text{(Involution)}.$$

In any Girard algebra the residual complement $\perp$: $G \to G$ and the dual operation $\uplus$: $G \times G \to G$ associated with $\circ$ can be defined by

$$a^{\perp} = a \setminus 0 \ \text{ and } \ a \uplus b = (a^{\perp} \circ b^{\perp})^{\perp}.$$

DEFINITION 2.2.

1. *A Girard algebra G is called an MV-algebra ("multi-valued") iff the following axiom is satisfied:*

$$a \circ (a \setminus b) = a \wedge b \ \text{(Divisibility)}.$$

2. *An MV-algebra is called a Boolean algebra iff $\circ = \wedge$.*

3. I-extension of Girard algebras

In [2] was introduced the "canonical" extension G_1 of a Girard algebra G, the Girard algebra $G_1 = \{\langle a_1, a_2\rangle \in G \times G \mid a_1 = a_2\}$, where G was identified with its diagonal $G_\triangle = \{\langle a, a\rangle \mid a \in G\}$. In this note we generalize this canonical extension of G to the I-extension defined as follows.

DEFINITION 3.1. *Given a Girard algebra G and the set of natural numbers $I := \{1, 2, \dots, n\}$ for $n \geqslant 2$, by I-extension G^I of G we shall understand the bounded lattice of all isotone functions $a\colon I \to G$. We write $a \leqslant b$ to mean that $a_i \leqslant b_i$ for all $i = 1, 2, \dots, n$. The lattice-theoretic operations and universal bounds are given by, for all $i \in I$,*

$$(a \wedge b)_i = a_i \wedge b_i, (a \vee b)_i = a_i \vee b_i, 0_i = 0 \text{ and } 1_i = 1.$$

If we identify G with all constant functions of G^I, then G is a sublattice of G^I.

THEOREM 3.2. *Let G be a Girard algebra. Then G^I is also a Girard algebra with structure (denoted by the same symbols) given by:*

1. $(a \circ b)_i = \bigvee_{j,k=1}^{n} \{a_j \circ b_k \mid j + k = i + 1\}$,
2. $(a \setminus b)_i = \bigwedge_{j=1}^{n-i+1} a_j \setminus b_{i+j-1}$,
3. $(a^\perp)_i = (a_{n-i+1})^\perp$,
4. $(a \uplus b)_i = \bigwedge_{j,k=1}^{n} \{a_j \uplus b_k \mid j + k = n + i\}$.

COROLLARY 3.2. *Let G be Girard algebra, and G^I be its I-extension. Then the following assertions are equivalent:*
1. *G^I is an MV-algebra.*
2. *G is a Boolean algebra.*

4. Conditioning and mean value generation

In [1]–[2] were proposed the following axioms for conditioning operators and mean value functions in Girard algebras:

DEFINITION 4.1. *Let G be a Girard algebra. A binary operation $|\colon G \times G \to G$ is called a conditioning operator on G iff $|$ satisfies the following axioms:*

1. $a|1 = a$,
2. $(b \circ (b \setminus a))|b = a|b$,

3. $a \leqslant b \Rightarrow a|c \leqslant b|c$,
4. $b \leqslant c$ *and* $c \circ (c \setminus a) \leqslant b \circ (b \setminus a) \Rightarrow a|c \leqslant a|b$
5. $(a|b)^{\perp} = (a^{\perp} \circ b)|b$, *particularly,* $(0|0)^{\perp} = 0|0$.

LEMMA 4.2 (Lemma 4.3 [2. *)] Every conditioning operator fulfils the following property:*

$$b \circ (b \setminus a) \leqslant a|b \leqslant b \setminus a.$$

DEFINITION 4.3. *Let G be a Girard algebra. A mean value function is an isotone, idempotent binary (not necessarily commutative) operation on G, i.e., a map $C\colon G \times G \to G$ satisfying the following axioms*

1. $C_{a,a} = a$ *(idempotency),*
2. $C_{a,b} \leqslant C_{c,d}$ *whenever* $a \leqslant c$, $b \leqslant d$ *(isotonocity).*
   A mean value function C is said to be compatible with the residual complement in G iff C satisfies the following additional condition
   $(C_{a,b})^{\perp} = C_{b^{\perp},a^{\perp}}$.

PROPOSITION 4.4 (Theorem 4.5 [2. *)] Let G be a Girard algebra and C be a mean value function on G which is compatible with the residual complement in G. Then C induces a conditioning operator $|$ on G by*

$$a|b = C_{b \circ (b \setminus a), b \setminus a}.$$

PROPOSITION 4.4 (Theorem 4.6 [2. *)] Let G be an MV-algebra and $|$ be a conditioning operator. Then the following assertions are equivalent:*

1. $|$ *is a mean value based conditioning operator;*
2. $|$ *satisfies the condition* $a \leqslant b \Rightarrow (a \wedge c)|(c \setminus a) \leqslant (b \wedge c)|(c \setminus b)$.

5. Conditioning operators and mean value functions on I-extensions of Girard algebras

LEMMA 5.1. *Let G be a Girard algebra and G^I be the I-extension of G, where $I = \{1, 2, \ldots, n, n+1, \ldots, 2n\}$ with $n \geqslant 1$. Every mean value function B on G induces a mean value function C on G^I by:*

$$(C_{a,b})_i = \begin{cases} B_{a_i, a_{i+1} \wedge b_i} & \text{if } i = 1, \ldots, n, \\ (B_{(b_i)^{\perp}, (b_{i-1} \vee a_i)^{\perp}})^{\perp} & \text{if } i = n+1, \ldots, 2n. \end{cases}$$

Moreover, C on G^I is compatible with the residual complement in G^I and satisfies the further property:

$$(C_{a,b})_i = \begin{cases} a_1 & \text{if } i = 1, \ldots, n, \\ b_1 & \text{if } i = n+1, \ldots, 2n, \end{cases}$$

whenever a and b are constant functions, $a \leqslant b$, $a = a_1$, $b = b_1$.

REMARK 5.2 (existence of mean value functions). *Let G be a Girard algebra. Then the following maps B^1 and B^2 defined by*

$$B^1_{a,b} = a \text{ and } B^2_{a,b} = b$$

are mean value functions on G. Further the corresponding mean value functions C^1 and C^2 (in the sense of the preceding lemma) are given by

$$(C^1_{a,b})_i = \begin{cases} a_i & \text{if } i = 1, \dots, n, \\ b_i & \text{if } i = n+1, \dots, 2n, \end{cases}$$

$$(C^2_{a,b})_i = \begin{cases} a_{i+1} \wedge b_i & \text{if } i = 1, \dots, n, \\ b_{i-1} \vee a_i & \text{if } i = n+1, \dots, 2n. \end{cases}$$

PROPOSITION 5.3. *Let G^I be the I-extension of a Girard algebra G with $I = \{1, 2, \dots, 2n\}$. Then there exists a mean value function C on G^I which is compatible with the residual complement in G^I and satisfies the condition in the preceding lemma. Further the mean value based conditioning operator $|$ corresponding to C satisfies the additional property:*

$$a|b = \begin{cases} b_1 \circ (b_1 \setminus a_1), & \text{if } i = 1, \dots, n, \\ b_1 \setminus a_1, & \text{if } i = n+1, \dots, 2n, \end{cases}$$

whenever a and b are constant functions, $a \leqslant b$, $a = a_1$, $b = b_1$.

6. Uncertainty measures on MV-algebras

DEFINITION 6.1. ([2]) *Let G be an MV-algebra, and $^\perp$ and $\uplus$ be the residual complement and the dual operation associated with $\circ$. A map $m\colon G \to [0, 1]$ is called an uncertainty measure iff m satisfies the following conditions:*

1. $m(0) = 0$, $m(1) = 1$,
2. $a \leqslant b \Rightarrow m(a) \leqslant m(b)$.

   An uncertainty measure m is said to be additive iff it satisfies the axiom:
3. $a \circ b = 0 \Rightarrow m(a \uplus b) = m(a) + m(b)$.

Now we are going to present the last important result of this note (generalizing Theorem 6.5 [2]), to establish the existence of additive measure extension in the Boolean case. For this, we need the following observation

LEMMA 6.2. *Let G be a Boolean algebra, and m be a probability measure on it. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ (with $n \geqslant 2$) be elements of G such that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Then for every $i = 1, \dots, n-1$ the equality holds:*

$$m\Big(\bigwedge_{j,k=1}^{n}\big\{a_j \vee b_k \mid j+k=n+i\big\}\Big)$$
$$= \sum_{j,k=1}^{n}\big\{m(a_j \vee b_k) \mid j+k=n+i\big\} - \sum_{j,k=1}^{n}\big\{m(a_j \vee b_k) \mid j+k=n+i+1\big\}.$$

Finally, we arrive at

THEOREM 6.3. *Let G be a Boolean algebra, and m be a probability measure on G. Let G^I be the MV-algebra I-extension of G. Then m has a unique extension to an additive uncertainty measure $\tilde{m}$ on G^I, i.e., there exists a unique additive uncertainty measure $\tilde{m}$ on G^I such that the restriction of $\tilde{m}$ to G coincides with m. In particular, $\tilde{m}$ is given by*

$$\tilde{m}(a) = \frac{1}{n}\sum_{i=1}^{n} m(a_i).$$

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REZIUMĖ

R.P. Gylys. Apie Žiraro algebrų plėtinį

Aprašomas Žiraro algebrų kanoninio plėtinio apibendrinimas.