

On approximation of stochastic integrals with respect to a fractional Brownian motion

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Introduction

In the past few years, a fractional Brownian motion has been the subject of numerous investigations. The fBm B^H , $H > 1/2$, has the following kernel representation with respect to the standard Brownian motion:

$$B_t^H = \int_0^t K_H(t, s) dW_s,$$

where K_H is the deterministic kernel

$$K_H(t, s) = c_H \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du$$

with the normalizing constant c_H .

Let $\mathcal{A}^n = \{t_k^n: 0 \leq k \leq n\}$ be the sequence of partitions of the interval $[0, 1]$ with $t_k^n = k/n$. Define

$$A_t^n = \sum_{k=1}^{[nt]} n \int_{t_{k-1}^n}^{t_k^n} K_H\left(\frac{[nt]}{n}, u\right) du \cdot \frac{\xi_k^n}{\sqrt{n}},$$

where $[nt]$ denotes the integer part of nt , $\{\xi_k^n\}$ are i.i.d. random variables with $\mathbf{E}\xi_1^n = 0$ and $\mathbf{D}\xi_1^n = 1$. Sottinen [4] has proved that $A^n \xrightarrow{\mathcal{D}} B^H$ as $n \rightarrow \infty$.

Denote by $D([0, 1], \mathbb{R}^d)$ the space of all mappings $x: [0, 1] \rightarrow \mathbb{R}^d$ that are right-continuous and admit left-hand limits equipped with the Skorokhod \mathcal{J}_1 topology.

The aim of this note is to find conditions under which the convergence

$$\left(X^n, A^n, \int_0^\cdot X_{s-}^n dA_s^n \right) \xrightarrow{\mathcal{D}} \left(X, B^H, \int_0^\cdot X_s dB_s^H \right)$$

holds, where $\{X^n\}$ is a sequence of càdlàg processes such that

$$(X^n, A^n) \xrightarrow{\mathcal{D}} (X, B^H) \quad \text{in } D([0, 1], \mathbb{R}^2).$$

Denote by $C^\alpha(\mathbb{R})$ the space of all continuous functions $g: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\sup_x |g(x)| + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\alpha} < \infty.$$

THEOREM 1. *Let $0 < \alpha < 1$, $q > 2$, and $1 < p < 2$ be such that $\frac{\alpha}{q} + \frac{1}{p} > 1$. Let $\{X^n\}$ be a sequence of càdlàg processes with q -bounded variation, and let $f \in C^\alpha(\mathbb{R})$. Assume that $\sup_n \mathbf{E}v_q(X^n; [0, 1]) < +\infty$. If*

$$(X^n, A^n) \xrightarrow{\mathcal{D}} (X, B^H) \quad \text{in } D(\mathbb{R}_+, \mathbb{R}^2), \tag{1}$$

where X is a continuous process, then

$$\left(X^n, A^n, \int_0^\cdot f(X_{s-}^n) dA_s^n \right) \xrightarrow{\mathcal{D}} \left(X, B^H, \int_0^\cdot f(X_s) dB_s^H \right) \quad \text{in } D([0, 1], \mathbb{R}^3).$$

1. Basic notions and auxiliary results

The p -variation, $0 < p < \infty$, of a real-valued function f on $[a, b]$ is defined as

$$v_p(f; [a, b]) = \sup_{\varkappa} \sum_{k=1}^n |f(x_k) - f(x_{k-1})|^p,$$

where the supremum is taken over all subdivisions $\varkappa = \{x_i: i = 0, \dots, n\}$ of $[a, b]$ such that $a = x_0 < x_1 < \dots < x_n = b$. Denote by $\mathcal{W}_p([a, b])$ the class of all functions on $[a, b]$ with bounded p -variation.

Let $f \in \mathcal{W}_q([a, b])$ and $h \in \mathcal{W}_p([a, b])$ with $0 < p < \infty$, $q > 0$, $1/p + 1/q > 1$. Then the integral $\int_a^b f dh$ exists as the refinement Riemann–Stieltjes (RRS) integral if f and h have no common discontinuities on the same side at the same point. In particular, this necessary condition is satisfied if f is left-continuous and h is right-continuous or vice versa. If the integral exists, the Love–Young inequality

$$\left| \int_a^b f dh - f(y)[h(b) - h(a)] \right| \leq C_{p,q} V_q(f; [a, b]) V_p(h; [a, b]) \tag{2}$$

holds for all $y \in [a, b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n \geq 1} n^{-s}$.

Let $f \in \mathcal{W}_p([a, b])$ and $p_1 > p > 0$. Then

$$V_{p_1}(f; [a, b]) \leq \text{Osc}(f; [a, b])^{(p_1-p)/p_1} V_p^{p/p_1}(f; [a, b]), \tag{3}$$

where $\text{Osc}(f; [a, b]) = \sup\{|f(x) - f(y)|: x, y \in [a, b]\}$.

2. Proof of the theorem

Let $A_1, \dots, A_n, B_1, \dots, B_n$ be random variables such that $\mathbf{E}|A_i|^p, \mathbf{E}|B_i|^q < +\infty$, $i = 1, \dots, n$. Let $S_{p,q}$ be the largest value of the products

$$\left(\sum_{k=1}^m \mathbf{E}|\bar{A}_k|^p \right)^{1/p} \left(\sum_{k=1}^m \mathbf{E}|\bar{B}_k|^q \right)^{1/q}$$

for which $\bar{A}_k = A_{i_k+1} + \dots + A_{i_{k+1}}$ and $\bar{B}_k = B_{i_k+1} + \dots + B_{i_{k+1}}$, $1 = i_1 < \dots < i_k < \dots < i_{m+1} = m$, $m \leq n$, are the corresponding sums of successive random variables A_i and B_i , respectively. A version of this lemma was proved in [2].

LEMMA 2. Assume that $\frac{1}{p} + \frac{1}{q} > 1$. Then

(i) there exists an index k ($1 \leq k \leq n$) such that

$$\mathbf{E}|A_k B_k| \leq \left(\frac{1}{n} \sum_{k=1}^m \mathbf{E}|A_k|^p \right)^{1/p} \left(\frac{1}{n} \sum_{k=1}^m \mathbf{E}|B_k|^q \right)^{1/q},$$

(ii)

$$\mathbf{E} \left| - \sum_{1 \leq i < j \leq k} A_i B_j + \sum_{k < i \leq j \leq n} A_i B_j \right| \leq C_{p,q} S_{p,q},$$

where $C_{p,q} = \zeta(p^{-1} + q^{-1})$.

The prove of this lemma is the same as that of the Love–Young inequality in [5], [3].

Proof of Theorem 1. Sottinen [4] has proved that, for all $s < t$,

$$\mathbf{E}|A^n(t) - A^n(s)|^2 \leq \left(\frac{[nt]}{n} - \frac{[ns]}{n} \right)^{2H} \quad \text{as } n \rightarrow \infty. \tag{4}$$

We show that

$$\sum_{k=1}^{[n]} f(X^n(t_{k-1}^n))(A^n(t_k^n) - A^n(t_{k-1}^n)) \xrightarrow{\mathcal{D}} \int_0^t f(X(s)) dB^H(s), \quad n \rightarrow \infty. \tag{5}$$

Let us fix a partition of the interval $[0, 1]$. Denote it by \varkappa^m . By (1) we get

$$(X^n(t_0^m), A^n(t_0^m), \dots, X^n(t_m^m), A^n(t_m^m)) \xrightarrow{d} (X(t_0^m), B^H(t_0^m), \dots, X(t_m^m), B^H(t_m^m))$$

as $n \rightarrow \infty$. Thus,

$$\begin{aligned} & \sum_{k=1}^m f(X^n(t_{k-1}^m))(A^n(t_k^m) - A^n(t_{k-1}^m)) \\ & \xrightarrow{d} \sum_{k=1}^m f(X(t_{k-1}^m))(B^H(t_k^m) - B^H(t_{k-1}^m)) \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{6}$$

Assume that $m \leq n$ and set $\varkappa = \varkappa^m \cup \varkappa^n = \{t_k: t_k \in \varkappa^m \cup \varkappa^n\}$, $r(t) = \max\{k: t_k \leq t, t_k \in \varkappa\}$. Then

$$\left| \sum_{k=1}^{[mt]} f(X^n(t_{k-1}^m))(A^n(t_k^m) - A^n(t_{k-1}^m)) - \sum_{j=1}^{[nt]} f(X^n(t_{j-1}^n))(A^n(t_j^n) - A^n(t_{j-1}^n)) \right|$$

$$\begin{aligned} &\leq \left| \sum_{k=1}^{\lfloor mt \rfloor} f(X^n(t_{k-1}^m))(A^n(t_k^m) - A^n(t_{k-1}^m)) - \sum_{j=1}^{r(t)} f(X^n(t_{j-1})) (A^n(t_j) - A^n(t_{j-1})) \right| \\ &\quad + \left| \sum_{j=1}^{r(t)} f(X^n(t_{j-1})) (A^n(t_j) - A^n(t_{j-1})) - \sum_{k=1}^{\lfloor nt \rfloor} f(X^n(t_{k-1}^n))(A^n(t_k^n) - A^n(t_{k-1}^n)) \right| \\ &= I_1^{m,n}(t) + I_2^{n,n}(t). \end{aligned}$$

We estimate only $I_1^{m,n}$, since the proof of $I_2^{n,n}$ is the same.

Note that

$$\begin{aligned} I_1^{m,n}(t) &= \left| \sum_{k=1}^{\lfloor mt \rfloor} \sum_{j \in \Delta_k^m} (f(X^n(t_{k-1}^m)) - f(X^n(t_{j-1}))) (A^n(t_j) - A^n(t_{j-1})) \right| \\ &= \left| - \sum_{k=1}^{\lfloor mt \rfloor} \sum_{\substack{i < j \\ i, j \in \Delta_k^m}} (f(X^n(t_i)) - f(X^n(t_{i-1}))) (A^n(t_j) - A^n(t_{j-1})) \right|, \end{aligned}$$

where $\Delta_k^m = \{j: t_j \in \mathcal{I}, t_{k-1}^m < t_j \leq t_k^m\}$. Thus, by Lemma 2 we have

$$\begin{aligned} \mathbf{E} \sup_{t \leq 1} I_1^{m,n}(t) &\leq \sum_{k=1}^m \mathbf{E} \left| - \sum_{\substack{i < j \\ i, j \in \Delta_k^m}} (f(X^n(t_i)) - f(X^n(t_{i-1}))) (A^n(t_j) - A^n(t_{j-1})) \right| \\ &\leq C_{p,q/\alpha} \sum_{k=1}^m S_{p,q/\alpha}^{m,n,k}, \end{aligned}$$

where $S_{p,q/\alpha}^{m,n,k}$ is the largest value of the products

$$\left(\sum_{j=1}^r \mathbf{E} |f(X^n(s_j)) - f(X^n(s_{j-1}))|^{q/\alpha} \right)^{\alpha/q} \left(\sum_{j=1}^r \mathbf{E} |A^n(s_j) - A^n(s_{j-1})|^p \right)^{1/p} \tag{7}$$

for $t_{k-1}^m = s_0 \leq \dots \leq s_r = t_k^m, s_j \in \mathcal{I}, r \in \mathbb{N}$. Further, by (4), for $p > 1/H$, we have

$$\begin{aligned} &\left(\sum_{j=1}^r \mathbf{E} |f(X^n(s_j)) - f(X^n(s_{j-1}))|^{q/\alpha} \right)^{\alpha/q} \left(\sum_{j=1}^r \mathbf{E} |A^n(s_j) - A^n(s_{j-1})|^p \right)^{1/p} \\ &\leq |f|_\alpha (\mathbf{E} v_q(X^n; [t_{k-1}^m, t_k^m]))^{\alpha/q} (t_k^m - t_{k-1}^m)^H \end{aligned}$$

and by the Hölder inequality

$$\sum_{k=1}^m S_{p,q/\alpha}^{m,n,k} \leq |f|_\alpha \left(\sum_{k=1}^m \mathbf{E} v_q(X^n; [t_{k-1}^m, t_k^m]) \right)^{\alpha/q} \left(\sum_{k=1}^m (t_k^m - t_{k-1}^m)^{pH} \right)^{1/p}$$

$$\leq |f|_\alpha (\mathbf{E} v_q(X^n; [0, 1]))^{1/q} \max_{1 \leq k \leq m} |t_k^m - t_{k-1}^m|^{H-1/p}.$$

Thus,

$$\mathbf{E} \sup_{t \leq 1} I_1^{m,n}(t) + \mathbf{E} \sup_{t \leq 1} I_2^{n,n}(t) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty. \tag{8}$$

Denote

$$\widehat{X}^m(t) := X(t_k^m) \quad \text{for } t \in [t_{k-1}^m, t_k^m), \quad 1 \leq k \leq m - 1, \quad \text{or } t \in [t_{m-1}^m, t_m^m],$$

and

$$H^m(X) = \sup_{0 \leq t \leq 1} |X(t) - \widehat{X}^m(t)|.$$

Let $q_1 > q$ and $\frac{\alpha}{q_1} + \frac{1}{p} > 1$. By inequality (3) we get

$$\begin{aligned} & \sup_{t \leq 1} \left| \sum_{k=1}^{[mt]} f(X(t_{k-1}^m)) (B^H(t_k^m) - B^H(t_{k-1}^m)) - \int_0^t f(X_s) dB_s^H \right| \\ & \leq \sup_{t \leq 1} \left| \sum_{k=1}^{[mt]} f(X(t_{k-1}^m)) (B^H(t_k^m) - B^H(t_{k-1}^m)) - \int_0^{[mt]/m} f(X_s) dB_s^H \right| \\ & \quad + \sup_{t \leq 1} |f(X_t)| H^m(B^H) \\ & \leq 4C_{p,q_1/\alpha} |f|_\alpha (H^m(X))^{\alpha(q_1-q)/q_1} V_q^{\alpha q/q_1}(X; [0, 1]) V_p(B^H; [0, 1]) \\ & \quad + |f|_\alpha V_q^\alpha(X; [0, 1]) H^m(B^H). \end{aligned} \tag{9}$$

Now by Lemma 4.2 of [1] and estimates (6), (8), and (9) we get (5).

References

1. P. Billingsley, *Convergence of Probability Measures*, Wiley, New York (1968).
2. F. Coquet, L. Słomiński, On the convergence of Dirichlet processes, *Bernoulli*, **5**(4), 615–639 (1999).
3. R.M. Dudley, R. Norvaiša, An introduction to p -variation and Young integrals, *Lecture Notes*, **1**, Aarhus University (1998).
4. T. Sottinen, Fractional Brownian motion, random walks and binary market models, *Finance Stochast.*, **5**, 343–355 (2001).
5. L.C. Young, An inequality of the Hölder type, connected with Stieltjes integration, *Acta Math.*, **67**, 251–282 (1936).

REZIUMĖ

K. Kubilius. Atsitiktinių integralų trupmeninio Brauno judesio atžvilgiu aproksimacijos klausimu

Nustatytos sąlygos, kada stochastinis integralas pasirinktos trupmeninio Brauno judesio aproksimacijos atžvilgiu silpnai konverguoja į stochastinį integralą trupmeninio Brauno judesio atžvilgiu.