

# Weighted empirical FCLT for weakly dependent processes

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**Abstract.** This paper provides a weak convergence theorem for weighted empirical processes of a stationary sequence under a new weak dependence condition introduced by P. Doukhan and S. Louhichi.

*Keywords:* weak convergence, empirical process, weak dependence, Brownian Bridge.

## 1. Introduction

Let  $(U_n)_{n \in \mathbb{Z}}$  be a stationary sequence of random variables with common distribution function  $F$ . Assume without loss of generality that  $F$  is a uniform law on  $[0, 1]$ . Then the empirical distribution function  $E_n$  of  $U_1, \dots, U_n$  is defined by

$$E_n(x) = n^{-1} \sum_{i=1}^n I(U_i \leq x), \quad -\infty < x < \infty,$$

where  $I$  is the usual indicator function. Define the empirical process

$$\alpha_n(t) = n^{1/2}(E_n(t) - t), \quad 0 < t < 1.$$

Let  $q$  be a positive weight function on  $(0, 1)$ . That is,  $\inf_{\delta \leq t \leq 1-\delta} q(t) > 0$  for all  $0 < \delta < 1/2$ . And consider the following convergence result in the Skorohod space  $D[0, 1]$  when the sample size converges to infinity

$$\frac{1}{q} \alpha_n \rightarrow^D \frac{1}{q} B^*. \quad (1)$$

Where  $B^*$  is the dependent analogue of a Brownian Bridge, that is  $B^*$  denotes the centered Gaussian process specified by  $B^*(0) = B^*(1) = 1$  and

$$E B^*(x) B^*(y) = \sum_{k=-\infty}^{\infty} \text{Cov}(I(U_0 \leq x) I(U_{|k|} \leq y)). \quad (2)$$

Note that for independent sequences  $\{U_n\}$   $B^*$  coincides with the standard Brownian Bridge. For other important properties of  $B^*$  we refer to Shao, Yu [5]. The following theorem is the well known result from Chibisov (1964) and O'Reilly (1974).

**THEOREM 1.1.** *Let  $\{U_i, i \in Z\}$  be i.i.d. r.v's. Assume that  $q$  is a positive and continuous weight function on  $(0, 1)$ , and is nondecreasing in a neighborhood of 0 and non increasing in a neighborhood of 1. Then*

$$I(q, \lambda) := \int_0^1 \frac{1}{t(1-t)} \exp\left(-\frac{\lambda q^2(t)}{t(1-t)}\right) dt < \infty \tag{3}$$

for all  $\lambda > 0$  if and only if (1) holds in  $D[0, 1]$  with the Skorohod  $J_1$ -topology.

Function  $q$  which satisfies (3) for all  $\lambda > 0$  is called a Chibisov–O'Reilly weight function. The critical function in this case is  $\sqrt{t(1-t)}$ , which doesn't satisfy relation (3).

The case of dependent random variables is considered in the paper by Q.M. Shao, H. Yu [5]. They proved the weighted Empirical CLT (1) for mixing and associated processes. More general dependent processes, in the context of an empirical limit theorem, are provided in the paper by P. Doukhan and S. Louhichi [2].

This work provides the weighted Empirical CLT (1) when a sequence  $(U_n)_{n \in Z}$  is weakly dependent. With the help of Rozenthal-type inequalities under weak dependence it is shown that weighted ECLT holds for the weight functions having the following form

$$q(t) \geq C(t(1-t))^\mu \left(\log \frac{1}{t(1-t)}\right)^\beta, \quad t \in (0, 1), \tag{4}$$

where  $C > 0$ ,  $\beta > 1/2$ ,  $0 < \mu \leq 1/2$  and  $\mu$  depends on the form of weak dependence.

## 2. Dependence

### 2.1. Weak Dependence

Let  $L_n^\infty$  be the set of real valued bounded measurable functions on the Euclidean space  $R^n$  and let  $\|\cdot\|_\infty$  be the norm on  $L_n^\infty$ . Consider a function  $h: R^n \rightarrow R$  where  $R^n$  is equipped with the  $l^1$ -norm (i.e.,  $\|(z_1, \dots, z_n)\|_1 = |z_1| + \dots + |z_n|$ ) and define the Lipschitz modulus  $Lip(h) = \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_1}$ . Let  $L = \{h \in \bigcup_{i=1}^\infty L_i^\infty: \|h\|_\infty \leq 1, Lip(h) < \infty\}$ .

**DEFINITION 2.1** (P. Doukhan and S. Louhichi [2]). A sequence  $(X_n)_{n \in Z}$  of random variables is called  $(\theta, L, \psi)$ -weakly dependent if there exists a sequence  $\theta = (\theta_r)_{r \in N}$  decreasing to zero at infinity and a real valued function  $\psi$  with arguments  $(h, k, u, v) \in L \times L \times N \times N$  such that for any  $u$ -tuple  $(i_1, \dots, i_u)$  and any  $v$ -tuple  $(j_1, \dots, j_v)$  with  $i_1 \leq \dots \leq i_u < i_u + r \leq j_1 \leq \dots \leq j_v$  one has

$$|Cov(h(X_{i_1}, \dots, X_{i_u}), k(X_{j_1}, \dots, X_{j_v}))| \leq \psi(h, k, u, v)\theta_r \quad \forall r \in N,$$

and the functions  $h$  and  $k$  are defined respectively on  $R^u$  and  $R^v$ .

Examples of interest include  $\psi(h, k, u, v) = c(u, v)\mu(Lip(h), Lip(k))$  for some locally bounded functions  $\mu$  and  $c$ . For example if we take  $\mu(x, y) = c(x, y) = xy$  then

we get a weak dependence condition typical for associated processes. An interesting case is a so called  $s$ -weak dependence where function  $\psi(h, k, u, v) = v\|h\|_\infty \text{Lip}(k)$ .

A wide class of weakly dependent sequences describes the following definition.

**DEFINITION 2.2** Let  $(\xi_i)_{i \in \mathbb{Z}}$  be a sequence of i.i.d. real valued r.v.'s and let a function  $H: R^{\mathbb{Z}} \rightarrow R$  be measurable. A sequence  $(X_n)_{n \in \mathbb{Z}}$  of r.v. is called a Bernoulli shift if it is defined by  $X_n = H(\xi_{n-j}, j \in \mathbb{Z})$ . A sequence  $(X_n)_{n \in \mathbb{Z}}$  of r.v. is called causal Bernoulli shift if  $X_n = H(\xi_n, \xi_{n-1}, \dots, \xi_0, \xi_{-1}, \dots)$ ,  $H: R^{\mathbb{N}} \rightarrow R$ .

For any integer  $k > 0$ , denote  $\delta_k = \sup_{i \in \mathbb{Z}} E|H(\xi_{i-j}, j \in \mathbb{Z}) - H(\xi_{i-j} I_{|j| < k}, j \in \mathbb{Z})|$ . The following lemma describes the dependence structure of Bernoulli shifts.

**LEMMA 2.1** (P. Doukhan, S. Louhichi [2]). *Mean zero Bernoulli shifts are  $(\theta, L, \psi)$ -weakly dependent with*

$$\psi(h, k, u, v) = 4(u\|k\|_\infty \text{Lip}(h) + v\|h\|_\infty \text{Lip}(k)), \quad \theta_r = \delta_{r/2}.$$

*Under causality, this holds with  $\theta_r = \delta_r$  and  $\psi(h, k, u, v) = 2v\text{Lip}(k)\|h\|_\infty$ .*

### 2.2. ECLT under dependence

The following theorem of Q.M. Shao, H. Yu ([5]) is essential for the proofs of this paper.

**THEOREM 2.2.** *Let  $\{U_n, n \geq 1\}$  be a stationary sequence of uniform  $[0, 1]$  random variables. Assume that for all  $0 \leq s, t \leq 1$  and  $n \geq 1$  we have the following conditions*

$$\text{a) } E|\alpha_n(t) - \alpha_n(s)|^p \leq C_1(|t - s|^{p_1} + n^{-p_2/2}|t - s|^{r_1})$$

*for some  $C_1 > 0, p > 2, p_1 > 1, 0 \leq r_1 \leq 1$  and  $p_2 > 1 - r_1$ ,*

$$\text{b) } E|\alpha_n(t) - \alpha_n(s)|^2 \leq C_2|t - s|^{r_2}$$

*for some  $C_2 > 0$  and  $0 < r_2 \leq 1$ ,*

$$\text{c) } \alpha_n \rightarrow^D B^*.$$

*Then (1) holds with  $q$  an arbitrary weight function satisfying (4) for some  $C > 0$  and  $\beta > 1/2$  and*

$$\mu = \min\left(\frac{p_1}{p}, \frac{r_1 + p_2}{p + p_2}, \frac{r_2}{2}\right). \tag{5}$$

Note that if a Chibisov–O’Reilly weight function has the following form

$$q(t) = \left(t(1 - t) \log \log (1/(t(1 - t)))\right)^{1/2} f(t)$$

then, necessarily,  $f(t) \rightarrow \infty$  as  $t \rightarrow 0$  or  $t \rightarrow 1$ . Thus the weight function  $q$  from (4) can be compared to a Chibisov–O’Reilly weight function by taking  $\mu$  in (5) close to  $1/2$  or exactly  $1/2$ .

### 3. Weighted ECLT under weak dependence

Let's assume that  $\{U_n, n \geq 1\}$  is a stationary sequence of uniform  $[0, 1]$  random variables satisfying the following condition

$$\sup_{f \in \Phi} \left| Cov \left( \prod_{i=1}^2 f(U_{t_i}), \prod_{i=3}^4 f(U_{t_i}) \right) \right| \leq \theta_r, \tag{6}$$

where  $\Phi = \{x \rightarrow I(s < x \leq t), s, t \in [0, 1]\}$ ,  $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$  and  $r = t_3 - t_2$ .

Note that if condition (6) is satisfied and  $\theta_r = O(r^{-\frac{5}{2}-\nu})$ ,  $\nu > 0$ , then the sequence of processes  $\alpha_n$  is tight in the Skorohod space  $D[0, 1]$ (see [1]).

The idea of this section is a use of  $(\theta, L, \psi)$ -weak dependence together with the condition (6). Indeed the first one implies the second under the assumption of concentration of marginal distributions(in our case marginal distribution is uniform).

LEMMA 3.1. *Let  $\{U_n, n \geq 1\}$  be a stationary sequence such that (6) holds. Assume that  $\theta_r = O(r^{-a})$ ,  $a > 5/2$  then for some constant  $C > 0$*

$$E|\alpha_n(t) - \alpha_n(s)|^2 \leq C|t - s|^{\frac{a-1}{a}}.$$

*Proof.* Define  $Y_n = I(s < U_n \leq t) - (t - s)$ ,  $S_n = \sum_{i=1}^n Y_i$ . By lemma 3 in P. Doukhan, S. Louhichi [2] it follows

$$E|S_n|^2 \leq Cn \sum_{r=1}^n \min(r^{-a}, |t - s|) \leq Cn \left( \sum_{r \geq |t-s|^{-\frac{1}{a}}} r^{-a} + \sum_{r < |t-s|^{-\frac{1}{a}}} |t - s| \right).$$

We can approximate the first sum by the integral

$$\int_{|t-s|^{-\frac{1}{a}}}^{n-1} \frac{du}{u^a} \leq C|t - s|^{\frac{a-1}{a}}.$$

the second sum also satisfies the bound  $\leq C|t - s|^{\frac{a-1}{a}}$ .

THEOREM 3.1. *Let  $\{U_n, n \geq 1\}$  be a stationary sequence of uniform  $[0, 1]$  random variables. In addition assume that  $\{U_n, n \geq 1\}$  is  $(\theta, L, \psi_1)$ -weakly dependent, with*

$$\theta_r = O(r^{-5-\nu}), \quad \psi_1(h, k, u, v) = (Lip(h) \vee Lip(k))(u + v), \quad \nu > 0,$$

*then (1) holds with an arbitrary weight function  $q$  satisfying (4) for some  $C > 0$ ,  $\beta > 1/2$  and  $\mu = 1/2 - 1/(10 + 2\nu)$ .*

*Proof.* Using Rosenthal type inequalities under weak dependence obtained by P. Doukhan, S. Louhichi [2] it follows that for  $a = 5 + \nu$

$$E|\alpha_n(t) - \alpha_n(s)|^4 \leq C \left( \left( \sum_{r=0}^{n-1} r^{-a} \wedge |t - s| \right)^2 + \left( \frac{1}{n} \sum_{r=0}^{n-1} (r + 1)^2 \theta_r \right) \right)$$

$$\begin{aligned} &\leq C \left( \left( \sum_{r \geq |t-s|^{-\frac{1}{a}}} r^{-a} \right)^2 + \left( \sum_{r < |t-s|^{-\frac{1}{a}}} |t-s| \right)^2 + n^{2-a} \right) \\ &\leq C \left( |t-s|^{\frac{2(a-1)}{a}} + n^{2-a} \right). \end{aligned}$$

By Lemma 3.1 it follows that

$$E|\alpha_n(t) - \alpha_n(s)|^2 \leq C|t-s|^{\frac{a-1}{a}}.$$

Note that from [2] it follows  $\alpha_n \rightarrow^D B$ . Now it is possible to use Theorem 2.2

$$p_1 = \frac{2(a-1)}{a}, \quad r_1 = 0, \quad p_2 = 2(a-2), \quad r_2 = \frac{a-1}{a}, \quad p = 4.$$

Thus

$$\mu = \min \left( \frac{p_1}{p}, \frac{r_1 + p_2}{p + p_2}, \frac{r_2}{2} \right) = \min \left( \frac{a-2}{a}, \frac{a-1}{2a} \right) = \frac{a-1}{2a} = 1/2 - 1/(10 + 2v).$$

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## REZIUMĒ

### **M. Juodis. Empirinė centrinė ribinē teorema su svoriais silpnai priklausomiems procesams**

Darbe įrodoma centrinė ribinē teorema empiriniams procesams silpnai priklausomiems dydžiams. Nagrinėjamas empirinis procesas normuotas standartinėmis svorio funkcijomis.