Numerical Analysis of the Age-Sex-Structured Population Dynamics Taking into Account Spatial Diffusion

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Abstract. We present results of the numerical investigation of the homogenous Dirichlet and Neumann problems to an age-sex-structured population dynamics deterministic model taking into account random mating, female's pregnancy, and spatial diffusion. We prove the existence of separable solutions to the non-dispersing population model and, by using the numerical experiment, corroborate their local stability.

Keywords: age-sex-structured population, child care, spatial diffusion.

1 Introduction

To describe dynamics of the one-sex age-structured population with or without spatial diffusion the Sharpe-Lotka-McKendrick-von-Förster model [1] or its Gurtin-MacCamy [2] generalization is usually used. The survey of the numerical methods of solving of these models is given in [3]. In paper [4] a model of the two-sex age-structured population was proposed taking into account spatial diffusion and Cauchy problem was examined. It takes into account random mating of sexes and females' pregnancy. The model involves pairs that exist for period of mating only and uses the mating function of the simplified harmonic mean type. In the present paper we use a more general harmonic mean-type mating function (see equations (3)). To decrease the dimension of the problem in the computer simulation we consider the case where all vital rates do not depend on time. We also assume that ν_3 is independent of age of male and consider the homogeneous one-dimensional Dirichlet and Neumann problems.

The paper is organized as follows. In Section 3 we formulate the problem. In Section 4 we prove the existence of separable solutions to the non-dispersing population model. Results of the numerical investigation are presented in Section 5. Section 6 concludes the paper.

2 Notation

$x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$: the position in $\mathbb{R}^n$;

τ_1, τ_2, τ_3: the age of male, female, and embryo;

$\sigma_1 = (\tau_{11}, \tau_{12}), 0 < \tau_{11} < \tau_{12} < \infty$: the male sexual activity age-interval;

$\sigma_2(\tau_3) = (\tau_{21} + \tau_3, \tau_{22} + \tau_3), 0 < \tau_{21} < \tau_{22} < \infty$;

$\sigma_2(0)$: the female gestation age-interval;

$\sigma_2(T)$: the female delivery age-interval;

T: the gestation period;

$u_1(t, \tau_1, x)$: the density of males aged τ_1 at time t at the position x;

$u_2(t, \tau_2, x)$: the density of single (unfertilized) females aged τ_2 at time t at the position x;

$u_3(t, \tau_1, \tau_2, \tau_3, x)$: the density of fertilized females aged τ_2 at time t at the position x who were fertilized by males aged τ_1 (at the mating moment) and whose embryos are aged τ_3 at the same time t;

$\nu_k(t, \tau_k, x)$: the death rate of individuals ($k = 1$ for males, $k = 2$ for single females) aged τ_k at time t at the position x;

$\nu_3(t, \tau_1, \tau_2, \tau_3, x)$: the death rate of fertilized females aged τ_2 at time t at the location x who were fertilized by males aged τ_1 at the mating moment and whose embryos are aged τ_3 at the same time t;

$h_k(\tau_k)$: the preference functions;

$p(t, \tau_1, \tau_2, x)dt$: the fertilization probability;

$Q_1 = (0, \infty), Q_2 = (0, \infty)\backslash\{\tau_{21}, \tau_{21} + T, \tau_{22}, \tau_{22} + T\}$;

$Q_3 = \sigma_1 \times Q_3'$;

$Q_3' = \{(\tau_2, \tau_3) : \tau_2 \in \sigma_2(\tau_3), \tau_3 \in (0, T)\}$;
$\mathfrak{B} = l_1\partial_\nu + l_2, l_s(x) \geq 0, l_1 + l_2 > 0; \partial_\nu$: the outward normal derivative on $\partial\Omega$;
$b_k(t, \tau_1, \tau_2, x)$: the birth moduli ($k = 1$ for males, $k = 2$ for females);
$[u_2|_{\tau=\tau}]$: a jump discontinuity of u at the $\tau = \tau_{21}, \tau_{21} + T, \tau_{22}, \tau_{22} + T$;
κ_k: the diffusion coefficient;
$u_1^0(\tau_1, x), u_2^0(\tau_2, x), u_3^0(\tau_1, \tau_2, \tau_3, x)$: the initial distributions.

3 The model

We consider the initial-boundary value problem with the homogeneous Robin condition on the boundary Ω. We assume that the pair formation can be described by a harmonic mean-type function. The model consists of the equations for u_1, u_2, and u_3,

$$\begin{cases} \partial u_1/\partial t + \partial u_1/\partial \tau_1 = -\nu_1 u_1 + \kappa_1 \Delta u_1, \quad t > 0,\ \tau_1 \in Q_1,\ x \in \Omega, \\ u_1|_{t=0} = u_1^0, \\ u_1|_{\tau_1=0} = \int\limits_{\sigma_1} d\tau_1 \int\limits_{\sigma_2(T)} b_1 u_3\big|_{\tau_3=T} d\tau_2, \\ (\mathfrak{B}u_1)|_{\partial\Omega} = 0; \end{cases} \tag{1}$$

$$\begin{cases} \partial u_2/\partial t + \partial u_2/\partial \tau_2 \\ \quad = -\nu_2 u_2 - X_l + X_g + \kappa_2 \Delta u_2, \quad t > 0,\ \tau_2 \in Q_2,\ x \in \Omega, \\ u_2|_{t=0} = u_2^0, [u_2|_{\tau_2=\tau}] = 0, \\ u_2|_{\tau_2=0} = \int\limits_{\sigma_1} d\tau_1 \int\limits_{\sigma_2(T)} b_2 u_3|_{\tau_3=T} d\tau_2, \\ (\mathfrak{B}u_2)|_{\partial\Omega} = 0, \\ X_l = \begin{cases} 0, & \tau_2 \notin \sigma_2(0), \\ \int\limits_{\sigma_1} u_3|_{\tau_3=0} d\tau_1, & \tau_2 \in \sigma_2(0), \end{cases} \\ X_g = \begin{cases} 0, & \tau_2 \notin \sigma_2(T), \\ \int\limits_{\sigma_1} u_3|_{\tau_3=T} d\tau_1, & \tau_2 \in \sigma_2(T); \end{cases} \end{cases} \tag{2}$$

$$
\begin{cases}
\partial u_3/\partial t + \partial u_3/\partial \tau_2 + \partial u_3/\partial \tau_3 \\
\quad = -\nu_3 u_3 + \kappa_3 \Delta u_3, \quad t > 0,\ (\tau_1, \tau_2, \tau_3) \in Q_3,\ x \in \Omega, \\
u_3|_{t=0} = u_3^0, \\
u_3|_{\tau_3=0} = p u_1 u_2 / \Big(\int\limits_{\sigma_1} h_1 u_1 d\tau_1' + \int\limits_{\sigma_2(0)} h_2 u_2 d\tau_2' \Big), \\
(\mathfrak{B} u_3)|_{\partial\Omega} = 0.
\end{cases}
\tag{3}
$$

Here $\triangle$ is the Laplace operator in $\mathbb{R}^m$. All given functions $\nu_1, \nu_2, \nu_3, b_1, b_2,$ $h_1, h_2, l_1, l_2, u_1^0, u_2^0, u_3^0,$ and p and the unknown ones $u_1, u_2,$ and u_3 must be positive supported otherwise they have no biological significance. We assume that $\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22},$ and T are positive given constants and formulate the following compatibility conditions:

$$
\begin{aligned}
&u_k^0|_{\tau_k=0} = \int\limits_{\sigma_1} d\tau_1 \int\limits_{\sigma_2(T)} u_3^0\Big|_{\tau_3=T} d\tau_2, k = 1, 2; \\
&u_3^0|_{\tau_3=0} = p|_{t=0} u_1^0 u_2^0 / \Big(\int\limits_{\sigma_1} h_1|_{t=0} u_1^0 d\tau_1' + \int\limits_{\sigma_2(0)} h_2|_{t=0} u_2^0 d\tau_2' \Big), \\
&[u_2^0|_{\tau_2=\tau}] = 0, \\
&\mathfrak{B} u_s^0|_{\partial\Omega=0}, s = 1, 2, 3.
\end{aligned}
$$

4 Separable solutions to the non-dispersing population model

In this section, by using the method used in [5], we examine separable solutions to non-dispersing population dynamics model.

Let all vital rates $\nu_1, \nu_2, \nu_3, b_1, b_2,$ and p and initial distributions $u_1^0, u_2^0,$ and u_3^0 do not depend on x. Then the non-dispersing population model can be derived from model (1)–(3) by letting $\kappa_1, \kappa_2,$ and κ_3 be zero and dropping conditions on $\partial\Omega$. Assume, in addition, that $\nu_1, \nu_2, \nu_3, b_1, b_2,$ and p do not depend on t and seek solutions to the non-dispersing population model in the form

$$
\begin{cases}
u_1(t,\tau_1) = a_1 v_1(\tau_1)\exp\big\{\lambda(t-\tau_1)\big\}, \quad v_1(0)=1,\\
u_2(t,\tau_2) = a_2 v_2^\lambda(\tau_2)\exp\big\{\lambda(t-\tau_2)\big\}, \quad v_1^\lambda(0)=1,\\
u_3(t,\tau_1,\tau_2,\tau_3)\\
\quad = \dfrac{a_1a_2}{\gamma} v_1(\tau_1)v_2^\lambda(\tau_2-\tau_3)v_3(\tau_1,\tau_3,\tau_3)\exp\big\{\lambda(t-\tau_1-\tau_2)\big\},\\
v_3(\tau_1,\tau_2,0) = p(\tau_1,\tau_2),\\
\gamma = a_1 \displaystyle\int_{\sigma_1} h_1(\tau_1)v_1(\tau_1)\exp\{-\lambda\tau_1\}d\tau_1\\
\qquad + a_2 \displaystyle\int_{\sigma_2(0)} h_2(\tau_2)v_2^\lambda(\tau_2)\exp\{-\lambda\tau_2\}d\tau_2
\end{cases}
\tag{4}
$$

where functions v_1, v_2^λ, v_3 and the constants γ and λ are to be determined. Set $y_s = a_s/\gamma, s = 1,2$. Substituting functions (4) into the model of non-dispersing population we get the following equations:

$$
\begin{cases}
\partial\tau_1 v_1 = -\nu_1 v_1, \quad v_1(0)=1,\\
\partial\tau_2 v_2^\lambda = -\nu_2^\lambda - \begin{cases} 0, & \tau_2 \notin \sigma_2(0),\\ A(\tau_2,\lambda,y_1)v_2^\lambda, & \tau_2\in\sigma_2(0)\end{cases}\\
\qquad\qquad + \begin{cases} 0, & \tau_2\notin\sigma_2(T), v_2^\lambda(0)=1,\\ B(\tau_2,\lambda,y_1)v_2^\lambda(\tau_2-T), & \tau_2\in\sigma_2(T), [v_2^\lambda|_{\tau_2=\tau}]=0,\end{cases}\\
\partial\tau_2 v_3 + \partial\tau_3 v_3 = -\nu_3 v_3, \quad v_3|_{\tau_3=0} = p,
\end{cases}
\tag{5}
$$

$$y_1 = 1/q_1^\lambda(y_1), \tag{6}$$

$$y_2 = 1/q_2^\lambda(y_1), \tag{7}$$

$$r(\lambda) = 1 \tag{8}$$

where

$$q_1^\lambda(y_1) = \int_{\sigma_1} d\tau_1 \int_{\sigma_2(T)} b_2(\tau_1,\tau_2)v_1(\tau_1)v_2^\lambda(\tau_2-T)v_3(\tau_1,\tau_2,T)\exp\big\{-\lambda(\tau_1+\tau_2)\big\}d\tau_2,$$

$$q_2^\lambda(y_1) = \int_{\sigma_1} d\tau_1 \int_{\sigma_2(T)} b_1(\tau_1,\tau_2)v_1(\tau_1)v_2^\lambda(\tau_2-T)v_3(\tau_1,\tau_2,T)\exp\big\{-\lambda(\tau_1+\tau_2)\big\}d\tau_2,$$

$$r(x) = y_1\int_{\sigma_1} h_1(\tau_1)v_1(\tau_1)\exp\{-\lambda\tau_1\}d\tau_1 + y_2\int_{\sigma_2(0)} h_2(\tau_2)v_2^\lambda(\tau_2)\exp\{-\lambda\tau_2\}d\tau_2$$

and

$$A(\tau_2,\lambda,y_1)=y_1\int\limits_{\sigma_1}p(\tau_1,\tau_2)v_1(\tau_1)\exp\{-\lambda\tau_1\}d\tau_1,$$
$$B(\tau_2,\lambda,y_1)=y_1\int\limits_{\sigma_1}v_1(\tau_1)v_3(\tau_1,\tau_2,T)\exp\{-\lambda\tau_1\}d\tau_1.$$

From equations (5) by formal integration it follows that

$$v_1(\tau_1)=\exp\Big\{-\int\limits_0^{\tau_1}\nu_1(\xi)d\xi\Big\},$$
$$v_3(\tau_1,\tau_2,\tau_3)=p(\tau_1,\tau_2-\tau_3)\exp\Big\{-\int\limits_0^{\tau_3}\nu_3(\tau_1,\xi+\tau_2-\tau_3,\xi)d\xi\Big\},$$
$$v_2^\lambda(\tau_2)=v_{2\star}(\tau_2):=\exp\Big\{-\int\limits_0^{\tau_2}\nu_2(\xi)d\xi\Big\},\quad \tau_2\le\tau_{21},$$

and

$$v_2^\lambda(\tau_2)=v_{2\star}(\tau_2):=\exp\Big\{-\int\limits_0^{\tau_2}\nu_2(\xi)d\xi-\int\limits_{\tau_{21}}^{\tau_2}\sup_{\tau_2}\big(p(\tau_1,\tau_2)/h_1(\tau_1)\big)d\xi\Big\},$$
$$\tau_2\ge\tau_{21},$$

since, by (8), $y_1\int\limits_{\sigma_1}h_1(\tau_1)v_1(\tau_1)\exp\{-\lambda\tau_1\}d\tau_1<1$.

Therefore

$$0<1/q_1^\lambda(y_1)<1/q_1(\lambda)\quad\forall y_1\ge0 \tag{9}$$

where

$$0<q_1(\lambda):=\int\limits_{\sigma_1}d\tau_1\int\limits_{\sigma_2(T)}b_2(\tau_1,\tau_2)v_1(\tau_1)v_{2\star}(\tau_2-T)v_3(\tau_1,\tau_2,T)$$
$$\times\exp\{-\lambda(\tau_1+\tau_2)\}d\tau_2\le q_1^\lambda(y_1).$$

Now, by estimate (9),

$$\big(y_1-1/q_1^\lambda(y_1)\big)\big|_{y_1=0}<0\quad\text{and}\quad\big(y_1-1/q_1^\lambda(y_1)\big)\big|_{y_1=1/q_1(\lambda)}>0.$$

Hence, (6) has at least one root $y_1(\lambda)\in(0,1/q_1(\lambda))$. It is easy to see that $y_1(\lambda)$ and $y_2(\lambda)=1/q_2^\lambda(y_1(\lambda))$ are continuous in λ. Substituting $y_s(\lambda), s=1,2$ into

(8) we get a characteristic equation for λ. It remains to prove that it has at least one real root λ. Setting

$$\begin{aligned} C(\tau_1) &= \sup_{\tau_2} \big(v_3(\tau_1,\tau_2,T)b_1(\tau_1,\tau_2)/h_2(\tau_2-T)\big), \\ D(\tau_1) &= \inf_{\tau_2} \big(v_3(\tau_1,\tau_2,T)b_1(\tau_1,\tau_2)/h_2(\tau_2-T)\big), \\ E(\tau_2) &= \inf_{\tau_1} \big(v_3(\tau_1,\tau_2,T)b_2(\tau_1,\tau_2)/h_1(\tau_1)\big) \end{aligned}$$

we have

$$\begin{aligned} r(\lambda) > \frac{1}{q_2^\lambda(y_1)} &\int\limits_{\sigma_2(0)} h_2(\tau_2)v_2^\lambda(\tau_2)\exp\{-\lambda\tau_2\}d\tau_2 \\ &\geq \int\limits_{\sigma_2(0)} h_2(\tau_2)v_2^\lambda(\tau_2)\exp\{-\lambda\tau_2\}d\tau_2 \\ &\quad \times 1/\int\limits_{\sigma_1} v_1(\tau_1)C(\tau_1)\exp\big\{-\lambda(\tau_1+T)\big\}d\tau_1 \\ &\quad \times 1/\int\limits_{\sigma_2(T)} h_2(\tau_2-T)v_2^\lambda(\tau_2-T)\exp\big\{-\lambda(\tau_2-T)\big\}d\tau_2 \\ &= r_1(\lambda) := 1/\int\limits_{\sigma_1} v_1(\tau_1)C(\tau_1)\exp\{-\lambda(\tau_1+T)\}d\tau_1 \end{aligned}$$

and similarly

$$\begin{aligned} r(\lambda) \leq r_2(\lambda) :=& 1/\int\limits_{\sigma_2(T)} v_{2\star}(\tau_2-T)E(\tau_2)\exp\{-\lambda\tau_2\}d\tau_2 \\ &+ 1/\int\limits_{\sigma_1} v_1(\tau_1)D(\tau_1)\exp\{-\lambda(\tau_1+T)\}d\tau_1. \end{aligned}$$

Since $r_1(\lambda) \leq r(\lambda) \leq r_2(\lambda), r_1(\lambda), r_2(\lambda) \to 0$ as $\lambda \to -\infty$, and $r_1(\lambda)$, $r_2(\lambda) \to \infty$ as $\lambda \to \infty$, (8) has at least one real root λ. We note that the positive parameter γ is not determined. As a result we formulate the following proposition.

Theorem 1. *Let constants* $T, \tau_{11} < \tau_{12}, \tau_{21} < \tau_{22}$, *functions* $\nu_1, \nu_2, \nu_3, p, b_1$, *and* b_2 *be positive and* $\inf h_i > 0, i = 1, 2$. *Assume that functions* $\nu_1, \nu_2, \nu_3, p, b_1$ *and* b_2 *do not depend on* t *and* $\nu_1, \nu_2 \in C^0(0,\infty), b_1, b_2 \in C^0(\overline{\sigma}_1 \times \overline{\sigma}_2(T)), h_1 \in C^0(\overline{\sigma}_1), h_2 \in C^0(\overline{\sigma}_2(0)), p \in C^{0,1}(\overline{\sigma}_1 \times \overline{\sigma}_2(0)), \nu_3 \in C^{0,1,0}(\overline{Q}_3)$. *Then the non-dispersing population model has at least one one-parameter class of positive solutions of type* (4).

5 Numerical results

We consider the case where all vital rates are stacionary and excluding ν_1 and p do not depend on τ_1. This enables us to determine $\overline{u}_3(t,\tau_2,\tau_3) = \int\limits_{\sigma_1} u_3 d\tau_1$ and decrease the dimension of problem.

By using $\overline{u}_3$ from (1)–(3) we get the system

$$\begin{cases} \partial u_1/\partial t + \partial u_1/\partial \tau_1 = -\nu_1 u_1 + \kappa_1 \Delta u_1, \quad t > 0,\ \tau_1 \in Q_1,\ x \in \Omega, \\ u_1|_{t=0} = u_1^0, \\ u_1|_{\tau_1=0} = \displaystyle\int\limits_{\sigma_2(T)} b_1 \overline{u}_3\big|_{\tau_3=T} d\tau_2, \\ (\mathfrak{B} u_1)|_{\partial\Omega} = 0; \end{cases} \tag{10}$$

$$\begin{cases} \partial u_2/\partial t + \partial u_2/\partial \tau_2 \\ \quad = -\nu_2 u_2 - X_l + X_g + \kappa_2 \Delta u_2, \quad t > 0,\ \tau_2 \in Q_2,\ x \in \Omega, \\ u_2|_{t=0} = u_2^0, [u_2|_{\tau_2=\tau}] = 0, \\ u_2|_{\tau_2=0} = \displaystyle\int\limits_{\sigma_2(T)} b_2 \overline{u}_3\big|_{\tau_3=T} d\tau_2, \\ (\mathfrak{B} u_2)|_{\partial\Omega} = 0, \\ X_l = \begin{cases} 0, & \tau_2 \notin \sigma_2(0), \\ \overline{u}_3|_{\tau_3=0}, & \tau_2 \in \sigma_2(0), \end{cases} \\ X_g = \begin{cases} 0, & \tau_2 \notin \sigma_2(T), \\ \overline{u}_3|_{\tau_3=T}, & \tau_2 \in \sigma_2(T); \end{cases} \end{cases} \tag{11}$$

$$\begin{cases} \partial \overline{u}_3/\partial t + \partial \overline{u}_3/\partial \tau_2 + \partial \overline{u}_3/\partial \tau_3 \\ \quad = -\nu_3 u_3 + \kappa_3 \Delta \overline{u}_3, \quad t > 0,\ (\tau_1,\tau_2,\tau_3) \in Q_3,\ x \in \Omega, \\ \overline{u}_3|_{t=0} = \overline{u}_3^0, \\ \overline{u}_3|_{\tau_3=0} = \int\limits_{\sigma_1} p u_1 d\tau_1 u_2 / \Big(\displaystyle\int\limits_{\sigma_1} h_1 u_1 d\tau_1' + \int\limits_{\sigma_2(0)} h_2 u_2 d\tau_2' \Big), \\ (\mathfrak{B} \overline{u}_3)|_{\partial\Omega} = 0. \end{cases} \tag{12}$$

In what follows we analyze the case where $\Omega = (0;1)$ and use the following initial functions:

$$
\begin{aligned}
&u_k^0(\tau_k, x) = f(x)U_k(\tau_k, x), \quad U_k(\tau_k, x) = \alpha_{k3}(\tau_k + \alpha_{k2})\exp(-\alpha_{k1}\tau_k),\\
&k = 1, 2,\\
&p(\tau_1, \tau_2) = p_1(\tau_1)p_2(\tau_2),\\
&p_k(\tau_k) = p_{k1} + (p_{k2} - p_{k1})(\tau_k - \tau_{k1})/(\tau_{k2} - \tau_{k1}), \quad k = 1, 2,\\
&\overline{u}_3^0(\tau_2, \tau_3, x) = f(x)q(\tau_3)U_3(\tau_2 - \tau_3, x), q(\tau_3) = 1 + (q(T) - 1)\tau_3/T,\\
&U_3(\tau_2, x) = p_2(\tau_2)U_2(\tau_2, x)\frac{\int\limits_{\sigma_1} p(\tau_1)U_1(\tau_1, x)d\tau_1}{\int\limits_{\sigma_1} U_1(\tau_1, x)d\tau_1 + \int\limits_{\sigma_2(0)} U_2(\tau_2', x)d\tau_2'},\\
&b_k(\tau_2, x) = \xi(x)\beta_k \sin^\alpha\big(\pi(\tau_2 - \tau_{21} - T)/(\tau_{22} - \tau_{21})\big),\\
&U_k(0, x) = \int\limits_{\sigma_2(T)} b_k(\tau_2, x)U_3(\tau_2 - T, x)d\tau_2 q(T), \quad k = 1, 2,\\
&\nu_k(\tau_k) = \mu_{k1}|\tau_k - \tau_{k0}|^{\alpha_k} + \mu_{k2}, \quad k = 1, 2,\\
&\nu_3(\tau_2, \tau_3) = \mu_{31}(\tau_3)|\tau_2 - \tau_{20}|^{\alpha_2} + \mu_{32}(\tau_3).
\end{aligned}
\tag{13}
$$

From conditions (13) it follows that

$$
\frac{\alpha_{23}(x)}{\alpha_{13}} = \frac{\alpha_{12}}{\alpha_{22}(x)}\frac{\beta_2}{\beta_1}
\tag{14}
$$

and

$$
\begin{aligned}
&\alpha_{22}\Big(\int\limits_{\sigma_1}(\tau_1 + \alpha_{12})\exp(-\alpha_{11}\tau_1)d\tau_1\\
&\qquad + \frac{\alpha_{12}}{\alpha_{22}(x)}\frac{\beta_2}{\beta_1}\int\limits_{\sigma_2(0)}(\tau_2 + \alpha_{22})\exp(-\alpha_{21}\tau_2)d\tau_2\Big)\\
&= q(T)\beta_2\xi(x)\int\limits_{\sigma_2(0)} sin^\alpha\pi\frac{(\tau_2 - \tau_{21})}{(\tau_{22} - \tau_{21})}(\tau_2 + \alpha_{22})\exp(-\alpha_{21}\tau_2)p_2(\tau_2)d\tau_2\\
&\quad \times\int\limits_{\sigma_1} p_1(\tau_1)(\tau_1 + \alpha_{12})\exp(-\alpha_{11}\tau_1)d\tau_1.
\end{aligned}
\tag{15}
$$

Let $\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, q(T), \alpha, \alpha_1, \alpha_2, \xi(x), p_{11}, p_{12}, p_{21}, p_{22}, \mu_{11}, \mu_{12}$, $\mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}$ be free. From (15) we determine $\alpha_{22}(x)$. Then from (14) we find $\alpha_{23}(x)$ and use the following restrictions:

$\partial f/\partial x|_{x=0;1} = 0$ for the homogenous Neumann problem

and

$f|_{x=0;1} = 0$ for the homogenous Dirichlet problem

and the functions:

$$\xi(x) = 1 \quad \forall x \in [0;1] \quad \text{or} \quad \xi(x) = \begin{cases} 0, & x \in (0.5;1], \\ x(0,5-x), & x \in [0;0.5] \end{cases}$$

for the Dirichlet problem and $\xi(x) = 1 \ \forall x \in [0;1]$ for the Neumann problem.

In what follows $\alpha_{12}(x) = \widetilde{\alpha}_{12} + x(1-x), x \in [0;1]$ where $\widetilde{\alpha}_{12}$ is a constant.

Now we describe numerical schemes for both Neumann and Dirichlet problems. We use the Douglas and Milner [6] method which they applied to the Gurtin MacCamy [2] model. To do this we write both models on the characteristic lines and then discretize them by using the same step for time and ages. As a result we get two corresponding discrete nonlinear systems at each time level. To determine u_2 from these nonlinear systems we apply the Zeidel and the Crank-Nicolson schemes [7] and at each time level use the following procedure:

determine u_1 except $u_1|_{\tau_1=T_{24}}$,

determine $\overline{u}_3$ except $\overline{u}_3|_{\tau_3=0}$,

determine u_2 for $\tau_2 \le \tau_{21}$ and $\tau_2 > \tau_{22} + T$,

determine u_2 for $\tau_2 \in (\tau_{21}, \tau_{22} + T)$ by the Zeidel method,

determine $u_1|_{\tau_1=T_{24}}, \overline{u}_3|_{\tau_3=0}$.

Results of the numerical calculations are displayed in Figs. 1–12 for the following values of constants:

$$\begin{aligned} & h_1 = h_2 = 1,\ \alpha_{11} = 0.1, \widetilde{\alpha}_{12} = 4,\ \alpha_{13} = 2, \alpha_{21} = 0.1, \\ & q(T) = 0.85,\ \beta_2 = 0.80, \\ & \tau_{11} = 4,\ \tau_{12} = 9, \\ & p_{11} = 0.95,\ p_{12} = 0.99,\ p_{22} = 0.99, \\ & \alpha = 1.5,\ \alpha_1 = 1.5,\ \alpha_2 = 1.5, \\ & \mu_{11} = \mu_{21} = \mu_{31} = 0.0001,\ \mu_{12} = \mu_{22} = \mu_{32} = 0.01. \end{aligned} \tag{16}$$

The other constants and functions are given in Table 1.

Table 1. Constants and functions used in all calculations

Figs.	τ_{12}	τ_{22}	T	β_1	$\kappa \cdot 10^{-2}$	p_{21}	$f(x)$	$\xi(x)$
1	9	9	1	0.85	0.5	0.95	f_2	ξ_2
2	9	9	1	0.85	0.5	0.95	f_2	ξ_2
3	9	10	1	0.85	0.5	0.95	f_2	ξ_2
4	9	10	1	0.85	0.5	0.95	f_1	ξ_1
5	9	9	1	0.85	0.5	0.95	f_1	ξ_1
6	9	9	1	0.85	0.5	0.95	f_3	ξ_1
7	9	9	1	0.85	0.5	0.95	f_3	ξ_1
8	9	10	1	0.85	0.5	0.95	f_3	ξ_1
9	9	9	1	0.85	0.5	0.95	f_3	ξ_1
10	[5;11]	[8;14]	1	0.85	0.5	0.95	f_3	ξ_1
11	9	9	[0.6; 1.8]	0.85	0.1 1	0.95	f_3	ξ_1
12	9	9	1	0.95	0.5	[0.6;0.95]	f_3	ξ_1

Here

$$f_1(x) = f(x) = x(1-x), \quad x \in [0;1],$$

$$f_2(x) = f(x) = \begin{cases} 0, & x \in (0.5;1], \\ x(0,5-x), & x \in [0;0.5], \end{cases}$$

$$f_3(x) = f(x) = 2 + x^{3/2}(1-x)^{3/2}, \quad x \in [0;1],$$

$$\xi_1(x) = \xi(x) = 1, x \in [0;1],$$

$$\xi_2(x) = \xi(x) = \begin{cases} 0, & x \in (0.5;1], \\ x(0,5-x), & x \in [0;0.5]. \end{cases}$$

In addition to set (16) we use constants and functions from Table 1.

Figs. 1, 2, and 3 represent numerical solution to the Dirichlet problem for $\xi(x) = \xi_2(x)$.

Fig. 1 shows that the total population

$$N(x,t) = \int_0^\infty u_1 d\tau_1 + \int_0^\infty u_2 d\tau_2 + \int_0^T d\tau_3 \int_{\tau_{21}+\tau_3}^{\tau_{22}+\tau_3} \overline{u}_3 d\tau_2$$

with initial support $[0;0.5]$ spreads in time over all interval $[0;1]$.

In Fig. 2 and 3 graphs of functions $u_1(t,\tau_1,x)$, and $u_2(t,\tau_2,x)$ are exhibited for $t = 0, t = 2, t = 4$.

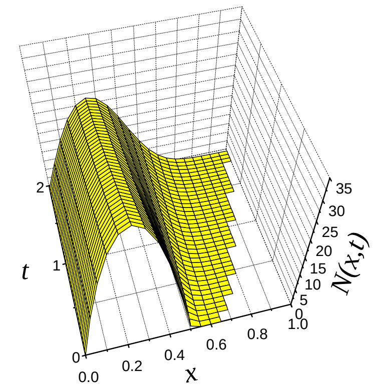

Fig. 1. The behavior of $N(x,t)$ for the solution of the Dirichlet problem.

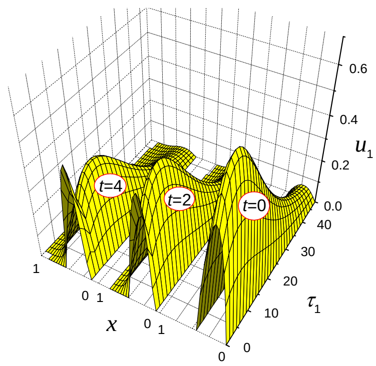

Fig. 2. The graph of $u_1(t,\tau_1,x)$ for the solution of the Dirichlet problem with f_2 and $\xi_2, t=0;2;4$.

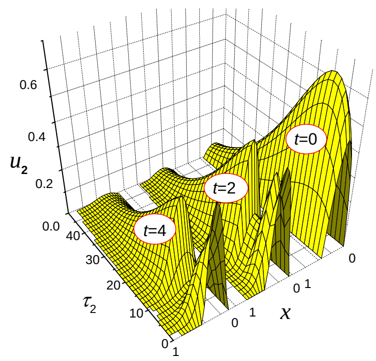

Fig. 3. The graph of $u_2(t,\tau_2,x)$ for the solution of the Dirichlet problem with f_2 and $\xi_2, t=0;2;4$.

In Figs. 4 and 5 we illustrate the numerical solution to the Dirichlet problem taking $\xi(x) = \xi_1(x), x \in [0;1]$. Figs. 4 and 5 represents functions u_1 and u_2.

Figs. 6, 7, 8, and 9 illustrate the numerical solution of the Neumann problem for $f(x) = f_3(x)$ and $\xi(x) = \xi_1(x), x \in [0;1]$. Fig. 6 represents the graph of the function N(x,t) while graphs 7 and 8 exhibit functions u_1 and u_2. We see that function N(x,t) loses the dependence on x in time very fast for $\kappa = 0.1$.

In Fig. 9 we illustrate the graph of the function

$$N_1(x,t) = \int\limits_0^T d\tau_3 \int\limits_{\tau_{21}+\tau_3}^{\tau_{22}+\tau_3} \overline{u}_3 d\tau_2.$$

Function $N_1(x,t)$ means the total number of fertilized females. We see that function N_1 loses the dependence on x in time very slowly for $\kappa = 0.001$

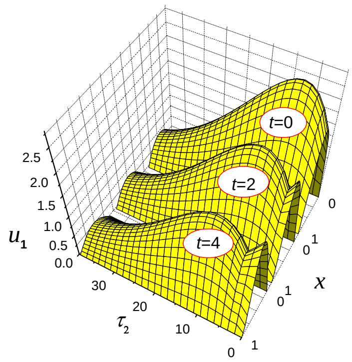

Fig. 4. The graph of $u_1(t,\tau_1,x)$ for the solution of the Dirichlet problem with f_1 and $\xi_1, t = 0;2;4$.

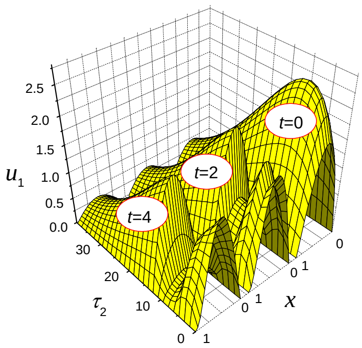

Fig. 5. The graph of $u_2(t,\tau_2,x)$ for the solution of the Dirichlet problem with f_1 and $\xi_1, t = 0;2;4$.

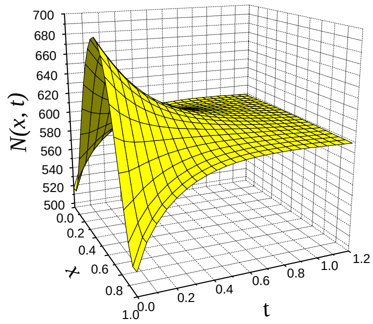

Fig. 6. The behavior of $N(x, t)$ for the solution of the Neumann problem.

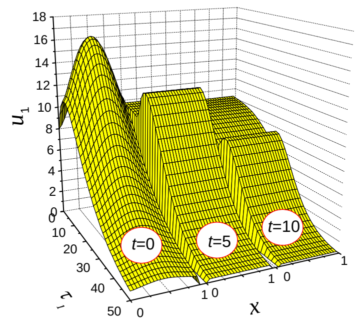

Fig. 7. The graph of $u_1(t, \tau_1, x)$ for the solution of the Neumann problem.

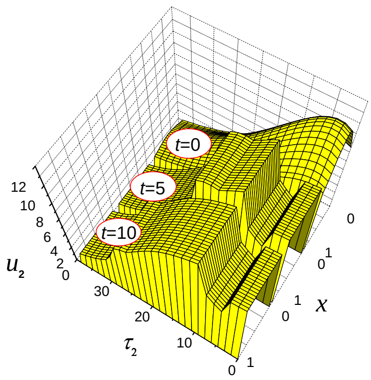

Fig. 8. The graph of $u_2(t, \tau_2, x)$ for the solution of the Neumann problem.

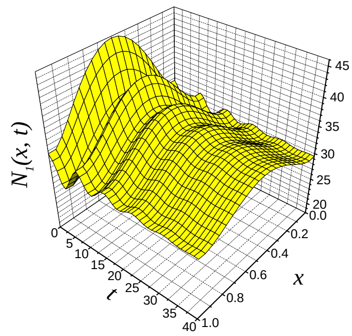

Fig. 9. The behavior of $N_1(x, t)$ for the solution of the Neumann problem.

In Figs. 10–12 the behavior of $\lambda = \lim_{t_1 < t_2, t_1 \to \infty} \frac{1}{t_2 - t_1} \ln N(t_2, x)/N(t_1, x)$ is demonstrated. Note that λ is independent of x.

In Fig. 10 functions $\lambda(\tau_{12})$ (curve 1) and $\lambda(\tau_{22})$ (curve 2) are exhibited. We can see that the behavior of these curves is different.

Fig. 11 shows that the dependence of function $\lambda(\kappa, T)$ on κ is weak.

The last Fig. 12 shows graphs of functions $\lambda(p_{21})$ and $\lambda(\beta_2)$ where $p_{21} = p_2(\tau_{21})$.

The existence of λ shows the local stability of separable solutions given in Section 4.

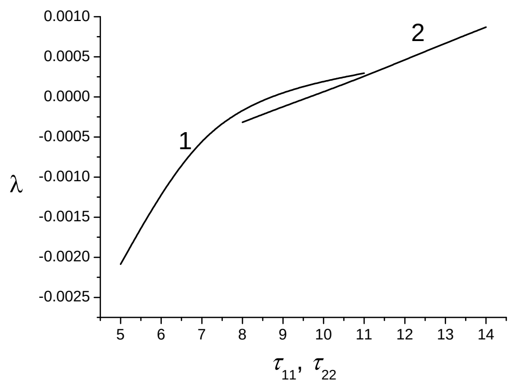

Fig. 10. Plot of λ. Curve 1 represents $\lambda(\tau_{12})$. Curve 2 illustrates $\lambda(\tau_{22})$.

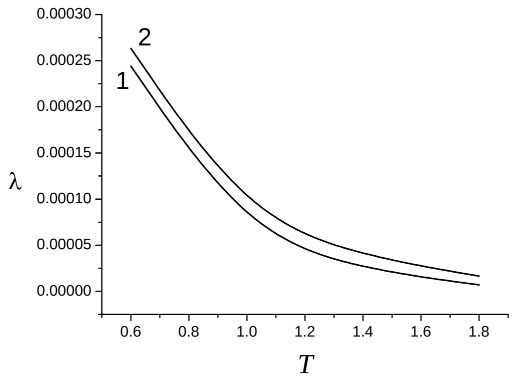

Fig. 11. Plot of $\lambda(T)$. Curve 1 for $\kappa = 0.1$. Curve 2 for $\kappa = 0.001$.

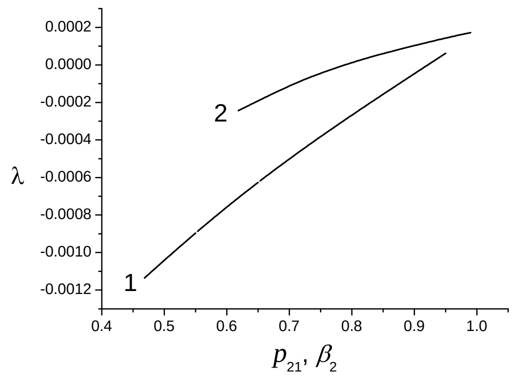

Fig. 12. Plot of $\lambda(p_{21})$ (curve 1) and $\lambda(\beta_2)$ (curve 2).

6 Concluding remarks

The numerical analysis of the two-sex age-structual population dynamics model is presented taking into account formation of temporal pairs and spatial dispersal. The homogenous Dirichlet and Neumann problems are examined. The numerical experiment shows that the solution of the homogeneous Neumann problem tends in time to the solution of the non-dispersing population model.

The existence of at least one one-parameter class of separable solution is proved to the non-dispersing population model. Numerical results show the local stability of a separable solutions.

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