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Sequent calculus for propositional likelihood logic

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1. Introduction

The logic of likelihood LL was introduced in 1987 by Halpern and Rabin [3]. As a motivation, in the paper, it is shown that the logic can be used for protocol verification of data transfer.

LL has three modal operators: $\mathbf{L}$, $\mathbf{L}^*$ and $\mathbf{G}$. $\mathbf{L}p$ can be roughly read as "p is reasonably likely to be a consistent hypothesis." The degree of likelihood can be changed by nesting multiple $\mathbf{L}$ operators. E.g., $\mathbf{LL}p$ expresses a less likely statement than $\mathbf{L}p$ and can be read as "p is somewhat likely to be a consistent hypothesis." Adding even more $\mathbf{L}$ would make the statement less and less likely. Let $\mathbf{L}^n = \underbrace{\mathbf{L}\ldots\mathbf{L}}_{n}$. If $m < n$, then $L^m p \Rightarrow L^n p$ is valid in LL but $L^n p \Rightarrow L^m p$ is not. Thus, although numeric values are not used for likelihood, various degrees of likelihood still can be expressed in LL.

$\mathbf{L}^* p$ denotes the limit of the sequence $\mathbf{L}p, \mathbf{L}^2 p, \mathbf{L}^3 p, \ldots$

The operator $\mathbf{G}$ is used to denote the necessity: $\mathbf{G}p$ is read as "necessarily p."

For more information on the logic of likelihood, we refer to [3], [4], and [2].

The logic without operator $\mathbf{L}^*$ is called star free likelihood logic and denoted by LL^-. Sequent calculi for LL^- are considered in [4] and [1].

In this paper, we introduce and consider a classical sequent calculus for propositional likelihood logic. The admissibility of structural rules and cut rule, invertibility of rules, correctness and completeness of the calculus with respect to the given semantics are proved.

The paper is organized as follows. Syntax and semantics are presented in Section 2. The admissibility of structural and cut rules and invertibility of rules of the introduced sequent calculus are proved in Section 3. The correctness and completeness of the calculus with respect to the given semantics are proved in Sections 4 and 5, respectively.

2. Syntax and semantics

In this section, we present the syntax and semantics of LL, which, except the deduction system, are taken mainly from [3].

2.1. Formulas and sequents

The formula definition is common, and sequents are of multiset type (see, e.g. [1]).

2.2. Semantics

LL is a modal logic. As usual for modal logics, the semantics of LL is given by means of Kripke models.

Let Φ denote the set of all propositional variables. An LL model M is a quadruple $\langle S, \mathcal{L}, \mathcal{G}, \pi \rangle$ with the following properties:

$S \neq \emptyset$ is a set of states, $\mathcal{L} \subseteq S \times S$ is a binary reflexive relation, $\mathcal{G} \subseteq S \times S$ is a binary relation, π is a function such that, for all $s \in S$ and $P \in \Phi$, $\pi(s, P) \in \{\mathbf{true}, \mathbf{false}\}$, i.e., each propositional variable is set to **true** or **false** in each state.

Let $\langle S, \mathcal{L}, \mathcal{G}, \pi \rangle$ be a model. We say that $t \in S$ is reachable from $s \in S$ if and only if there is a finite chain $s_1, \ldots, s_n$ such that 1) $s_1 = s$, 2) $s_n = t$, and 3) $(s_i, s_{i+1}) \in \mathcal{L} \cup \mathcal{G}$, $1 \leqslant i \leqslant (n-1)$. If $(s_i, s_{i+1}) \in \mathcal{L}$, for all $1 \leqslant i \leqslant (n-1)$, then we say that t is $\mathcal{L}$-reachable from s.

Note that the sets $A_s = \{t\colon t$ is $\mathcal{L}$-reachable from $s\}$ and $B_s = \{t\colon t$ is reachable from $s\}$ are never empty, since $s \in A_s$ and $s \in B_s$ by the reflexivity of $\mathcal{L}$.

Now we define the binary relation $\models$ between pairs (M, s) and LL formulas, where $s \in S$, and S is the set of states in the model M:

$(M, s) \models P$, where $P \in \Phi$ if and only if $\pi(s, P) = \mathbf{true}$;

$(M, s) \models \mathbf{L}^* A$ if and only if there exists $t \in S$ such that t is $\mathcal{L}$-reachable from s and $(M, t) \models A$;

For the other cases, see [1].

A more comprehensive discussion of the semantics of LL can be found in [3] and [4].

We extend the relation $\models$ for sequents: $(M, s) \models A_1, \ldots, A_n \to B_1, \ldots, B_m$ if and only if there is i such that $(M, s) \not\models A_i$ or $(M, s) \models B_i$.

Note that $(M, s) \models A_1, \ldots, A_n \to B_1, \ldots, B_m$ implies that $(M, s) \models (A_1 \wedge \ldots \wedge A_n) \supset (B_1 \vee \ldots \vee B_m)$, and vice versa. In particular, $(M, s) \models \to B$ if and only if $(M, s) \models B$.

If, for all possible pairs (M, s), $(M, s) \models \lambda$, where λ is a formula or a sequent, then we write $\models \lambda$ and call λ valid.

2.3. Deduction system LLK

Calculus LLK is obtained from a variant of the classical propositional Gentzen-type sequent calculus LK_0 by adding rules for modal operators.

See [1] for the calculus LK_0 (axiom formulas are atomic).

Rules for modal operators $\mathbf{L}$, $\mathbf{L}^*$, and $\mathbf{G}$:

$$\frac{\mathbf{G}\Gamma, \Theta \to \Delta, \mathbf{L}^*\Lambda}{\Pi, \mathbf{G}\Gamma, \mathbf{L}\Theta \to \mathbf{L}\Delta, \mathbf{L}^*\Lambda, \Sigma}\ (\mathbf{L} \to), \qquad \frac{\Gamma \to A, \mathbf{L}A, \Delta}{\Gamma \to \mathbf{L}A, \Delta}\ (\to \mathbf{L}),$$

$$\frac{\mathbf{G}\Gamma, A \to \mathbf{L}^*\Delta}{\Pi, \mathbf{G}\Gamma, \mathbf{L}^*A \to \mathbf{L}^*\Delta, \Lambda}\ (\mathbf{L}^* \to), \qquad \frac{\Gamma \to A, \mathbf{L}^*A, \Delta}{\Gamma \to \mathbf{L}^*A, \Delta}\ (\to \mathbf{L}^*),$$

$$\frac{A, \mathbf{G}A, \Gamma \to \Delta}{\mathbf{G}A, \Gamma \to \Delta}\ (\mathbf{G} \to), \qquad \frac{\mathbf{G}\Gamma \to A}{\Pi, \mathbf{G}\Gamma \to \mathbf{G}A, \Delta}\ (\to \mathbf{G}).$$

Here: A denotes an arbitrary formula; Ω, Γ, Π, Δ, Λ, and Σ denote finite, possibly empty, multisets of formulas; if an arbitrary multiset $\Gamma = A_1, A_2, \ldots, A_n$, then, as usual, $\sigma\Gamma = \sigma A_1, \sigma A_2, \ldots, \sigma A_n$, where $\sigma \in \{\mathbf{L}, \mathbf{L}^*, \mathbf{G}\}$; $\Theta \in \{A, \emptyset\}$.

The definition of derivation is common (see, e.g., [5]). We usually denote a derivation and the height of the derivation by V and $h(V)$, respectively. $h(V)$ is equal to the height of the longest branch in V, and the height of a branch is measured by the number of rule applications in it.

We say that a formula B is derivable in a sequent calculus if and only if the sequent $\to B$ is derivable in it.

3. Some properties of *LLK*

3.1. Admissibility of weakening and rule invertibility

We say that a rule

$$\frac{\Gamma \to \Delta}{\Gamma' \to \Delta'}\ (r)$$

is admissible in LLK if and only if $LLK \vdash^{V} \Gamma \to \Delta$ implies $LLK \vdash^{V'} \Gamma' \to \Delta'$. The rule is strongly admissible if and only if $LLK \vdash^{V} \Gamma \to \Delta$ implies $LLK \vdash^{V'} \Gamma' \to \Delta'$ and $h(V') \leqslant h(V)$.

LEMMA 3.1. *The rule of weakening*

$$\frac{\Gamma \to \Delta}{\Gamma', \Gamma \to \Delta, \Delta'}\ (W)$$

is strongly admissible in LLK. Here Γ' and Δ' are arbitrary multisets.

The lemma is proved by induction on $h(V)$, where $h(V)$ denotes the height of derivation of the weakening premise.

LEMMA 3.2. *Any sequent of the shape $\Gamma, D \to D, \Delta$, where D is an arbitrary formula, is derivable in LLK.*

The lemma is proved by induction on $\mathcal{G}(D)$, where $\mathcal{G}(D)$ denotes the number of logical connectives and modal operators in D.

Having this lemma in mind, we reckon that a derivation is obtained if all the derivation tree leaves are of the shape $\Gamma, D \to D, \Delta$, where D is not necessarily atomic. We 'remember' that all the axioms (except for $\mathcal{F}, \Gamma \to \Delta$) must be of the shape $\Gamma, E \to \Delta, E$ (with E atomic) only when it is convenient for our investigation.

We say that a rule

$$\frac{\Gamma_1 \to \Delta_1; \ldots; \Gamma_m \to \Delta_m}{\Gamma \to \Delta}$$

($m \geqslant 1$) of LLK is invertible if and only if

$$(LLK \vdash^{V} \Gamma \to \Delta) \Rightarrow (LLK \vdash^{V_i} \Gamma_i \to \Delta_i),\quad 1 \leqslant i \leqslant m.$$

The rule is called strongly invertible if and only if

$$(LLK \vdash^{V} \Gamma \to \Delta) \Rightarrow \big(LLK \vdash^{V_i} \Gamma_i \to \Delta_i \text{ and } h(V_i) \leqslant h(V)\big),\ 1 \leqslant i \leqslant m.$$

LEMMA 3.3. *All LLK rules, except $(\boldsymbol{L} \to)$, $(\boldsymbol{L}^* \to)$, and $(\to \boldsymbol{G})$, are strongly invertible.*

Proof. The lemma is proved by induction on $h(V)$, where V is a derivation of the conclusion of a considered rule.

3.2. Admissibility of contraction

LEMMA 3.4. *The rules of contraction*

$$\frac{C, C, \Gamma \to \Delta}{C, \Gamma \to \Delta}\,(C \to) \quad \text{and} \quad \frac{\Gamma \to \Delta, C, C}{\Gamma \to \Delta, C}\,(\to C)$$

are strongly admissible in LLK. Here C is an arbitrary formula.

Proof. The lemma is proved by induction on $h(V)$, where $h(V)$ denotes the height of derivation of the contraction premise.

3.3. Admissibility of cut

THEOREM 3.5. *The rule of cut*

$$\frac{V_1\left\{\dfrac{}{\Pi \to C, \Lambda}\,(i);\quad V_2\left\{\dfrac{}{C, \Gamma \to \Delta}\,(k)\right.\right.}{\Pi, \Gamma \to \Lambda, \Delta}\,(cut).$$

is admissible in LLK.

Proof. The theorem is proved by induction on the ordered pair $\langle \mathcal{G}(C), H\rangle$, where $H = h(V_1) + h(V_2)$, using Lemmas 3.1 and 3.4.

4. Correctness of *LLK*

Calculus LLK is correct with respect to the given semantics:

THEOREM 4.1 (correctness).

$$(LLK \vdash^{V} \Gamma \to \Delta) \Rightarrow (\models \Gamma \to \Delta).$$

Proof. The theorem is proved by induction on $h(V)$. We write $(M, s) \models \Gamma = A_1, \ldots, A_n$ if and only if $(M, s) \models A_i$, $1 \leqslant i \leqslant n$.

If $h(V) = 0$, then $\Gamma \to \Delta$ is an axiom, and it is easy to see that it is valid.

$h(V) > 0$:

$$\frac{}{\Gamma \to \Delta}\,(i).$$

We consider only case when $(i) = (\mathbf{L}^* \to)$. For the other cases, we refer to [1].

$$\frac{\mathbf{G}\Gamma, A \to \mathbf{L}^*\Delta}{S = \Pi, \mathbf{G}\Gamma, \mathbf{L}^*A \to \mathbf{L}^*\Delta, \Lambda}\ (\mathbf{L}^* \to).$$

Let $(M, s) \models \Pi, \mathbf{G}\Gamma, \mathbf{L}^*A$ (otherwise, $(M, s) \models S$ by definition; for simplicity, we further write $s \models$ instead of $(M, s) \models$). Then there is $t \in S$ such that t is $\mathcal{L}$-reachable from s, and $t \models A$. $t \models \mathbf{G}\Gamma$ since $s \models \mathbf{G}\Pi$ and t is $\mathcal{L}$-reachable from s. By the inductive hypothesis, there is $\mathbf{L}^*D \in \mathbf{L}^*\Delta$ such that $t \models \mathbf{L}^*D$. But then $s \models \mathbf{L}^*D$, as well.

5. Completeness of *LLK*

Further we will need the following lemma:

LEMMA 5.1. *If $LLK \vdash \boldsymbol{L}A \to A$, then $LLK \vdash^V \Gamma, \boldsymbol{L}A \to \Delta$ implies that $LLK \vdash \Gamma, \boldsymbol{L}^*A \to \Delta$.*

Proof. We prove the lemma by induction on the ordered pair $\langle n, h(V)\rangle$, where n stands for the number of '**L**' occurrences in Γ and Δ.

First note that if $LLK \vdash \mathbf{L}A \to A$, then the rule

$$\frac{\Pi, A \to \Lambda}{\Pi, \mathbf{L}A \to \Lambda}\ (R)$$

is admissible in LLK. Indeed,

$$\frac{\mathbf{L}A \to A; \Pi, A \to \Lambda}{\Pi, \mathbf{L}A \to \Lambda}\ (cut)$$

and, by Lemma 3.5, cut is admissible in LLK.

Base case: $n = 0$ and $h(V) = 0$. This case is obvious.

Inductive case 1: $n = 0$ and $h(V) > 0$. All the cases are easy, but we chose to consider the following one:

$$\frac{\mathbf{G}\Gamma, A \to \mathbf{L}^*\Lambda}{\Pi, \mathbf{G}\Gamma, \mathbf{L}A \to \mathbf{L}^*\Lambda, \Sigma}\ (\mathbf{L} \to).$$

It remains to apply $(\mathbf{L}^* \to)$ to the premise.

Inductive case 2: $n > 1$ and $h(V) > 0$. Again, we consider only case, this time a bit more complex:

$$\frac{\mathbf{G}\Gamma, A \to \Delta, \mathbf{L}^*\Lambda}{\Pi, \mathbf{G}\Gamma, \mathbf{L}A \to \mathbf{L}\Delta, \mathbf{L}^*\Lambda, \Sigma}\ (\mathbf{L} \to).$$

If $\Delta = \emptyset$, then we argue as in Induction case 1. Let $\Delta \neq \emptyset$. Apply (R) to the premise and get $\mathbf{G}\Gamma, \mathbf{L}A \to \Delta, \mathbf{L}^*\Lambda$; apply the inductive hypothesis on n to this sequent and get $\mathbf{G}\Gamma, \mathbf{L}^*A \to \Delta, \mathbf{L}^*\Lambda$; apply the rule of weakening and Lemma 3.1 to this sequent and get $\Pi, \mathbf{G}\Gamma, \mathbf{L}^*A \to \mathbf{L}\Delta, \Delta, \mathbf{L}^*\Lambda, \Sigma$; apply $(\to \mathbf{L})$ to this sequent, and get $\Pi, \mathbf{G}\Gamma, \mathbf{L}^*A \to \mathbf{L}\Delta, \mathbf{L}^*\Lambda, \Sigma$.

THEOREM 5.2 (completeness).

$$(\models \Gamma \to \Delta) \Rightarrow (LLK \vdash \Gamma \to \Delta).$$

Proof. By [3], the following Hilbert type calculus is complete with respect to the given semantics:

Axioms:
propositional logic tautologies; $\mathbf{G}A \supset A$; $\mathbf{G}A \supset \mathbf{GG}A$; $\mathbf{G}A \supset \neg\mathbf{L}\neg A$;
$A \supset \mathbf{L}A$; $\big(\mathbf{L}(A \vee B) \supset (\mathbf{L}A \vee \mathbf{L}B)\big) \wedge \big((\mathbf{L}A \vee \mathbf{L}B) \supset \mathbf{L}(A \vee B)\big)$;
$\mathbf{G}(A \supset B) \supset (\mathbf{G}A \supset \mathbf{G}B)$; $\mathbf{G}(A \supset B) \supset (\mathbf{L}A \supset \mathbf{L}B)$;
$\mathbf{G}(A \supset B) \supset (\mathbf{L}^*A \supset \mathbf{L}^*B)$; $\mathbf{L}^*A \equiv (A \vee \mathbf{LL}^*A)$.

Inference rules:

$$\frac{A}{\mathbf{G}A}\ \text{(generalization)};\quad \frac{A,\ A \supset B}{B}\ \text{(modus ponens)};\quad \frac{\neg A \supset \neg\mathbf{L}A}{\neg A \supset \neg\mathbf{L}^*A}.$$

All these axioms are derivable and inference rules admissible in LLK.

LLK is obtained from LK_0 by adding some rules. It is well known that LK_0 is complete with respect to propositional logic tautologies. Thus, all the propositional logic tautologies are derivable in it and in LLK as well. Axiom $\mathbf{G}A \supset \neg\mathbf{L}\neg A$ in our notation is the following: $\mathbf{G}A \supset (\mathbf{L}(A \supset \mathcal{F}) \supset \mathcal{F})$. It is easy to see that it is derivable in LLK.

The modus ponens rule is admissible in LLK. To prove it, we use invertibility of the rule $(\to\supset)$ and Lemma 3.5. It is easy to see that the generalization rule is admissible in LLK as well.

Let a formula $\neg A \supset \neg\mathbf{L}A$ be derivable in LLK. Then the sequent $\mathbf{L}A \to A$ is also derivable in LLK due to the invertibility of the rules $(\to\supset)$ and $(\supset\to)$. Then it follows from Lemma 5.1 that the sequent $\mathbf{L}^*A \to A$ is derivable in LLK, as well (take $\Gamma = \emptyset$ and $\Delta = A$). Then the formula $(A \supset \mathcal{F}) \supset (\mathbf{L}^*A \supset \mathcal{F})$, which is the same as $\neg A \supset \neg\mathbf{L}^*A$, is derivable in LLK.

6. Concluding remarks

As we have seen, the structural and cut rules are admissible in the sequent calculus for the propositional likelihood logic. This enables us to have convenient deductive system for further investigation of the proof-theoretical properties of the logic.

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REZIUMĖ

R. Alonderis. Sekvencinis skaičiavimas propozicinei tikėtinumo logikai

Darbe yra pateikiamas klasikinis sekvencinis skaičiavimas propozicinei tikėtinumo logikai. Įrodoma, kad šiame skaičiavime yra leistinos struktūrinės bei pjūvio taisyklės. Taip pat įrodomi pateiktojo skaičiavimo korektiškumas bei pilnumas duotos semantikos atžvilgiu.