Liet. matem. rink, **46**, *spec. nr.*, 2006, 242–246

Specialization of derivations in modal logic $S5$

Aida PLIUŠKEVIČIENĖ (MII)
e-mail: aida@ktl.mii.lt

Abstract. Loop-check-free decidable specialization of sequent calculus for modal logic $S5$ is presented. Soundness and completness of this calculus is proved.

Keywords: modal logic, sequent calculus, decidable calculus, loop-check.

1. Introduction

Modal logics are widely used in artificial intelligence and computer science. Therefore one of the main objective is to design effective and simple to use decision procedures for these logics.

A proof that suitable logical calculus (e,g., sequent or tableaux calculus) for a modal logic allows us to get a decision procedure is crucial but it is not enough. Check of termination of a decision procedure is very important problem and require to keep an information on previous part of a derivation.

Traditional techniques used for test termination of a decision procedure in modal (e.g., knowledge-based) sequent (and tableau) calculi is based on *loop-check* [4]. Namely, before applying any rule it is checked if this rule was already applied to "essentially the same" sequent; if this is the case we block the application of the rule. However, loop-check method often leads to an inefficient implementation (see,e.g., [2]). Therefore loop-check is often considered as useless. In [5], [10] efficient loop-check for modal logics KT, $K4$, $S4$, tense logic K_t, and a fragment of intuitionistic logic was presented using sequents extended by notion of history. For modal logic T loop-check-free sequent calculus is presented in [5] using sequents with two halves. In [3], [6] a contraction-free calculus for propositional intuitionistic logic was proposed. A contraction-free calculus entirely excludes loop-check in derivations. In [7], a contraction-free calculus for $S4$ was constructed, however only for sequents in a certain normal form. Alternative approach [1] is to translate a modal logic into a more simple logic. Interesting approach is proposed in [9] allowing us to decrease a complexity of loop-check for various modal logics. A decision procedure for a fragment of mutual belief logic with quantified agent variables (with loop-check only for sequents derivations of which require induction-like axiom) is proposed in [12].

In this paper the modal logic $S5$ is considered. This modal logic is considered as the logic of idealized knowledge. The aim of this paper is to get a specialization of derivations for the modal logic $S5$ that allows us to present loop-check-free decision procedure. This procedure is carried out by means of invertible loop-check-free sequent calculus.

2. Kanger-style sequent calculus $KS5$

To get cut-free sequent calculus for $S5$ S.Kanger proposed to use indexed propositional symbols along with usual propositional ones [8]. Let P be a propositional symbol. Then P^k, where k is a natural number, is an indexed propositional symbol. Below we present a slightly different from Kanger procedure of indexation of formula of $S5$:

1. $(P^k)^l = P^{k+l}$, where $k,\ l$ are natural numbers; it is assumed that $P^0 = P$.
2. $(A \odot B)^l = A^l \odot B^l$, where $\odot \in \{\supset, \wedge, \vee\}$.
3. $(\neg A)^l = \neg A^l$.
4. $(\Box A)^l = \Box A$.

A sequent is a formal expression $\Gamma \to \Delta$, where $\Gamma,\ \Delta$ are multisets of formulas.

Let $KS5$ be a calculus obtained from Kanger-style logical calculus [8] adding the following modal rules:

$$\frac{\Gamma \to \Delta, A^i}{\Gamma \to \Delta, \Box A}\ (\to \Box), \qquad \frac{A^k, \Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to).$$

where i is a natural number not entering in the conclusion of the rule $(\to \Box)$; $k \in \{0, 1, 2, \ldots\}$.

In [11] the following theorems are proved:

THEOREM 1. *The calculus $KS5$ is a conservative extension of a traditional Hilbert-style calculus $HS5$, i.e., the calculus $KS5$ is sound and complete.*

THEOREM 2. *The structural rules of weakening and cut are admissible in $KS5$.*

From the admissibility of the weakening rules it follows the invertibility of the rule $(\Box \to)$.

A derivation in a calculus I is called an atomic one if the main formula of an axiom is a propositional symbol. It is obvious that backward applying rules of $KS5$ each derivation in $KS5$ can be reduced to an atomic one with the same end-sequent.

LEMMA 1. *Let S (S_1) be a conclusion (premise, correspondingly) of a rule (i) where (i) is a logical rule. Let $KS5 \vdash^V S$ where V is an atomic derivation of S in $KS5$ and $h(V)$ is a height of this derivation. Then $KS5 \vdash^{V^*} S_1$ and $h(V^*) \leqslant h(V)$.*

Proof. By induction on $h(V)$.

A backward proof search in the calculus $KS5$ is not terminative, in general. Indeed, let S be a sequent $\Box(P \vee Q) \to P$. Then the backward proof search contains an infinitive branch because we repeatedly get almost the same sequents $S_m = \overbrace{Q, \ldots, Q}^{m\ times}, \Box(P \vee Q) \to P, m \in \{1, 2, \ldots\}$.

To prune the infinite branch the method of loop-check [4] is used. Since the sequents S_1 and S_2 are almost the same we block applications of the rules $(\Box \to)$ and $(\vee \to)$ and conclude that $KS5 \nvdash S$.

3. Loop-check-free sequent calculus K_1S5

To avoid loop-check in $KS5$ we shall show that applications of the rule ($\Box\rightarrow$) can be restricted in such way that it is not possible to apply this rule repeatedly using the same occurence of formula as the main formula of the rule.

Along with usual modality $\Box$ let us introduce a marked modality $\Box^*$ which has the same semantical meaning as non-marked modality $\Box$ and serves as a stopping device for a backward application of the rule ($\Box\rightarrow$).

Let K_1S5 be a calculus obtained from the initial calculus $KS5$ replacing the rule ($\Box\rightarrow$) by the following one:

$$\frac{A^k, \Box^* A, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta}(\Box^* \rightarrow),$$

where in the conclusion of the rule the modality $\Box$ in the main formula $\Box A$ is not marked (the restriction on main formula).

It is obvious that if $K_1S5 \vdash S$ then $KS5 \vdash S$ $(*)$.

To prove the inverse implication let us introduce some auxiliary calculi.

Let $K_1^C S5$ be a calculus obtained from the calculus K_1S5 adding the following structural rule of contraction:

$$\frac{\Box A, \Box^\sigma A, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta}\ (C_\Box \rightarrow),$$

where $\sigma \in \{\varnothing, *\}$.

Let $K_1^{dC} S5$ be a calculus obtained from the calculus $K_1^C S5$ adding the rule ($\Box^{*d}\rightarrow$) which is obtained from the rule ($\Box^*\rightarrow$) by dropping the restriction on main formula.

It is obvious that if $KS5 \vdash S$ then $K_1^{dC} S5 \vdash S$ $(**)$.

Let us prove the following

LEMMA 2. *If* $K_1^{dC} S5 \vdash^V S$ *then* $K_1^C S5 \vdash S$, *where* S *does not contain marked modality.*

Proof. The proof is carried out using induction on the number of applications of the rule ($\Box^{*d}\rightarrow$) in V denoted by $n(V)$. If $n(V) = 0$ then V^* coincides with V.

Let $n(V) > 0$. Then let us consider the lowest application of the rule ($\Box^{*d}\rightarrow$) in V. Let $\Box^* A$ be the main formula of this lowest application of the rule ($\Box^{*d}\rightarrow$). Let $S_1 = A^k, \Box^* A, \Gamma \rightarrow \Delta$ be the premise of this lowest application of the rule ($\Box^{*d}\rightarrow$). Since the end-sequent S of V does not contain marked modality $\Box^*$, below this lowest application of ($\Box^{*d}\rightarrow$) must be an application of the rule ($\Box^*\rightarrow$) with the main formula $\Box A$. Let us replace this application of ($\Box^*\rightarrow$) with the applications of the rules ($C_\Box\rightarrow$) and ($\Box^*\rightarrow$) (with the main formula $\Box A$). So, instead of the lowest application of the rule ($\Box^{*d}\rightarrow$) we get the application of the rule ($\Box^*\rightarrow$) with the main formula $\Box A$ and with the premise $S_1' = A^k, \Box A, \Box^* A, \Gamma \rightarrow \Delta$. A derivation of the sequent S_1' can be get from the derivation of the sequent S_1 using admissibility of weakening. Thus, instead of the derivation V we get a derivation V' such that $n(V') < n(V)$. Therefore, by induction assumption we can get $K_1^C S5 \vdash S$.

LEMMA 3. *If* $K_1S5 \vdash^V \Box A, \Box^\sigma A, \Gamma \to \Delta$, *where* $\sigma \in \{\varnothing, *\}$, *then* $K_1S5 \vdash \Box A, \Gamma \to \Delta$.

Proof. The proof is carried out using induction on $h(V)$.

LEMMA 4. *If* $KS5 \vdash S$ *then* $K_1S5 \vdash S$.

Proof. The proof follows from Lemmas 2, 3, and (∗∗).

LEMMA 5. $KS5 \vdash S$ *if and only if* $K_1S5 \vdash S$

Proof. The proof follows from Lemma 4, and (∗).

It is easy to see that the restriction on main formula in the rule $(\Box^* \to)$ destroys invertibility of the rule $(\Box^* \to)$. Indeed, it is obvious that $K_1S5 \vdash \Box P \to \Box(P \vee Q)$ but $K_1S5 \nvdash P^k, \Box^* P \to \Box(P \vee Q)$. To save the invertibility of $(\Box^* \to)$ let us introduce a notion of primary sequent. A sequent S is a *primary* one if $S = \Sigma_1, \Box\Gamma \to \Sigma_2$ and Σ_i $(i \in \{1, 2\})$ is empty or consists of propositional symbols, $\Box\Gamma$ is empty or consists of the formulas of the shape $\Box A$.

LEMMA 6 (reduction to primary sequents). *It is possible automatically construct a reduction of a sequent* S *to a set* $\{S_1, \ldots, S_m\}$, *where* S_j $(1 \leqslant j \leqslant m)$ *is a primary sequent. Moreover, if* $K_1S5 \vdash^V S$, *where* V *is an atomic derivation, then* $K_1S5 \vdash^{V_j} S_j$ $(j \in \{1, \ldots, m\})$ *and* $h(V_j) \leqslant h(V)$.

Proof. Follows from invertibility of logical rules and the rule $(\to \Box)$.

Let K_2S5 be a calculus obtained from the calculus K_1S5 replacing the rule $(\Box^* \to)$ by the following one:

$$\frac{\Sigma_1, A^k, \Box^* A, \Box\Gamma \to \Sigma_2}{\Sigma_1, \Box A, \Box\Gamma \to \Sigma_2}\ (\Box_p^* \to),$$

where Σ_i $(i \in \{1, 2\})$ is empty or consists of propositional symbols, moreover, $\Sigma_1 \cap \Sigma_2$ is empty; $\Box\Gamma$ is empty or consists of the formulas of the shape $\Box^\sigma B$, $\sigma \in \{\varnothing, *\}$. In the conclusion of this rule the modality $\Box$ in the main formula $\Box A$ is not marked (the restriction on main formula).

LEMMA 7. *Let* $K_2S5 \vdash^V \Sigma_1, \Box A, \Box\Gamma \to \Sigma_2$, *where* V *is an atomic derivation. Then* $K_2S5 \vdash^{V^*} \Sigma_1, A^k, \Box^* A, \Box\Gamma \to \Sigma_2$, *i.e. the rule* $(\Box_p^* \to)$ *is invertible.*

Proof. The proof is carried out by induction on $h(V)$ and using Lemma 6.

It is obvious that if $K_2S5 \vdash S$ then $K_1S5 \vdash S$. Using Lemma 6 we get that if $K_1S5 \vdash S$ then $K_2S5 \vdash S$. Thus, $K_1S5 \vdash S$ if and only if $K_2S5 \vdash S$. From this fact and relying on Lemma 5 and Theorem 1 we get

THEOREM 3. *The calculus K_2S5 is sound and complete.*

A primary sequent S of the shape $\Sigma_1, \Box^*\Gamma \rightarrow \Sigma_2$ is a *critical* one if $\Sigma_1 \cap \Sigma_2$ is empty. A derivation V of a sequent S in K_2S5 is *successful* if each branch of V ends with an axiom. In this case $K_2S5 \vdash S$. A derivation V of a sequent S in K_2S5 is *unsuccessful* if V contains a branch ending with a critical sequent. In this case $K_2S5 \nvdash S$.

From invertibility of the rules of K_2S5 and shape of these rules we get

THEOREM 4. *For a sequent S one can automatically construct a successful or unsuccessful derivation of the sequent S in K_2S5, i.e., K_2S5 is a loop-check-free decidable calculus.*

References

1. S. Cerrito, M. Cialdea Mayer, A polynomial translation of $S4$ into T and contraction-free tableaux for $S4$, *Journal of the Interest Gropup in Pure and Applied Logic*, **5**(2), 287–300 (1997).
2. S. Demri, Uniform and non uniform strategies for tableaux calculi for modal logics, *Journal of Applied Non-Classical Logics*, **5**(1), 77–96 (1995).
3. R. Dyckhoff, Contraction-free sequent calculi for intuitionistic logic, *Journal of Symbolic Logic*, **57**, 795–807 (1992).
4. M. Fitting, Proof methods for modal and intuitionistic logics, in: *Synthese Library*, vol. 169, D.Reidel, Dordrecht, Holland (1983).
5. A. Heuerding, M. Seyfried, H. Zimmermann, Efficient loop-check for backward proof search in some non-classical propositional logics, *Lecture Notes in Computer Science*, **1071**, 210–225 (1996).
6. J. Hudelmaier, Bounds for cut-elimination in intuitionistic propositional logic, *Arch. Math. Logic*, **31**, 331–353 (1992).
7. J. Hudelmaier, A contraction-free sequent calculus for $S4$, in: H. Wansing (Ed.), *Proof Theory for Modal Logic*, Kluwer Academic Publishers, Dordrechts, Boston/London (1996), pp. 3–16.
8. S. Kanger, *Provability in Logic*, Almgvist & Wiksell, Stockholm (1957).
9. F. Massacci, Single step tableaux for modal logics, *Journal of Automated Reasoning*, **24**(3), 319–364 (2000).
10. M. Mouri, Constracting counter-models for modal logic K4 from refutation trees, *Bulletin of the Section of Logic*, **31**(2), 81–90 (2002).
11. R. Pliuškevičius, On the conservative extension of quantifier modal logic $S5$, *Mathematical Logic and its Applications*, **4**, 25–38 (1985).
12. R. Pliuškevičius, A. Pliuškevičienė, Decision procedure for a fragment of mutual belief logic with quantified agent variables, *Lecture Notes in Artificial Intelligence*, **3900**, 112–128 (2006).

REZIUMĖ

A. Pliuškevičienė. Įrodymų specializacija modalumo logikai $S5$

Pateikta korektiška ir pilna specializacija sekvenciniam skaičiavimui modalumo logikai $S5$. Pasiūlytas specializuotas skaičiavimas įgalina gauti išsprendžiamąją procedūrą, kurios realizacijoje nėra ciklų.