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# Logic of knowledge with infinitely many agents

Regimantas PLIUŠKEVIČIUS (MII)
e-mail: regis@ktl.mii.lt

**Abstract.** Cut-free sequent calculus for logic of knowledge with infinitely many agents, based on multi-modul $S5_n$.

*Keywords:* modal logic, logic of knowledge, sequent calculus.

## 1. Introduction

Reasoning about knowledge has been shown to be widely applicable in computer science and artificial intelligence (see, e.g., [1], [3]). Complete calculi for logics of knowledge are well known in the case of finite set of agents (see, e.g., [1]). However, in many applications the set of agents is not known in advance. As it is indicated in [4] it is often easiest to model the set of agents as an infinite set.

In [4] complete Hilbert-style calculus for common knowledge logic with infinitely many agents is presented. This logic is obtained from logic of knowledge $KS5_n$, i.e., from multi-modal logic $S5_n$ with arbitrary n, by adding common knowledge operator (restricted in some way).

It is well-known that Hilbert-style calculi allow us to reflect semantics of considered logic. However, for automatization of reasoning Gentzen-style (sequent) cut-free calculi are more appropriate. It is desirable that the rules of a considered calculus would be invertible. This property allows us to preserve derivability in a backward proof search. The cut-free calculus for the logic $S5$ had been constructed in the various papers (an exhaustive exposition of these results is presented in [8]).

The aim of this paper is to construct sequent calculus for the subset of logic of knowledge considered in [4]. This subset does not contain common knowledge operator, i.e., is the logic $KS5_n$. The constructed sequent calculus instead of cut rule contains an effective analytic cut rule which allows to construct the premise of the rule from its conclusion automatically. All rules of constructed sequent calculus are invertible.

## 2. Ohnishi–Matsumoto-style sequent calculus for $KS5_n$

The *language* of $KS5_n$ contains: (1) a set of propositional symbols $P$, $P_1, \ldots$, $Q$, $Q_1$, $\ldots$; (2) a set of agent constants $i, i_1, i_2, \ldots$ $(i, i_j \in \{1, 2, \ldots\})$; (3) a set of knowledge modalities of the shape $\mathbf{K}(i)$, where $i$ is an agent constant; (4) logical symbols: $\supset$, $\wedge$, $\vee$, $\neg$.

Modalities $\mathbf{K}(i)$ satisfy equivalence relation.

Formula of $KS5_n$ is defined in a traditional way.

The formula $\mathbf{K}(i)A$ means: "agent $i$ knows that $A$". Along with formulas we consider sequents, i.e., formal expressions $\Gamma \to \Delta$ where $\Gamma,\ \Delta$ are multisets of formulas.

Let us introduce Ohnishi–Matsumoto-style sequent calculus $GS5_n$. The calculus $GS5_n$ is obtained from Kanger-style calculus for propositional logic [5] adding (Cut) rule and the following rules for knowledge modalities:

$$\frac{A,\ \mathbf{K}(i)A, \Gamma \to \Delta}{\mathbf{K}(i)A, \Gamma \to \Delta}\ (\mathbf{K}_i \to), \qquad \frac{\mathbf{K}(i)\Gamma_1 \to \mathbf{K}(i)\Delta_1, A}{\mathbf{K}(i)\Gamma_1, \Gamma \to \Delta,\ \mathbf{K}(i)\Delta_1,\ \mathbf{K}(i)A}\ (\to \mathbf{K}_i),$$

where $\mathbf{K}(i)\Gamma_1$ and $\mathbf{K}(i)\Delta_1$ are empty or consists of formulas of the shape $\mathbf{K}(i)B$,

The calculus $GS5_n$ with $n = 1$ was introduced and founded in [6], [7]. As in [6], [7] we can prove soundness and completeness of $GS5_n$ with any $n$.

Without the rule (Cut) the calculus $GS5_n$ is incomplete. Indeed, let $GS5'_n$ be a calculus obtained from $GS5_n$ by dropping (Cut). Let $S$ be a sequent $P \to \mathbf{K}(1)\neg\mathbf{K}(1)\neg P$. It is easy to verify that $GS5_n \vdash S$ using the formula $\neg\mathbf{K}(1)\neg P$ as the cut formula, but $GS5'_n \nvdash S$.

## 3. Cut-free sequent calculus for $KS5_n$

In this section we present a cut-free sequent calculus $G_1S5_n$ for $KS5_n$. Instead of the (Cut) rule the calculus $G_1S5_n$ contains an analytic-cut-style rule which destroys subformula property.

The calculus $G_1S5_n$ is obtained from the calculus $GS5_n$ by dropping (Cut) and replacing the rule $(\to \mathbf{K}_i)$ by the following rule:

$$\frac{\Gamma_{1i},\ \mathbf{K}(i)\Gamma_{1i} \to \mathbf{K}(i)B,\ \mathbf{K}(i)\Gamma_{2i}, A}{\Sigma_1,\ \mathcal{K}\Gamma_1 \to \Sigma_2,\ \mathcal{K}\Gamma_2,\ \mathbf{K}(i)A}\ (\to \mathbf{K}_i^C),$$

where $\Sigma_j$ $(j \in \{1,2\})$ is empty or consists of propositional symbols and $\Sigma_1 \cap \Sigma_2$ is empty;

$\mathcal{K}\Gamma_j$ $(j \in \{1,2\})$ is empty or consists of formulas of the shape $\mathbf{K}(l)A$;

$\mathbf{K}(i)\Gamma_{j1}$ $(j \in \{1,2\})$ is empty or consists of formulas of the shape $\mathbf{K}(i)B$, $B = \neg\Sigma_1^{\vee} \vee \neg\mathbf{K}(l)\Gamma_{1l}^{\vee} \vee \Sigma_2^{\vee} \vee \mathbf{K}(l)\Gamma_{2l}^{\vee}$, where $l \neq i$; $\mathbf{K}(l)\Gamma_{jl}^{\vee}$ $(j \in \{1,2\})$ is obtained from $\mathcal{K}\Gamma_j$ deleting all formulas of the shape $\mathbf{K}(i)B_{ji}$; here and below $\rho\nabla^{\vee} = \bigvee_{i=1}^{m} \rho A_i$ where $\rho \in \{\varnothing, \neg\}$ and $\nabla = A_1, \ldots, A_m$.

Thought the rule $(\to \mathbf{K}_i^C)$ destroys a subformula property the premise of this rule is constructed automatically from the conclusion and depends on the choice of the main formula of this rule.

EXAMPLE 1. (a) *Let $S$ be a sequent $P \to \mathbf{K}(1)\neg\mathbf{K}(1)\neg(P \vee Q)$. Then bottom-up applying $(\to \mathbf{K}_i^C)$ to $S$ we get $S_1 = \to \mathbf{K}(1)\neg P, \neg\mathbf{K}(1)\neg(P \vee Q)$. Bottom-up applying $(\to \neg)$ to $S_1$ we get $S_2 = \mathbf{K}(1)\neg(P \vee Q) \to \mathbf{K}(1)\neg P$. Bottom-up applying $(\to \mathbf{K}_i^C)$, $(\to \neg)$, $(\neg \to)$, $(\to \vee)$ from $S_2$ we get an axiom. Hence $G_1S5_n \vdash S$.*

(b) *Let $S$ be a sequent $\to \mathbf{K}(2)P,\ \mathbf{K}(1)\neg\mathbf{K}(1)\neg(\neg\mathbf{K}(2)P \vee Q)$. We have two possibilities.*

(1) *Let us choose* $\mathbf{K}(1)\neg\mathbf{K}(1)\neg(\neg\mathbf{K}(2)P \vee Q)$ *as the main formula of the application of the rule* $(\rightarrow \mathbf{K}_i^C)$. *Then bottom-up applying* $(\rightarrow \mathbf{K}_i^C)$, $(\rightarrow \neg)$ *from* $S$ *we get* $S_1 = \mathbf{K}(1)\neg(\neg\mathbf{K}(2)P \vee Q) \rightarrow \mathbf{K}(1)\mathbf{K}(2)P$. *Bottom-up applying* $(\rightarrow \mathbf{K}_i^C)$, $(\neg\rightarrow)$, $(\rightarrow\vee)$ *from* $S_1$ *we get* $S_2 = \mathbf{K}(1)\neg(\neg\mathbf{K}(2)P \vee Q) \rightarrow \neg\mathbf{K}(2)P, Q, \mathbf{K}(2)P$. *Bottom-up applying* $(\rightarrow\neg)$ *to* $S_2$ *we get an axiom with the main formula* $\mathbf{K}(2)P$. *Therefore* $G_1S5_n \vdash S$.

(2) *Let us choose* $\mathbf{K}(2)P$ *as the main formula of the application of the rule* $(\rightarrow \mathbf{K}_i^C)$. *Then bottom-up applying* $(\rightarrow \mathbf{K}_i^C)$ *to* $S$ *we get* $S_1^* = \rightarrow \mathbf{K}(2)\mathbf{K}(1)\neg\mathbf{K}(1)\neg(\neg\mathbf{K}(2)P \vee Q), P$. *Bottom-up applying* $(\rightarrow \mathbf{K}_i^C)$ *to* $S_1^*$ *we get the initial sequent. Using method of loop-check [2] we return to S, block the bottom-up application of the rule* $(\rightarrow \mathbf{K}_i^C)$ *with the main formula* $\mathbf{K}(2)P$. *As in the case (1) we get* $G_1S5_n \vdash S$.

It is obvious that bottom-up applying rules of $G_1S5_n$ each derivation in $G_1S5_n$ can be reconstructed into an *atomic* one (i.e., the main formula of any axiom is a propositional symbol) with the same end-sequent.

LEMMA 1. *Let* $i$ *be a logical rule of* $G_\sigma S5_n$ $(\sigma \in \{\varnothing, 1\})$ *and* $G_\sigma S5_n \vdash^V S$ *where* $V$ *is an atomic derivation of* $S$ *and* $h(V)$ *is a height of this derivation. Then* $G_\sigma S5_n \vdash^{V^*} S^*$ *where* $S^*$ *is a premise of a rule* $i$, *moreover,* $h(V^*) < h(V)$.

*Proof.* By induction on $h(V)$.

Using induction on the height of a derivation we can prove admissibility of the structural rule of weakening in $G_\sigma S5_n$ $(\sigma \in \{\varnothing, 1\})$. From this fact we get invertibility of the rule $(\mathbf{K}_i \rightarrow)$. To get invertibility of the rule $(\rightarrow \mathbf{K}_i^C)$ in $G_1S5_n$ we shall prove invertibility of this rule in the calculus $GS5_n$. Having proved (in next section) that $GS5_n \vdash S$ only and if only $G_1S5_n \vdash S$ we get invertibility of $(\rightarrow \mathbf{K}_i^C)$ in $G_1S5_n$.

LEMMA 2. *The rule* $(\rightarrow \mathbf{K}_i^C)$ *is invertible in* $GS5_n$.

*Proof.* Let $S$ be a sequent $\Sigma_1, \mathcal{K}\Gamma_1, \rightarrow \Sigma_2, \mathcal{K}\Gamma_2, \mathbf{K}(i)A$ and $GS5_n \vdash S$. Applying logical rules $(\rightarrow\neg)$, $(\rightarrow\vee)$ to $S$ we get $S_1 = \mathbf{K}(i)\Gamma_{1i} \rightarrow B, \mathbf{K}(i)\Gamma_{2i}, \mathbf{K}(i)A$, where $\mathbf{K}(i)\Gamma_{j1}$ $(j \in \{1,2\})$ and $B$ are the same as in definition of the rule $(\rightarrow \mathbf{K}_i^C)$. Applying $(\rightarrow \mathbf{K}_i)$ to $S_1$ we get $S_2 = \mathbf{K}(i)\Gamma_{1i} \rightarrow \mathbf{K}(i)B, \mathbf{K}(i)\Gamma_{2i}, \mathbf{K}(i)A$. It is obvious that $GS5_n \vdash S_3 = \mathbf{K}(i)A \rightarrow A$, where $A$ is arbitrary formula. Applying (Cut) to $S_2$ and $S_3$ and using admissibility of weakening we get $G_1S5_n \vdash \Gamma_{1i}, \mathbf{K}(i)\Gamma_{1i} \rightarrow \mathbf{K}(i)B, \mathbf{K}(i)\Gamma_{2i}, A$, i.e., the premise of $(\rightarrow \mathbf{K}_i^C)$. Thus, $(\rightarrow \mathbf{K}_i^C)$ is invertible in $GS5_n$.

Let $G_1^C S5_n$ be a calculus obtained from the calculus $G_1S5_n$ by adding (Cut). Let us prove that $G_1^C S5_n \vdash S$ only and if only $GS5_n \vdash S$. First let us prove the following lemma.

LEMMA 3. *The rule* $(\rightarrow \mathbf{K}_i^C)$ *is admissible in* $GS5_n$.

*Proof.* Let $S^*$ be a sequent $\Gamma_{1i}, \mathbf{K}(i)\Gamma_{1i} \to \mathbf{K}(i)B, \mathbf{K}(i)\Gamma_{2i}, A$ and $GS5_n \vdash S^*$. Applying $(\mathbf{K}_i \to)$ to $S^*$ we get $S_1 = \mathbf{K}(i)\Gamma_{1i} \to \mathbf{K}(i)B, \mathbf{K}(i)\Gamma_{2i}, \mathbf{K}(i)A$, where $\mathbf{K}(i)\Gamma_{j1}$ $(j \in \{1,2\})$ and $B$ are the same as in definition of the rule $(\to \mathbf{K}_i^C)$. Since $GS5_n \vdash S_3 = \mathbf{K}(i)B \to B$, applying (Cut) to $S_2$ and $S_3$ we get $GS5_n \vdash \mathbf{K}(i)\Gamma_{1i} \to B, \mathbf{K}(i)\Gamma_{2i}, \mathbf{K}(i)A$. Using Lemma 1 (i.e., invertibility of logical rules) we get $GS5_n \vdash S = \Sigma_1, \mathcal{K}\Gamma_1, \to \Sigma_2, \mathcal{K}\Gamma_2, \mathbf{K}(i)A$, i. e., the rule $(\to \mathbf{K}_i^C)$ is admissible in $GS5_n$.

Using Lemma 3 we get

LEMMA 4. *If $G_1S5_n \vdash S$ then $GS5_n \vdash S$.*

LEMMA 5. *The rule $(\to \mathbf{K}_i)$ is admissible in $G_1S5_n$.*

*Proof.* Let $S^*$ be a sequent $\mathbf{K}(i)\Gamma_1 \to \mathbf{K}(i)\Gamma_2, A$ and $G_1S5_n \vdash S^*$. Using admissibility of weakening we get $G_1S5_n \vdash \mathbf{K}(i)\Gamma_1 \to \mathbf{K}(i)B, \mathbf{K}(i)\Gamma_2, A$, where $B$ is the formula from definition of the rule $(\to \mathbf{K}_i^C)$. Applying $(\to \mathbf{K}_i^C)$ to $S_1$ we get $G_1S5_n \vdash \Sigma_1, \mathcal{K}\Gamma_1, \to \Sigma_2, \mathcal{K}\Gamma_2, \mathbf{K}(i)A$, i. e., the rule $(\to \mathbf{K}_i)$ is admissible in $G_1S5_n$.

Using Lemma 5 we get

LEMMA 6. *If $GS5_n \vdash S$ then $G_1^CS5_n \vdash S$.*

From Lemmas 2 and 6 we get

LEMMA 7. *$GS5_n \vdash S$ if and only if $G_1^CS5_n \vdash S$.*

## 4. Admissibility of rule (Cut) in $G_1S5_n$

To prove admissibility of rule (Cut) in calculus $G_1S5_n$ let us state some additional lemmas.

Let $S(\mathbf{K}(i)A^+)$ means that the formula $\mathbf{K}(i)A$ occurs positively in $S$.

LEMMA 8. *Let $G_1S5_n \vdash^V S(\mathbf{K}(i)A^+)$, where $V$ is an atomic derivation of $S$ and $h(V)$ is a height of this derivation. Then $G_1S5_n \vdash^{V^*} S(A)$ and $h(V^*) \leqslant h(V)$.*

*Proof.* Using proof-theoretical considerations.

LEMMA 9 (admissibility of contraction rules). *Let $G_1S5_n \vdash^V A, A, \Gamma \to \Delta$ $(G_1S5_n \vdash^V \Gamma \to \Delta, A, A)$, where $V$ is an atomic derivation and $h(V)$ is a height of this derivation. Then $G_1S5_n \vdash A, \Gamma \to \Delta$ $(G_1S5_n \vdash \Gamma \to \Delta, A$, respectively).*

*Proof.* By induction on $h(V)$ and using Lemma 8.

LEMMA 10 (admissibility of (Cut)). *Let $G_1S5_n \vdash^{V_1} \Gamma \to \Delta, A$ and $G_1S5_n \vdash^{V_2} A, \Pi \to \Theta$ where $V_1$, $V_2$ are atomic derivations. Then $G_1S5_n \vdash \Gamma, \Pi \to \Delta, \Theta$.*

*Proof.* Lemma is proved using double induction $< g(A), h(V_1) + h(V_2) >$, where $g(A)$ is complexity of the formula $A$ defined in a traditional way. Let $(i)$ and $(j)$ are rules applied last in derivations $V_1$ and $V_2$, correspondingly. Let us consider only the case when $(i)$ is the rule $(\to \mathbf{K}_i^C)$, $(j)$ is $(\mathbf{K}_i \to)$, $A = \mathbf{K}(i)M$, and $A = \mathbf{K}(i)M$ is the main formula of the applications of the rules $(\to \mathbf{K}_i^C)$, $(\mathbf{K}_i \to)$. In this case the end of $V_1$ has the following shape:

$$\frac{S_1' = \Gamma_{1i},\, \mathbf{K}(i)\Gamma_{1i} \to \mathbf{K}(i)B,\, \mathbf{K}(i)\Gamma_{2i},\, M}{S_1 = \Sigma_1,\, \mathcal{K}\Gamma_1 \to \Sigma_2,\, \mathcal{K}\Gamma_2,\, \mathbf{K}(i)M}\ (\to \mathbf{K}_i^C),$$

where all notations are the same as in definition of the rule $(\to \mathbf{K}_i^C)$.

The end of $V_2$ has the following shape:

$$\frac{S_2' = M,\, \mathbf{K}(i)M, \Pi \to \Theta}{S_2 = \mathbf{K}(i)M, \Pi \to \Theta}\ (\mathbf{K}_i \to),$$

Applying $(Cut)$ to $S_1$ and $S_2'$ (with $\mathbf{K}(i)M$ as the cut formula) and using induction on the height, we get $G_1S5_n \vdash S_1^* = \Sigma_1,\, \mathcal{K}\Gamma_1, M, \Pi \to \Sigma_2,\, \mathcal{K}\Gamma_2, \Theta$. Applying $(Cut)$ to $S_1'$ and $S_1^*$ (with $M$ as the cut formula) and using induction on complexity of cut formula, we get $G_1S5_n \vdash S_2^* = \Gamma_{1i}, \mathbf{K}(i)\Gamma_{1i}, \Sigma_1,\, \mathcal{K}\Gamma_1, \Pi \to \mathbf{K}(i)B,\, \mathbf{K}(i)\Gamma_{2i}, \Sigma_2,\, \mathcal{K}\Gamma_2, \Theta$. Using Lemmas 8, 9, invertibility of logical rules, and the rule $(\mathbf{K}_i \to)$ we get $G_1S5_n \vdash \Sigma_1,\, \mathcal{K}\Gamma_1, \Pi \to \Sigma_2,\, \mathcal{K}\Gamma_2, \Theta$, i.e., we get a desired derivation.

From Lemmas 10, 7 we get

LEMMA 11. *$GS5_n \vdash S$ if and only if $G_1S5_n \vdash S$.*

In the previous section it was proved that all rules of $G_1S5_n$, except of $(\to \mathbf{K}_i^C)$, are invertible in $G_1S5_n$ and the rule $(\to \mathbf{K}_i^C)$ is invertible in $GS5_n$. From Lemma 11 invertibility of $(\to \mathbf{K}_i^C)$ in $G_1S5_n$ follows.

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REZIUMĖ

***R. Pliuškevičius. Žinojimo logika su begaliniu agentų skaičiumi***

Sukonstruotas sekvencinis skaičiavimas žinojimo logikai su begaliniu agentų skaičiumi. Sukonstruotas skaičiavimas neturi piūvio taisyklės ir visos jo taisyklės yra apverčiamos.