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Chebyshev inequalities for unimodal distributions

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Abstract. We provide precise upper bounds for the survival function of bounded unimodal random variables.

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1. Introduction and the result

A distribution function F of a random variable X is said to be unimodal with mode m if it can be written in the form:

$$F(x) = \begin{cases} \int_{-\infty}^{x} p(t)\,\mathrm{d}t & \text{for } x < m, \\ F_m + \int_{-\infty}^{x} p(t)\,\mathrm{d}t & \text{for } x \geqslant m. \end{cases}$$

Here the function $t \mapsto p(t)$ is non-decreasing for $t < m$, non-increasing for $t \geqslant m$ and $F_m = F(m+) - F(m-)$. According to Khinchin (1938), unimodal random variables have the representation $X = m + UY$, where m is the mode of X and U is a uniform random variable on [0, 1]. Furthermore, the random variables U and Y are independent. We shall obtain precise upper bounds for the survival function of bounded unimodal random variables with given and unknown mode.

THEOREM. *Let X be a unimodal random variable such that $\mathbb{E}X = 0$ and $\mathbb{P}(|X| \leqslant 1) = 1$. Then we have*

$$\sup \mathbb{P}(X \geqslant x) =: U(x) = \begin{cases} 1 & \textit{for } x < 0, \\ \dfrac{1-x}{1+x} & \textit{for } 0 \leqslant x \leqslant 1, \\ 0 & \textit{for } x > 1. \end{cases} \tag{1}$$

The supremum in (1) is taken over all unimodal X on $[-1, 1]$ with $\mathbb{E}X = 0$. Without unimodality we have $\sup \mathbb{P}(X \geqslant x) = \frac{1}{1+x} := B(x)$ for $0 \leqslant x \leqslant 1$. Fig. 1 shows how the unimodality asumption improves the bound.

2. The proof

First we show that in cases $x < 0$ and $x > 1$ trivially holds $U(x) = 1$ and $U(x) = 0$, respectively. Indeed, if $x < -1$ then $\mathbb{P}(X \geqslant x) = 1$ for $X \equiv 0$ and thus $U(x) = 1$. When

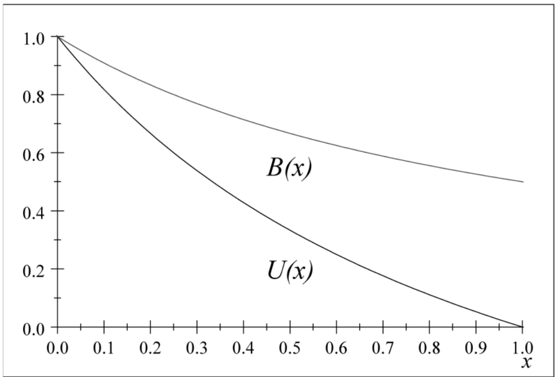

Fig. 1. Comparision of the bounds.

$x > 1$ we have $U(x) = 0$, because we assume $\mathbb{P}(|X| \leqslant 1) = 1$. Therefore henceforth we assume that $0 \leqslant x \leqslant 1$.

Khinchin's representation enables us to write the survival function in a form

$$\mathbb{P}(X \geqslant x) = \mathbb{E}\mathbb{I}\{m + UY \geqslant x\} = \mathbb{E}\mathbb{P}(Ut \geqslant x - m|Y = t) =: \mathbb{E}\Omega_m(Y),$$

where we write $\Omega_m(Y) = \mathbb{P}(Ut \geqslant x - m|Y = t)$. From $\mathbb{P}(|X| \leqslant 1) = 1$ we easily derive $\mathbb{P}(|Y + m| \leqslant 1) = 1$. Furthermore, $\mathbb{E}X = 0$ implies $\mathbb{E}Y = -2m$.

Let us define a function $U_m(x) = \sup E\Omega_m(Y)$, where the supremum is taken over all $X = m + UY$ with given m. The form of $U_m(x)$ depends on m. We consider three cases

$$\text{i)} \quad -1 \leqslant m \leqslant \frac{3x-1}{1+x}, \quad \text{ii)} \quad \frac{3x-1}{1+x} \leqslant m < x, \quad \text{iii)} \quad x \leqslant m \leqslant 1$$

separately. We will show that in the cases above we have:

$$\text{i)} \quad U_m(x) = \frac{1-x}{2}; \tag{2}$$

$$\text{ii)} \quad U_m(x) = \frac{1-m}{1-m+2x+2\sqrt{(x-m)(1+x)}}; \tag{3}$$

$$\text{iii)} \quad U_m(x) = \frac{1-x}{1+m}. \tag{4}$$

Let us now prove (2). Now we have $\Omega_m(t) = 1 - \frac{x-m}{t}$ for $t \in [x - m, 1 - m]$ and $\Omega_m(t) = 0$ for $t \in [-1 - m, x - m]$. Let us consider a linear function

$$Q_m(t) = \frac{1-x}{2(1-m)}(1 + m + t).$$

Then for $t \in [-1 - m, 1 - m]$ we have $\Omega_m(t) \leqslant Q_m(t)$. The inequality is easily checked by defining a function $H(t) = Q_m(t) - \Omega_m(t)$, for which $H(-1 - m) =$

$H(1-m)=0$ and $H'(t)\leqslant 0$ when $t\in[x-m,1-m]$. For $t\in[-1-m,x-m]$ the inequality holds trivially since then $\Omega_m(t)=0$ and $Q_m(t)\geqslant 0$. The inequality $\Omega_m(t)\leqslant Q_m(t)$ implies $\mathbb{E}\Omega_m(Y)\leqslant\mathbb{E}Q_m(Y)=\frac{1-x}{2}$.

Let us prove (3).We have the same Ω_m, but this time let us define

$$Q_m(t)=\frac{1+m+t}{1-m+2x+2\sqrt{(x-m)(1+x)}}.$$

For $t\in[-1-m;1-m]$ we again have $\Omega_m(t)\leqslant Q_m(t)$. This is similarly checked as in the previous case. Thus we derive

$$\mathbb{E}\Omega_m(Y)\leqslant\mathbb{E}Q_m(Y)=\frac{1-m}{1-m+2x+2\sqrt{(x-m)(1+x)}}.$$

Let us prove (4). This case is handled in the same manner and here $\Omega_m(t)=\frac{x-m}{t}$ for $t\in[-1-m,x-m)$ and $\Omega_m(t)=1$ for $t\in[x-m,1-m]$. The linear function this time is defined by

$$Q_m(t)=1+\frac{t-x+m}{1+m}$$

and then

$$\mathbb{E}Q_m(Y)=\frac{1-x}{1+m}.$$

Remark. The maximizing distributions in (2)–(4) are Bernoulli distributions concentrated in the two point set $\{t\colon\ \Omega_m(t)=Q_m(t)\}$.

To prove the theorem it suffices to note that

$$\sup_{-1\leqslant m\leqslant 1} U_m(x)=\frac{1-x}{1+x}.$$

The maximizing random variable in (1) has the form $X=U\varepsilon+x$, where ε is a Bernoulli random variable such that

$$\mathbb{P}\bigl(\varepsilon=-(1+x)\bigr)=\frac{2x}{1+x}\quad\text{and}\quad\mathbb{P}(\varepsilon=0)=\frac{1-x}{1+x}.$$

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REZIUMĖ

T. Juškevičius. Čebyšovo nelygybės unimodaliesiems skirstiniams

Gauti tikslūs tikimybių $\mathbb{P}(X\geqslant x)$ įverčiai iš viršaus, kai X yra aprėžtas unimodalusis atsitiktinis dydžis.