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Exposedness in Bernstein spaces

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Abstract. The Bernstein space B_σ^p, $\sigma > 0$, $1 \leqslant p \leqslant \infty$, consists of those $L^p(\mathbb{R})$-functions whose Fourier transforms are supported on $[-\sigma, \sigma]$. Every function in B_σ^p has an analytic extension onto the complex plane $\mathbb{C}$ which is an entire function of exponential type at most σ. Since B_σ^p is a conjugate Banach space, its closed unit ball $\mathcal{D}(B_\sigma^p)$ has nonempty sets of both extreme and exposed points. These sets are nontrivially arranged only in the cases $p = 1$ and $p = \infty$. In this paper, we investigate some properties of exposed functions in $\mathcal{D}(B_\sigma^1)$ and illustrate them by several examples.

Keywords: Bernstein spaces, entire functions of exponential type, sine-type functions, exposed points.

1. Introduction

An entire function f is said to be of exponential type at most σ $(0 \leqslant \sigma < \infty)$ if for every $\varepsilon > 0$, there exists an $M_\varepsilon > 0$ such that

$$|f(z)| \leqslant M_\varepsilon \mathrm{e}^{(\sigma+\varepsilon)|z|}, \quad z \in \mathbb{C}.$$

Certainly, the greatest lower bound of those σ coincides with the type σ_f of f. For $1 \leqslant p \leqslant \infty$ and $0 < \sigma < \infty$, the *Bernstein space* B_σ^p consists of all $f \in L^p(\mathbb{R})$ which can be extended from $\mathbb{R}$ onto $\mathbb{C}$ to an entire function of exponential type at most σ. The classes B_σ^p are Banach spaces under the $L^p(\mathbb{R})$-norm. From the Paley–Wiener–Schwartz theorem and its inversion it follows that functions in B_σ^p can be described as those $L^p(\mathbb{R})$-functions whose Fourier transform (considered as generalized functions) vanish outside $[-\sigma, \sigma]$. Therefore, B_σ^p consists of bandlimited functions: such functions are interpreted as signals with no frequencies outside "band" $[-\sigma, \sigma]$.

Let $\mathcal{D}(B_\sigma^p)$ denote the closed unit ball in B_σ^p. Recall that $f \in \mathcal{D}(B_\sigma^p)$ is *extreme function* (point) if for any $u, v \in \mathcal{D}(B_\sigma^p)$, $f = \frac{1}{2}(u+v)$, implies that $u = v = f$. We call $f \in \mathcal{D}(B_\sigma^p)$ *exposed* in $\mathcal{D}(B_\sigma^p)$ if there exists a functional Φ on B_σ^p with $\|\Phi\| = 1$ such that $\Phi(f) = 1$ and $\operatorname{Re} \Phi(g) < 1$ for all $g \in \mathcal{D}(B_\sigma^p)$, $g \neq f$. That Φ will be called an *exposing functional* for f. We shall denote by $\operatorname{extr} \mathcal{D}(B_\sigma^p)$ the set of extreme points in $\mathcal{D}(B_\sigma^p)$, and the set of exposed points will be denoted by $\exp \mathcal{D}(B_\sigma^p)$. It is obvious that an exposed point of $\mathcal{D}(B_\sigma^p)$ is necessarily extreme, but the converse need not hold in general (see Example 6).

The existence of extreme points in B_σ^p guarantees, by the Krein–Milman theorem, that B_σ^p are conjugate Banach spaces. Moreover, if $1 < p < \infty$, then B_σ^p is uniformly convex. In uniformly convex spaces, every point of the unit sphere is an extreme point of the unit ball. The cases B_σ^1 and B_σ^∞ are not so trivial. Consider the duality pair

$(C_0(\mathbb{R}), M(\mathbb{R}))$, where $C_0(\mathbb{R})$ is the usual normed space of complex continuous functions on $\mathbb{R}$ vanishing at infinity, and $M(\mathbb{R})$ is the Banach convolution algebra of all regular complex Borel measures on $\mathbb{R}$, equipped with the total variation norm. Let $\mathfrak{I}_\sigma$ be the closed ideal of $M(\mathbb{R})$ consisting of those $\mu \in M(\mathbb{R})$ for which the Fourier–Stieltjes transforms $\hat{\mu}$ vanish for $|t| \geqslant \sigma$. Set $C_{0,\sigma} = \left\{f \in C_0(\mathbb{R})\colon \int_{\mathbb{R}} f(x)\,\mathrm{d}\mu(x) = 0,\ \forall \mu \in \mathfrak{I}_\sigma\right\}$. Then B^1_σ is the dual space to the quatient space $C_0/C_{0,\sigma}$ (see [2]). Therefore, in contrast to the unit ball of $L^1(\mathbb{R})$ the set $\mathcal{D}(B^1_\sigma)$ has large both sets extr $\mathcal{D}(B^1_\sigma)$ and exp $\mathcal{D}(B^1_\sigma)$. Second of these statements follows from the following Phelps theorem [3]: in a separable dual Banach space the closed unit ball coincides with the closed convex hull of its strongly exposed points. The set extr $\mathcal{D}(B^1_\sigma)$ can be described in terms of zeros of entire functions (see [2]).

THEOREM A. *A function $f \in B^1_\sigma$, $\|f\| = 1$, belongs to* extr $\mathcal{D}(B^1_\sigma)$ *if and only if f is an entire function of type $\sigma_f = \sigma$ and has no conjugate complex zeros.*

Here we determine exp $\mathcal{D}(B^1_\sigma)$ and illustrate them by several examples. A criterion and a sufficient condition of exposedness in $\mathcal{D}(B^1_\sigma)$ are also given. Finally, we consider relations between the exposedness and sine-type function notion.

2. Exposed points of the unit ball in B^1_σ

Let Φ be a continuous linear functional on B^1_σ, i.e., $\Phi \in (B^1_\sigma)^*$. Suppose that Φ attains its norm. We shall call a $f \in B^1_\sigma$, $f \not\equiv 0$, an extremal for Φ if $\Phi(f) = \|\Phi\|\,\|f\|$. It may be noted that $f \in$ exp $\mathcal{D}(B^1_\sigma)$ if and only if there exists $\Phi \in (B^1_\sigma)^*$ such that Φ has in $\mathcal{D}(B^1_\sigma)$ an unique extremal with the unit norm. By the Hahn–Banach theorem, every nonzero $f \in B^1_\sigma$ is an extremal for some functional in $(B^1_\sigma)^*$. We select among such functionals the following

$$\Phi_f(g) = \int_{\mathbb{R}} g(x) u_f(x)\,\mathrm{d}x, \quad g \in B^1_\sigma,$$

where $u_f(x)$ is the function $\overline{f(x)}/|f(x)| \in L^\infty(\mathbb{R})$ defined for almost all $x \in \mathbb{R}$. Thus, if $f \in$ exp $\mathcal{D}(B^1_\sigma)$, then Φ_f is an exposing functional for f. Moreover, every $f \in$ exp $\mathcal{D}(B^1_\sigma)$ has the unique exposing functional. Indeed, assume that $\Phi \in (B^1_\sigma)^*$ expose $f \in$ exp $\mathcal{D}(B^1_\sigma)$. By the Hahn–Banach theorem, Φ can be continued up to a functional Ψ on $L^1(\mathbb{R})$ without increase of its norm. Then there is $\psi \in L^\infty(\mathbb{R})$ with $\|\psi\| = 1$ such that $\Psi(a) = \int_{\mathbb{R}} a(t)\overline{\psi}(t)\,\mathrm{d}t$ for all $a \in L^1(\mathbb{R})$. From this and from $\Psi(f) = \Phi(f) = 1$ it follows that $\psi(t)$ coincides with $f(t)/|f(t)|$ for almost all $t \in \mathbb{R}$. Therefore, $\Phi = \Phi_f$.

The theorem A shows that extremeness in $\mathcal{D}(B^1_\sigma)$ can be described in terms of zeros of entire functions. We shall now restrict these conditions up to necessary and sufficient ones for the exposedness. Let $f \in B^1_\sigma$. Recall that an entire function of exponential type ϱ, $\varrho \not\equiv const$, is called the *multiplier* for $f \in B^1_\sigma$, if $\varrho f \in B^1_\sigma$.

THEOREM 1. *A function* $f \in \mathcal{D}(B_\sigma^1)$ *belongs to* $\exp\ \mathcal{D}(B_\sigma^1)$ *if and only if:* (i) $\|f\| = 1$, *and* f *has no conjugate complex zeros.* (ii) *Every real zero of* f *is simple.* (iii) f *has no nonnegative on* $\mathbb{R}$ *multipliers.*

We say that a function $f \in B_\sigma^1$ is real if $f(z) = \overline{f(\bar{z})}$, $z \in \mathbb{C}$. A real function $f \in B_\sigma^p$ takes on $\mathbb{R}$ only real values, and every its complex zero necessarily is conjugate zero, i.e. if $f(z_0) = 0$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$, then $f(\bar{z}_0) = 0$.

COROLLARY 2. *Any real function in* $\exp\ \mathcal{D}(B_\sigma^1)$ *has only real and simple zeros.*

The following theorem gives a sufficient condition of exposedness in $\mathcal{D}(B_\sigma^1)$. It allows to determine the large set of exposed functions in $\mathcal{D}(B_\sigma^1)$ (see Examples 6–8).

THEOREM 3. *Let* $f \in \operatorname{extr} \mathcal{D}(B_\sigma^1)$. *Suppose there exist* $\tau \in (0, 3]$, *and* $y_0 \in \mathbb{R}$ *such that*

$$\inf_{x \in \mathbb{R}} \Big(|x + iy_0|^\tau \, \big| f(x + iy_0) \big| \Big) > 0. \tag{1}$$

If f *has no multiple real zeros, then* $f \in \exp\ \mathcal{D}(B_\sigma^1)$.

Remark 4. Suppose $g \in B_\sigma^1$. From the Plancherel–Polya theorem it follows that g belongs to $L^1(\mathbb{R})$ not only on $\mathbb{R}$, but also on each line $\mathbb{R} + ia = \{z \in \mathbb{C}\colon\ z = x + ia, x \in \mathbb{R}\}$, where $a \in \mathbb{R}$. Therefore, $g_a(x) := g(x + ia)$, $x \in \mathbb{R}$, belongs to B_σ^1 for all $a \in \mathbb{R}$. Thus $\lim_{x \to \pm\infty} g(x + ia) = 0$, $a \in \mathbb{R}$. Now if $g \in \exp\ \mathcal{D}(B_\sigma^1)$, then theorem 1 implies that $z^m g(z) \notin L^1(\mathbb{R} + ia)$ for all $a \in \mathbb{R}$, and $m = 2, \ldots$. This means that each $f \in \exp\ \mathcal{D}(B_\sigma^1)$ is a slowly decreasing function on every line $\mathbb{R} + ia$. Moreover, it is not difficult to show that if $f \in \exp\ \mathcal{D}(B_\sigma^1)$, and $\sup_{\mathbb{R}} |x|^s |f(x)| < \infty$, then $s < 3$. Next Theorem 5 shows that this estimation is exact. On the other hand, by this theorem, the requirement $\tau \leqslant 3$ in (1) is also exact.

We shall prove that each sine-type function determines a large set in $\exp\ \mathcal{D}(B_\sigma^1)$ in a sense defined below by Theorem 5. Recall that an entire function F of exponential type is called σ*-sine-type function* (or simple sine-type function), if there are positive numbers c_1, c_2, and K such that

$$c_1 \leqslant \big|F(x + iy)\big| \mathrm{e}^{-\sigma|y|} \leqslant c_2, \quad x, y \in \mathbb{R}, \ \ |y| \geqslant K,$$

(see [1]). These functions compose the wide class. For example, it contain any function

$$F(z) = \int_{-\sigma}^{\sigma} \mathrm{e}^{-itz}\, \mathrm{d}\mu(t),$$

where μ is any finite complex measure such that $\mu(\{-\sigma\}) \neq 0$, and $\mu(\{\sigma\}) \neq 0$. Finally, every σ-sine-type function F has the type $\sigma_F = \sigma$ and belongs to B_σ^∞. Let us denote by N_f the set of all zeros (roots) of $f \in B_\sigma^1$ in $\mathbb{C}$ with multiplicities counted.

THEOREM 5. *Let F be a σ-sine-type function, and let $F(z) \not\equiv c\mathrm{e}^{\pm i\sigma z}$, $c \in \mathbb{C}$. Suppose that F has neither complex-conjugate nor multiple real zeros. Let p be a polynomial such that $N_p \subset N_F$. Put*

$$f_p(z) = \alpha \frac{F(z)}{p(z)}, \tag{2}$$

where $\alpha \in \mathbb{C}$ is such that $\|f_p\|_{L^1} = 1$. If $\deg p \geqslant 2$, then $f_p \in \operatorname{extr} \mathcal{D}(B^1_\sigma)$. The function f_p belongs to $\exp \mathcal{D}(B^1_\sigma)$ if and only if $2 \leqslant \deg p \leqslant 3$.

We shall indicate a few examples, which explain relation between notion of the exposed function in $\mathcal{D}(B^1_\sigma)$ and certain other properties of entire functions.

EXAMPLE 6. We shall begin from an example, which proves that

$$\exp \mathcal{D}(B^1_\sigma) \subsetneqq \operatorname{extr} \mathcal{D}(B^1_\sigma).$$

To this end, we put

$$f(z) = \alpha \frac{\sin(\sigma z)}{(\sigma^2 z^2 - \pi^2)(\sigma^2 z^2 - 4\pi^2)}, \tag{3}$$

where α is a complex normalizing constant, i.e., α is such that $\|f\|_{L^1} = 1$. For example, it is easily verified that it is possible to take

$$\alpha = 3\sigma\pi^3 \big(3\mathrm{Si}(\pi) + 2\mathrm{Si}(2\pi) - \mathrm{Si}(3\pi) - \mathrm{Si}(4\pi)\big)^{-1}, \quad \mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, \mathrm{d}t.$$

Let $p(z) = (\sigma^2 z^2 - \pi^2)(\sigma^2 z^2 - 4\pi^2)$. Since $F(z) = \sin(\sigma z)$ is a σ-sine-type function, and $N_p \subset N_F$, then by virtue of theorem 5, we conclude that $f \in \operatorname{extr} \mathcal{D}(B^1_\sigma)$, but $f \notin \exp \mathcal{D}(B^1_\sigma)$.

Although the set $\operatorname{extr} \mathcal{D}(B^1_\sigma)$ is completely described in terms of zeros of entire functions, the following two examples show that the problem of such a description of exposedness can be rather difficult.

EXAMPLE 7. Let $c \in \mathbb{C} \setminus \mathbb{R}$, and let $f_p(z) = \beta(z-c)^2 f(z)$, where f is the function (3), and β is a normalizing constant. Then f_p may be represented as in (2), where

$$F(z) = \frac{(z-c)^2 \sin(\sigma z)}{\sigma^2 z^2 - \pi^2},$$

and $p(z) = \sigma^2 z^2 - 4\pi^2$. Since such F is σ-sine-type function, then $f_p \in \exp \mathcal{D}(B^1_\sigma)$ by Theorem 5. This example shows that a function in $\exp \mathcal{D}(B^1_\sigma)$ can have multiple complex zeros (in contrast to its real zeros).

The following example shows that there are functions in exp $\mathcal{D}(B^1_\sigma)$, which have not separated zeros. Recall that a sequence $\Lambda = \{\lambda_k\}$ of complex numbers is called separated, if there is $\delta > 0$ such that

$$\inf_{\substack{\lambda_k, \lambda_m \in \Lambda \\ \lambda_k \neq \lambda_m}} |\lambda_k - \lambda_m| \geqslant \delta.$$

EXAMPLE 8. Let

$$f_p(z) = \alpha \frac{\cos\left(\sigma \frac{z}{2}\right) \cos\sqrt{\left(\sigma \frac{z}{2}\right)^2 + \varepsilon^2}}{\sigma^2 z^2 - \pi^2},$$

where $0 < \varepsilon < \pi/2$, and α is a normalizing f_p in B^1_σ constant. From $0 < \varepsilon < \pi/2$ it follows that the σ-sine-type function $F(z) = \cos(\sigma \frac{z}{2}) \cos\sqrt{(\sigma \frac{z}{2})^2 + \varepsilon^2}$ has only real and simple zeros. Therefore, $f_p \in \exp\ \mathcal{D}(B^1_\sigma)$ by Theorem 5. The set of roots $N_{f_p} = \left\{\frac{2}{\sigma}(\frac{\pi}{2} + \pi k),\ k \in \mathbf{Z}\right\} \cup \left\{\pm \frac{2}{\sigma}\sqrt{(\frac{\pi}{2} + \pi l)^2 - \varepsilon^2},\ l \in \mathbf{Z}\right\}$, is obviously not separated.

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REZIUMĖ

S. Norvidas. Eksponavimas Bernšteino erdvėse

Bernšteino erdvę B^p_σ, $\sigma > 0$, $1 \leqslant p \leqslant \infty$, sudaro tokios $L^p(\mathbb{R})$ klasės funkcijos, kurių Furje transformacijų atramos priklauso $[-\sigma, \sigma]$. Kiekvieną funkciją iš B^p_σ galima pratęsti analiziškai į visą kompleksinę plokštumą $\mathbb{C}$, kur ji apibrėžia sveikąją eksponentinio tipo $\leqslant \sigma$ funkciją. Kadangi kiekviena B^p_σ yra jungtinė Banacho erdvė, tai jos uždarame vienetiniame rutulyje $\mathcal{D}(B^p_\sigma)$ egzistuoja netušti ekstreminių ir eksponuotųjų taškų poaibiai. Šios aibės yra netrivialios tik, kai $p = 1$ ir $p = \infty$. Šiame darbe mes nagrinėjame eksponuotąsias rutulio $\mathcal{D}(B^1_\sigma)$ funkcijas ir jų pavyzdžius.