Nonlinear Analysis: Modelling and Control, Vol. 25, No. 3, 509–522
https://doi.org/10.15388/namc.2020.25.16661
ISSN: 1392-5113
eISSN: 2335-8963

Regularly distributed randomly stopped sum, minimum, and maximum*

Jonas Sprindys, Jonas Šiaulys

Institute of Mathematics, Vilnius University,
Naugarduko 24, Vilnius LT-03225, Lithuania
jonas.sprindys@gmail.com; jonas.siaulys@mif.vu.lt

Received: October 31, 2019 / **Revised:** February 7, 2020 / **Published online:** May 1, 2020

Abstract. Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued, possibly nonidentically distributed, random variables, and let η be a nonnegative, nondegenerate at 0, and integer-valued random variable, which is independent of $\{\xi_1, \xi_2, \dots\}$. We consider conditions for $\{\xi_1, \xi_2, \dots\}$ and η under which the distributions of the randomly stopped minimum, maximum, and sum are regularly varying.

Keywords: regularly varying distribution, counting random variable, randomly stopped maximum, randomly stopped minimum, randomly stopped sum, closure property.

1 Introduction

Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \dots\}$, and let η be a counting random variable (c.r.v.), that is, a nonnegative, nondegenerate at 0, and integer-valued r.v. In addition, we suppose that the r.v. η and the sequence $\{\xi_1, \xi_2, \dots\}$ are independent.

Let $S_0 := 0$, $S_n := \xi_1 + \dots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the *randomly stopped sum* of the r.v.s $\xi_1, \xi_2, \dots$.

Next, let $\xi^{(0)} := 0$, $\xi^{(n)} := \max\{0, \xi_1, \dots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi^{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{0, \xi_1, \dots, \xi_\eta\} & \text{if } \eta \geqslant 1, \end{cases}$$

be the *randomly stopped maximum* of the r.v.s $\xi_1, \xi_2, \dots$.

*Authors are supported by the Research Council of Lithuania, grant No. S-MIP-17-72.

Similarly, let $\xi_{(0)} := 0$, $\xi_{(n)} := \min\{\xi_1, \ldots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi_{(\eta)} := \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{\xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

be the *randomly stopped minimum* of the r.v.s $\xi_1, \xi_2, \ldots$.

We denote by $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and F_{S_η} the d.f.s of $\xi_{(\eta)}$, $\xi^{(\eta)}$, and S_η, respectively. We denote by $\overline{F}$ the tail of a d.f. F, that is, $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. It is obvious that the following equalities hold for $x > 0$:

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}(\xi_{(n)} > x),$$
$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}(\xi^{(n)} > x),$$
$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\eta = n)\mathbf{P}(S_n > x).$$

We use the following three notations for the asymptotic relations of arbitrary positive functions f and g: $f(x) = o(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$; $f(x) \sim cg(x)$, $c > 0$, means that $\lim_{x\to\infty} f(x)/g(x) = c$; and $f(x) \asymp g(x)$ means that $0 < \liminf_{x\to\infty} f(x)/g(x) \leqslant \limsup_{x\to\infty} f(x)/g(x) < \infty$.

Before discussing the properties of the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and F_{S_η}, we recall the definition of a regularly varying distribution function.

A d.f. F is regularly varying $(F \in \mathcal{R}_\alpha)$ for some index $\alpha \geqslant 0$ if

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

for any $y > 0$. A r.v. X is regularly varying with index $\alpha \geqslant 0$ if its distribution F_X belongs to $\mathcal{R}_\alpha$.

An important property following from the definition of $\mathcal{R}_\alpha$ is that the tail function of an arbitrary regularly varying d.f. can be represented in the form $\overline{F}(x) = x^{-\alpha}L(x)$, where L is a *slowly varying function*, that is,

$$\lim_{x\to\infty} \frac{L(xy)}{L(x)} = 1$$

for any $y > 0$.

In this paper, we consider a sequence $\{\xi_1, \xi_2, \ldots\}$ of possibly nonidentically distributed r.v.s. We suppose that some of the d.f.s of these r.v.s belong either to the class $\mathcal{R}_\alpha$ with some $\alpha \geqslant 0$ or to the classes $\mathcal{R} = \bigcup_{\alpha\geqslant 0} \mathcal{R}_\alpha$, $\mathcal{R}_+ = \bigcup_{\alpha>0} \mathcal{R}_\alpha$. We find conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and F_{S_η} are regularly varying.

The class of regularly varying functions was introduced by Karamata [14] in the context of real analysis. The notion of regular variation was introduced in probability theory by Feller [11] when considering limit theorems for sums of i.i.d. r.v.s. Many analytical results on regularly varying functions can be found in the monograph by Bingham et al. [1]. Some applications of regularly varying distributions to finance and insurance are presented by Embrechts et al. [9]. We further present a few results on randomly stopped structures related to regularly varying distribution functions.

The following two results present sufficient (Theorem 1) and necessary (Theorem 2) conditions for the closure of random sum of regularly varying r.v.s, see [10], Propositions 4.1 and 4.8, respectively.

Theorem 1. *Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed (i.i.d.) nonnegative r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Assume that the d.f. F_{ξ_1} is regularly varying with index $\alpha > 0$, $\mathbf{E}\eta < \infty$, and $\overline{F}_\eta(x) = o(\overline{F}_{\xi_1}(x))$. Then the d.f. F_{S_η} belongs to the class $\mathcal{R}_\alpha$, and $\overline{F}_{S_\eta}(x) \sim \mathbf{E}\eta\overline{F}_{\xi_1}(x)$.*

Theorem 2. *Let $\xi_1, \xi_2, \ldots$ be i.i.d. nonnegative r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Assume that S_η is regularly varying with index $\alpha > 0$ and $\mathbf{E}\eta^{1\vee p} < \infty$ for some $p > \alpha$. Then the d.f. F_{ξ_1} belongs to the class $\mathcal{R}_\alpha$, and $\overline{F}_{S_\eta}(x) \sim \mathbf{E}\eta\overline{F}_{\xi_1}(x)$.*

The following result on sufficient and necessary conditions for the closure under random maximum of regularly varying r.v.s was obtained in [13] (see Lemma 5.1(i)).

Theorem 3. *Let $\xi_1, \xi_2, \ldots$ be i.i.d. real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbf{E}\eta < \infty$. Then $\overline{F}_{\xi^{(\eta)}}(x) \sim \mathbf{E}\eta\overline{F}_{\xi_1}(x)$, and hence $F_{\xi^{(\eta)}}$ belongs to the class $\mathcal{R}_\alpha$ if and only if F_{ξ_1} belongs to $\mathcal{R}_\alpha$, $\alpha \geqslant 0$.*

Motivated by the presented statements and results obtained in [2, 4, 5, 7, 15, 18–21, 24, 26], we continue to consider conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, and F_{S_η} belong to either the class $\mathcal{R}_\alpha$ with some $\alpha \geqslant 0$ or the class $\mathcal{R}$. As we mentioned before, we deal with the case where the sequence $\{\xi_1, \xi_2, \ldots\}$ consists of independent but possibly nonidentically distributed r.v.s.

The rest of the paper is organized as follows. In Section 2, we present our main results. Section 3 consists of some auxiliary lemmas. The proofs of the main results are given in Section 4. Finally, in Section 5, we present two examples to expose the usefulness of our results.

2 Main results

In this section, we present the main results of this paper. *In all the statements, we suppose that the sequence $\{\xi_1, \xi_2, \ldots\}$ and the c.r.v. η are independent.* Our first theorem describes properties of randomly stopped minima.

Theorem 4. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of real-valued r.v.s. Then $F_{\xi_k} \in \mathcal{R}$ for all $k \in \mathbb{N}$ if and only if $F_{\xi_{(\eta)}} \in \mathcal{R}$ for every c.r.v. η.*

The second theorem below describes sufficient conditions for the regularity of randomly stopped maxima and sums when the c.r.v. η has a finite support.

Theorem 5. *Let $\xi_1, \dots, \xi_m$ be independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \dots, \xi_m\}$ such that $\mathbf{P}(\eta \leqslant m) = 1$. Then the d.f.s F_{S_η} and $F_{\xi^{(\eta)}}$ belong to the class $\mathcal{R}_\alpha,\ \alpha \geqslant 0$, if the following two conditions are satisfied*:

(i) $F_{\xi_1} \in \mathcal{R}_\alpha$;
(ii) *For each $k \geqslant 2$, either $F_{\xi_k} \in \mathcal{R}_\alpha$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$.*

Furthermore, under conditions (i)–(ii)*, the following tail equivalences hold*:

$$\overline{F}_{\xi^{(\eta)}}(x) \sim \overline{F}_{S_\eta}(x) \sim x^{-\alpha} \sum_{n=1}^{m} \mathbf{P}(\eta = n) \sum_{k \in \mathcal{I}_n} L_k(x), \tag{1}$$

where $\mathcal{I}_n = \{k = 1, \dots, n\colon F_{\xi_k} \in \mathcal{R}_\alpha\}$, and L_k are slowly varying functions from the representations $\overline{F}_{\xi_k}(x) = x^{-\alpha} L_k(x)$.

The following theorem describes properties of randomly stopped sums and maxima when the c.r.v. has a finite support. Here we provide both necessary and sufficient conditions for $\{\xi_1, \xi_2, \dots\}$, but the initial conditions for the collection of the primary r.v.s are more restrictive than in the previous theorem.

Theorem 6. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$ for all $k \geqslant 2$. Then the following statements are equivalent*:

(i) *$F_{\xi_k} \in \mathcal{R}_+$ for all $k \in \mathbb{N}$*;
(ii) *$F_{S_\eta} \in \mathcal{R}_+$ for any c.r.v. η with finite support*;
(iii) *$F_{\xi^{(\eta)}} \in \mathcal{R}_+$ for any c.r.v. η with finite support.*

In the following theorem, we give sufficient conditions under which the randomly stopped sum is regularly varying in the case of a general c.r.v. η.

Theorem 7. *Let $\xi_1, \xi_2, \dots$ independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Assume the following conditions are satisfied*:

(i) *$F_{\xi_1} \in \mathcal{R}_\alpha$, $\alpha \geqslant 0$*;
(ii) *For a sequence of nonnegative constants $\{d_1, d_2, d_3, \ldots\}$ such that $d_1 = 1$ and $\limsup_{n\to\infty} (1/n) \sum_{k=1}^{n} d_k < \infty$, it holds that*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0;$$

(iii) *$\mathbf{E}\eta^{p+1} < \infty$ for some $p > \alpha$.*

Then the following tail equivalence holds:

$$\overline{F}_{S_\eta}(x) \sim \overline{F}_{\xi_1}(x) \sum_{n=1}^{\infty} \mathbf{P}(\eta = n) \sum_{k=1}^{n} d_k,$$

and hence $F_{S_\eta} \in \mathcal{R}_\alpha$.

The following result on sufficient and necessary conditions for the closure under random maximum of regularly varying r.v.s is a direct generalization of Theorem 3.

Theorem 8. *Let $\xi_1, \xi_2, \ldots$ be real-valued r.v.s such that $\overline{F}_{\xi_1}(x) > 0$ for all $x \in \mathbb{R}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. In addition, suppose that*

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| = 0 \quad \textit{and} \quad \max\left\{ \mathbf{E}\eta, \mathbf{E}\left(\sum_{k=1}^{\eta} d_k \right) \right\} < \infty$$

for a sequence of nonnegative constants $\{d_1 = 1, d_2, \ldots\}$. Then

$$\overline{F}_{\xi^{(\eta)}}(x) \sim \overline{F}_{\xi_1}(x)\, \mathbf{E}\left(\sum_{k=1}^{\eta} d_k \right),$$

and hence $F_{\xi^{(\eta)}}$ belongs to the class $\mathcal{R}_\alpha$ if and only if F_{ξ_1} belongs to $\mathcal{R}_\alpha$, $\alpha \geqslant 0$.

3 Auxiliary lemmas

In this section, we give several auxiliary lemmas. Some of these lemmas are originally stated for wider heavy-tailed distribution classes, which include the class $\mathcal{R}$ as a subclass. Here we restate these lemmas for regularly varying d.f.s. The first lemma follows from Theorem 3.1 of [3] (see also Theorem 2.1 from [25]).

Lemma 1. *Let $X_1, \ldots, X_n$ be independent real-valued r.v.s. If $F_{X_k} \in \mathcal{R}$ for $k \in \{1, \ldots, n\}$, then*

$$\mathbf{P}\left(\sum_{k=1}^{n} X_k > x \right) \sim \sum_{k=1}^{n} \overline{F}_{X_k}(x). \tag{2}$$

The next lemma is Theorem 4.1 from [23]. This lemma provides necessary and sufficient conditions for the max-sum equivalence of regularly varying distributions.

Lemma 2. *Let $X_1, \ldots, X_n$ be independent real-valued r.v.s. Then $F_{\Sigma_n} \in \mathcal{R}_\alpha$, $\alpha \geqslant 0$, if and only if $\max\{0, 1 - \sum_{k=1}^{n} \overline{F}_{X_k}\} \in \mathcal{R}_\alpha$, where F_{Σ_n} is d.f. of sum $\Sigma_n = X_1 + \cdots + X_n$. In this case, the asymptotic relation* (2) *holds.*

The next lemma follows from Theorems 3.10, 3.11, and 4.1 by Shimura [23]. It describes the decomposition property of regularly varying distributions.

Lemma 3. *Let X be a real-valued r.v., and suppose that $F_X \in \mathcal{R}_+$. Furthermore, suppose that X can be decomposed into independent r.v.s X_1 and X_2, that is, $X = X_1 + X_2$. If $F_{X_1} \in \mathcal{R}_+$ and $\overline{F}_{X_2}(x) \asymp \overline{F}_{X_1}(x)$, then $F_{X_2} \in \mathcal{R}_+$.*

The following statement was proved in Proposition 1 of [8] and later was generalized to a broader distribution class in Corollary 3.19 of [12].

Lemma 4. *Let $\{X_1, \ldots, X_n\}$ be a collection of independent real-valued r.v.s. Assume that $\overline{F}_{X_k}(x)/\overline{F}(x) \to b_k$ as $x \to \infty$ for some regularly varying d.f. F and some constants $b_i \geqslant 0$, $i \in \{1, \ldots n\}$. Then*

$$\frac{\mathbf{P}(\sum_{k=1}^{n} X_k > x)}{\overline{F}(x)} \underset{x\to\infty}{\to} \sum_{k=1}^{n} b_k.$$

In the next lemma, we show in which cases the d.f. F_{Σ_n} of the sum $\Sigma_n = X_1 + \cdots + X_n$ and the d.f. $F_{X^{(n)}}$ of the maximum $X^{(n)} = \max\{X_1, \ldots, X_n\}$ belong to the class $\mathcal{R}_\alpha$.

Lemma 5. *Let $X_1, \ldots, X_n$ be independent real-valued r.v.s. Then the d.f.s F_{Σ_n} and $F_{X^{(n)}}$ belong to the class $\mathcal{R}_\alpha$, $\alpha \geqslant 0$, if the following conditions are satisfied*:

(i) $F_{X_1} \in \mathcal{R}_\alpha$;
(ii) *For each $k = 2, \ldots, n$, either $F_{X_k} \in \mathcal{R}_\alpha$ or $\overline{F}_{X_k}(x) = o(\overline{F}_{X_1}(x))$.*

Furthermore, under these conditions,

$$\overline{F}_{X^{(n)}}(x) \sim \overline{F}_{\Sigma_n}(x) \sim x^{-\alpha} \sum_{k\in\widehat{\mathcal{I}}_n} L_k(x),$$

where L_k are slowly varying functions from representations $\overline{F}_{X_k}(x) = x^{-\alpha} L_k(x)$, and $\widehat{\mathcal{I}}_n = \{k = 1, \ldots, n\colon F_{X_k} \in \mathcal{R}_\alpha\}$.

Proof. We first consider the sum Σ_n. For $n = 2$, the statement is well known (see, e.g., p. 278 in [11], Lemma 1.3.4 in [17], Proposition 4.2.5 in [22] or the case $n = 2$ of Corollary 3.19 of [12]). We use induction. Suppose the statement of the lemma holds for $n = K$. This means that $F_{\Sigma_K} \in \mathcal{R}_\alpha$ and, due to Lemma 2,

$$\overline{F}_{\Sigma_K}(x) \sim x^{-\alpha} \sum_{k\in\widehat{\mathcal{I}}_K} L_k(x) \sim \sum_{k=1}^{K} \overline{F}_{X_k}(x).$$

According to the conditions of the lemma, either $F_{X_{K+1}} \in \mathcal{R}_\alpha$ or $\overline{F}_{X_{K+1}}(x) = o(\overline{F}_{\Sigma_K}(x))$. Since $\Sigma_{K+1} = \Sigma_K + X_{K+1}$, in both cases, we obtain that $F_{\Sigma_{K+1}} \in \mathcal{R}_\alpha$ and

$$\overline{F}_{\Sigma_{K+1}}(x) \sim \sum_{k=1}^{K+1} \overline{F}_{X_k}(x) \sim x^{-\alpha} \sum_{k\in\widehat{\mathcal{I}}_{K+1}} L_k(x)$$

by Proposition 4.2.5 from [22] and Proposition 1.3.6 from [1] on the properties of slowly varying functions. According to the induction principle, the statement of the lemma holds for all sums Σ_n.

The statement of the lemma for $X^{(n)}$ follows immediately from the following asymptotic relations:

$$\overline{F}_{X^{(n)}}(x) = \mathbf{P}\left(\bigcup_{k=1}^{n} \{X_k > x\}\right) \sim \sum_{k=1}^{n} \overline{F}_{X_k}(x) \sim x^{-\alpha} \sum_{k\in\widehat{\mathcal{I}}_n} L_k(x)$$

for each $n \in \mathbb{N}$ by the classical Bonferoni inequalities and properties of slowly varying functions. The lemma is proved. $\square$

The following statement follows from Lemma 3.2 of [6].

Lemma 6. *Let $X_1, \dots, X_n$ be independent real-valued r.v.s, and let $F_{X_\nu} \in \mathcal{R}_\alpha$ for some $\nu \geqslant 1$ and $\alpha \geqslant 0$. Suppose, in addition, that*

$$\limsup_{x\to\infty} \frac{1}{\overline{F}_{X_\nu}(x)} \sup_{n\geqslant\nu} \frac{1}{n} \sum_{k=1}^{n} \overline{F}_{X_k}(x) < \infty.$$

Then, for any $p > \alpha$, there exists a positive constant $c = c(p)$ such that

$$\overline{F}_{S_n}(x) \leqslant c\, n^{p+1} \overline{F}_{X_\nu}(x)$$

for all $n \geqslant \nu$ and $x \geqslant 0$.

In fact, Lemma 3.2 in [6] is proved for nonnegative r.v.s, but the statement remains valid for real-valued r.v.s. To see this, it suffices to observe that $\mathbf{P}(X_1 + \dots + X_n > x) \leqslant \mathbf{P}(X_1^+ \dots + X_n^+ > x)$ and $\mathbf{P}(X_k > x) = \mathbf{P}(X_k^+ > x)$ for $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$, $x \geqslant 0$, where a^+ denotes the positive part of a.

4 Proofs of main results

In this section, we give detailed proofs of our main results.

Proof of Theorem 4. Let η be an arbitrary c.r.v., and set

$$\varkappa := \min\bigl\{n \geqslant 1\colon\ \mathbf{P}(\eta = n) > 0\bigr\}.$$

Then for any $x > 0$, we have

$$\begin{aligned}
\overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbf{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbf{P}(\eta = n) \\
&= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa)\left(1 + \sum_{n=\varkappa+1}^{\infty} \left(\prod_{k=\varkappa+1}^{n} \overline{F}_{\xi_k}(x)\right) \frac{\mathbf{P}(\eta = n)}{\mathbf{P}(\eta = \varkappa)}\right) \\
&\leqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa)\left(1 + \overline{F}_{\xi_{\varkappa+1}}(x) \frac{\mathbf{P}(\eta \geqslant \varkappa + 1)}{\mathbf{P}(\eta = \varkappa)}\right)
\end{aligned}$$

and

$$\overline{F}_{\xi_{(\eta)}}(x) \geqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbf{P}(\eta = \varkappa).$$

Therefore we obtain

$$\overline{F}_{\xi_{(\eta)}}(x) \sim \mathbf{P}(\eta = \varkappa)\,\overline{F}_{\xi_{(\varkappa)}}(x). \tag{3}$$

Necessity. If $F_{\xi_k} \in \mathcal{R}$ for all $k \in \mathbb{N}$, then $F_{\xi_1} \in \mathcal{R}_{\alpha_1}$, $F_{\xi_2} \in \mathcal{R}_{\alpha_2}$, ... for some nonnegative parameters $\alpha_1, \alpha_2, \ldots$. This means that, for each $k \in \mathbb{N}$, $\overline{F}_{\xi_k}(x) = x^{-\alpha_k} L_k(x)$ with a slowly varying function L_k. Hence, for a finite nonrandom $\varkappa$,

$$F_{\xi_{(\varkappa)}} \in \mathcal{R}_{\alpha_1+\cdots+\alpha_\varkappa} \tag{4}$$

by the closure properties of slowly varying functions (see, e.g., Proposition 1.3.6 in Bingham et al. [1]) because

$$\overline{F}_{\xi_{(\varkappa)}}(x) = \prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x) = x^{-(\alpha_1+\cdots+\alpha_\varkappa)} \prod_{k=1}^{\varkappa} L_k(x)$$

for $x > 0$. Thus, it follows from (3) and (4) that

$$F_{\xi_{(\eta)}} \in \mathcal{R}_{\alpha_1+\cdots+\alpha_\varkappa} \subset \mathcal{R}$$

for any c.r.v. η.

Sufficiency. If $F_{\xi_{(\eta)}} \in \mathcal{R}$ for an arbitrary c.r.v. η, then from (3) it follows that $F_{\xi_{(n)}} \in \mathcal{R}$ for any fixed $n \in \mathbb{N}$. In addition, for all $x > 0$, we have that $\overline{F}_{\xi_1}(x) = \overline{F}_{\xi_{(1)}}(x)$ and

$$\overline{F}_{\xi_k}(x) = \frac{\overline{F}_{\xi_{(k)}}(x)}{\overline{F}_{\xi_{(k-1)}}(x)}, \quad k \in \{2, 3, \ldots\}.$$

Therefore, by the closure properties of slowly varying functions (see, e.g., Proposition 1.3.6 in Bingham et al. [1]), we obtain that $F_{\xi_k} \in \mathcal{R}$ for each $k \in \mathbb{N}$. Theorem 4 is proved. □

Proof of Theorem 5. To verify that $F_{S_\eta} \in \mathcal{R}_\alpha$, it suffices to prove that

$$\overline{F}_{S_\eta}(x) \sim x^{-\alpha} L(x) \tag{5}$$

for some slowly varying function L.

For all $x > 0$, we have

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{m} \mathbf{P}(\eta = n)\mathbf{P}(S_n > x).$$

By Lemma 5 we conclude that for each $n \in \{1, \ldots, m\}$,

$$\overline{F}_{S_n}(x) \sim x^{-\alpha} \sum_{k \in \mathcal{I}_n} L_k(x),$$

where L_k are slowly varying functions. Asymptotic relation (1) now immediately follows.

By the closure properties of slowly varying functions (see Proposition 1.3.6 in Bingham et al. [1]) we conclude that asymptotic relation (5) holds with slowly varying function

$$L(x) = \sum_{n=1}^{m} \mathbf{P}(\eta = n) \sum_{k \in \mathcal{I}_n} L_k(x).$$

Consequently, $F_{S_\eta} \in \mathcal{R}_\alpha$.

The proof of the theorem for the d.f. $F_{\xi^{(\eta)}}$ is identical to that for F_{S_η}, and hence we omit it. The theorem is proved. □

Proof of Theorem 6. The implication (i) ⇒ (iii) immediately follows from Theorem 5.

Suppose now assumption (iii) holds, that is, $F_{\xi^{(\eta)}} \in \mathcal{R}_+$ for any c.r.v. η with finite support. From this assumption it follows that $F_{\xi^{(n)}} \in \mathcal{R}_{\alpha_n}$ for each $n \in \mathbb{N}$ with some index $\alpha_n > 0$. Applying the classical Bonferroni inequality, we obtain

$$\mathbf{P}\big(\xi^{(n)} > x\big) = \mathbf{P}\left(\bigcup_{k=1}^{n} \{\xi_k > x\}\right) \sim \sum_{k=1}^{n} \mathbf{P}(\xi_k > x).$$

Therefore the d.f. $\max\{0, 1 - \sum_{k=1}^{n} \overline{F}_k\}$ belongs to the class $\mathcal{R}_{\alpha_n}$ as well. Lemma 2 and the last asymptotic relation imply that $F_{S_n} \in \mathcal{R}_{\alpha_n}$ and

$$\mathbf{P}(S_n > x) \sim \sum_{k=1}^{n} \mathbf{P}(\xi_k > x) \sim \mathbf{P}\big(\xi^{(n)} > x\big) \tag{6}$$

for $n \in \mathbb{N}$.

Let us consider a c.r.v. η with finite support $\{0, 1, \ldots, m\}$, $m \geqslant 1$. In such a case, by the asymptotic relation (6) we have

$$\begin{aligned}\overline{F}_{S_\eta}(x) &= \sum_{n=1}^{m} \mathbf{P}(\eta = n)\mathbf{P}(S_n > x) \sim \sum_{n=1}^{m} \mathbf{P}(\eta = n)\mathbf{P}\big(\xi^{(n)} > x\big) \\ &= \overline{F}_{\xi^{(\eta)}}(x).\end{aligned}$$

Consequently, $F_{S_\eta} \in \mathcal{R}_+$ for c.r.v. η. The implication (iii) ⇒ (ii) is proved.

Finally, we give a proof of the implication (ii) ⇒ (i). Since by assumption (ii) $F_{S_\eta} \in \mathcal{R}_+$ for every c.r.v. η with finite support, it follows that

$$F_{S_n} \in \mathcal{R}_+ \tag{7}$$

for each $n \in \mathbb{N}$. In particular, $F_{\xi_1} \in \mathcal{R}_+$ and $F_{\xi_1+\xi_2} \in \mathcal{R}_+$. Lemma 3 implies that $F_{\xi_1} \in \mathcal{R}_\alpha$ and $F_{\xi_2} \in \mathcal{R}_\alpha$ for some $\alpha > 0$ because $\overline{F}_{\xi_1}(x) \asymp \overline{F}_{\xi_2}(x)$ by the conditions of the theorem.

Let us continue by induction. Suppose that $F_{\xi_1} \in \mathcal{R}_\alpha$, $F_{\xi_2} \in \mathcal{R}_\alpha$, ..., $F_{\xi_K} \in \mathcal{R}_\alpha$ with $K \geqslant 2$. Lemma 1 implies that $F_{S_K} \in \mathcal{R}_\alpha \subset \mathcal{R}_+$ and

$$\overline{F}_{S_K}(x) \sim \sum_{k=1}^{K} \overline{F}_{\xi_k}(x). \tag{8}$$

The distribution function $F_{S_{K+1}} \in \mathcal{R}_+$ because of relation (7). In addition, the conditions of the theorem imply that $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x) \asymp \overline{F}_{\xi_{K+1}}(x)$ for each $k \in \{1, \ldots, K\}$. This, together with asymptotic relation (8), implies that $\overline{F}_{S_K}(x) \asymp \overline{F}_{\xi_{K+1}}(x)$. Using Lemma 3, we obtain that $F_{\xi_{K+1}} \in \mathcal{R}_\alpha \subset \mathcal{R}_+$. Now statement (i) of Theorem 6 follows by the induction principle. This completes the proof. $\square$

Proof of Theorem 7. As in Theorem 5, it suffices to prove the tail equivalence formula (5) with some slowly varying function L.

For any $K \in \mathbb{N}$ and all $x > 0$, define the function

$$L_K^*(x) = L_1(x) \sum_{n=1}^{K} \mathbf{P}(\eta = n) \sum_{k=1}^{n} d_k.$$

In addition, for all $x > 0$, define

$$L_\infty^*(x) := \lim_{K \to \infty} L_K^*(x).$$

We begin with the existence of this limit. First, for each fixed x, the sequence $L_K^*(x)$ is nondecreasing. Second, for each fixed x, the sequence $L_K^*(x)$ has an upper bound by conditions (ii) and (iii). Indeed, condition (ii) implies that

$$\sum_{k=1}^{n} d_k \leqslant c_1 n$$

for all $n \in \mathbb{N}$ and some positive constant c_1, and condition (iii) implies that

$$\sum_{n=1}^{K} \mathbf{P}(\eta = n) \sum_{k=1}^{n} d_k \leqslant c_1 \sum_{n=1}^{\infty} n \, \mathbf{P}(\eta = n) = c_1 \mathbf{E}\eta < \infty$$

for all $K \in \mathbb{N}$.

Besides that, the function $L_\infty^*(x)$ is slowly varying. Let us prove the asymptotic relation

$$\overline{F}_{S_\eta}(x) \sim x^{-\alpha} L_\infty^*(x),$$

which is analogous to (5). For all $K \in \mathbb{N}$ and $x > 0$, denote

$$\begin{aligned}
\mathcal{J} &:= \frac{\mathbf{P}(S_\eta > x)}{L_\infty^*(x) x^{-\alpha}} \\
&= \frac{\sum_{n=1}^{K} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n)}{L_\infty^*(x) x^{-\alpha}} + \frac{\sum_{n=K+1}^{\infty} \mathbf{P}(S_n > x)\mathbf{P}(\eta = n)}{L_\infty^*(x) x^{-\alpha}} \\
&=: \mathcal{J}_1(K) + \mathcal{J}_2(K).
\end{aligned}$$

We have to prove the inequalities

$$\liminf_{x \to \infty} \mathcal{J} \geqslant 1 \quad \text{and} \quad \limsup_{x \to \infty} \mathcal{J} \leqslant 1. \tag{9}$$

Condition (ii) implies that

$$\lim_{x\to\infty}\left|\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}-d_k\right|=0$$

for each fixed k. Consequently, either $\overline{F}_{\xi_k}(x)\sim d_k\overline{F}_{\xi_1}(x)$ for positive d_k, implying that $F_{\xi_k}\in\mathcal{R}_\alpha$, or $\overline{F}_{\xi_k}(x)=o(\overline{F}_{\xi_1}(x))$. By Lemma 5, for all $n\in\mathbb{N}$, we have $F_{S_n}\in\mathcal{R}_\alpha$ and

$$\overline{F}_{S_n}(x)\sim\sum_{k\in\mathcal{I}_n}\overline{F}_{\xi_k}(x)\sim x^{-\alpha}L_1(x)\sum_{k=1}^{n}d_k.$$

From these asymptotic relations we get that

$$\liminf_{x\to\infty}\mathcal{J}_1(K)=\limsup_{x\to\infty}\mathcal{J}_1(K)=\frac{\sum_{n=1}^{K}\mathbf{P}(\eta=n)\sum_{k=1}^{n}d_k}{\sum_{n=1}^{\infty}\mathbf{P}(\eta=n)\sum_{k=1}^{n}d_k}. \tag{10}$$

Using the obvious inequality $\liminf_{x\to\infty}\mathcal{J}\geqslant\liminf_{x\to\infty}\mathcal{J}_1(K)$ and letting K tend to infinity, we derive from (10) the first inequality in (9).

Since

$$\begin{aligned}&\limsup_{x\to\infty}\sup_{n\geqslant1}\frac{1}{n\overline{F}_{\xi_1}(x)}\sum_{k=1}^{n}\overline{F}_{\xi_k}(x)\\&\quad\leqslant\limsup_{x\to\infty}\sup_{n\geqslant1}\left(\frac{1}{n}\sum_{k=1}^{n}\left|\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}-d_k\right|+\frac{1}{n}\sum_{k=1}^{n}d_k\right)<\infty\end{aligned}$$

by condition (ii) of the theorem, we can use Lemma 6 for the numerator of $\mathcal{J}_2(K)$ to obtain

$$\sum_{n=K+1}^{\infty}\mathbf{P}(S_n>x)\mathbf{P}(\eta=n)\leqslant c_2\overline{F}_{\xi_1}(x)\sum_{n=K+1}^{\infty}n^{p+1}\mathbf{P}(\eta=n)$$

with some positive constant c_2. Therefore

$$\begin{aligned}\limsup_{x\to\infty}\mathcal{J}_2(K)&\leqslant c_2\limsup_{x\to\infty}\frac{L_1(x)}{L_\infty^*(x)}\sum_{n=K+1}^{\infty}n^{p+1}\mathbf{P}(\eta=n)\\&\leqslant c_3\sum_{n=K+1}^{\infty}n^{p+1}\mathbf{P}(\eta=n)\end{aligned}$$

with some positive constant c_3.

The last inequality together with (10) implies that

$$\begin{aligned}\limsup_{x\to\infty}\mathcal{J}&\leqslant\limsup_{x\to\infty}\mathcal{J}_1(K)+\limsup_{x\to\infty}\mathcal{J}_2(K)\\&\leqslant\frac{\sum_{n=1}^{K}\mathbf{P}(\eta=n)\sum_{k=1}^{n}d_k}{\sum_{n=1}^{\infty}\mathbf{P}(\eta=n)\sum_{k=1}^{n}d_k}+c_3\mathbf{E}\big(\eta^{p+1}\mathbf{1}_{\{\eta\geqslant K+1\}}\big).\end{aligned}$$

Letting K tend to infinity, we get the second desired inequality in (9) by condition (iii) of the theorem. This completes the proof of Theorem 7. □

Proof of Theorem 8. Note that

$$\overline{F}_{\xi^{(n)}}(x) = \mathbf{P}(\xi^{(n)} > x) = \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \prod_{j=1}^{k-1} F_{\xi_j}(x)$$

for all $x > 0$ and $n \in \mathbb{N}$. Therefore

$$\begin{aligned}
\frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} &= \sum_{n=1}^{K} \mathbf{P}(\eta = n) \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x) \\
&\quad + \sum_{n=K+1}^{\infty} \mathbf{P}(\eta = n) \sum_{k=1}^{n} \left(\frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right) \prod_{j=1}^{k-1} F_{\xi_j}(x) \\
&\quad + \sum_{n=K+1}^{\infty} \mathbf{P}(\eta = n) \sum_{k=1}^{n} d_k \prod_{j=1}^{k-1} F_{\xi_j}(x) \\
&=: \mathcal{L}_1(K) + \mathcal{L}_2(K) + \mathcal{L}_3(K)
\end{aligned} \tag{11}$$

with an arbitrary $K \geqslant 2$.

For the first term, we have

$$\lim_{x\to\infty} \mathcal{L}_1(K) = \sum_{n=1}^{K} \mathbf{P}(\eta = n) \sum_{k=1}^{n} d_k \tag{12}$$

because $\lim_{x\to\infty} \overline{F}_{\xi_k}(x)/\overline{F}_{\xi_1}(x) = d_k$ for each fixed k.

In addition,

$$\left|\mathcal{L}_2(K)\right| \leqslant \sup_{n>K} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} - d_k \right| \mathbf{E}\big(\eta \mathbf{1}_{\{\eta>K\}}\big), \tag{13}$$

$$\mathcal{L}_3(K) \leqslant \mathbf{E}\left(\sum_{k=1}^{\eta} d_k \mathbf{1}_{\{\eta>K\}} \right). \tag{14}$$

Theorem 8 now follows from equalities (11), (12) and estimates (13), (14). □

5 Examples

In this section, we present two examples, which demonstrate the applicability of Theorem 7.

Example 1. Consider a counting r.v. η and a sequence of i.i.d. real-valued r.v.s $\{\xi_1, \xi_2, \ldots\}$ such that $F_{\xi_1} \in \mathcal{R}_\alpha$.

In this case, conditions (i) and (ii) of Theorem 7 are satisfied with constants $d_1 = d_2 = \cdots = 1$. Hence the theorem implies that $\overline{F}_{S_\eta}(x) \sim \mathbf{E}\eta\,\overline{F}_{\xi_1}(x)$ if $\mathbf{E}\eta^{1+p} < \infty$ for some $p > \alpha$.

Note that this example deals with the same i.i.d. r.v.s as in Theorem 1. The difference is that Theorem 7 imposes stricter conditions on the c.r.v., which are sufficient for the d.f. of the random sum to be regularly varying as well as for real valued summands

Example 2. Consider an example similar to that in [16]. Suppose that η is an arbitrary counting r.v. and $\{\xi_1, \xi_2, \ldots\}$ is a sequence of independent r.v.s distributed according to the two-sided Pareto laws

$$F_{\xi_k}(x) = \frac{a_k^-}{|x|^\alpha}\mathbf{1}_{(-\infty,-1)} + \left(1 - b_k - a_k^+\right)\mathbf{1}_{[-1,1)}(x) + \left(1 - \frac{a_k^+}{x^\alpha}\right)\mathbf{1}_{[1,\infty)}(x),$$

where $\alpha > 0$, and a_k^-, a_k^+, and b_k are nonnegative constants such that $a_k^+ > 0$ and $a_k^- + b_k + a_k^+ \leqslant 1$ for all $k \in \mathbb{N}$.

In this case, if $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n a_k^+ < \infty$ and $\mathbf{E}\eta^{1+p} < \infty$ for some $p > \alpha$, then conditions (i)–(iii) of Theorem 7 are satisfied, and

$$\overline{F}_{S_\eta}(x) \sim \frac{1}{x^\alpha}\sum_{k=1}^{\infty} a_k^+ \mathbf{P}(\eta \geqslant k).$$

Particularly, if η is distributed according to the Poisson law with parameter $\lambda > 0$ and $a_k^+ = 1/(k(k+1))$, $k \geqslant 1$, then

$$\overline{F}_{S_\eta}(x) \sim \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda} + \lambda - 1\right)x^{-\alpha}.$$

Acknowledgment. The authors are deeply grateful to the anonymous referees for careful reading and valuable comments and suggestions, which helped to improve the earlier version of the paper.

References

1. N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Cambridge Univ. Press, Cambridge, 1987.
2. J. Cai, Q. Tang, On max-sum equivalence and convolution closure of heavy-tailed distributions and their applications, *J. Appl. Probab.*, **41**:117–130, 2004.
3. Y. Chen, K.C. Yuen, Sums of pairwise quasi-asymptotically independent random variables with consistent variation, *Stoch. Models*, **25**:76–89, 2009.
4. Z. Cui, E. Omey, W. Wang, Y. Wang, Asymptotics of convolution with the semi-regular-variation tail and its application to risk, *Extremes*, **21**:509–532, 2018.
5. S. Danilenko, J. Markevičiūtė, J. Šiaulys, Randomly stopped sums with exponential-type distributions, *Nonlinear Anal. Model. Control*, **22**:793–807, 2017.

6. S. Danilenko, J. Šiaulys, Randomly stopped sums of not identically distributed heavy tailed random variables, *Stat. Probab. Lett.*, **113**:84–93, 2016.
7. S. Danilenko, J. Šiaulys, G. Stepanauskas, Closure properties of O-exponential distributions, *Stat. Probab. Lett.*, **140**:63–70, 2018.
8. P. Embrechts, C.M. Goldie, N. Veraverbeke, Subexponentiality and infinite divisibility, *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, **49**:335–347, 1979.
9. P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, New York, 1997.
10. G. Faÿ, B. González-Arévalo, T. Mikosch, G. Samorodnitsky, Modeling teletraffic arrivals by a Poisson cluster process, *Queueing Syst.*, **54**:121–140, 2006.
11. W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 2*, John Wiley & Sons, New York, 1970.
12. S. Foss, D. Korshunov, S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions,* 2nd ed., Springer, New York, 2013.
13. A. Jessen, T. Mikosch, Regularly varying functions, *Publ. Inst. Math., Nouv. Sér.*, **94**:171–192, 2006.
14. J. Karamata, Sur un mode de croissance régulière. Théorèmes fondamentaux, *Bull. Soc. Math. Fr.*, **61**:55–62, 1933.
15. R. Leipus, J. Šiaulys, On the random max-closure for heavy-tailed random variables, *Lith. Math. J.*, **57**:208–221, 2017.
16. R. Leipus, J. Šiaulys, I. Vareikaitė, Tails of higher-order moments with dominatedly varying summands, *Lith. Math. J.*, **59**:389–407, 2019.
17. T. Mikosch, Regular variation, subexponentiality and their applications in probability theory, Technical Report 99-013, University of Groningen, the Netherlands, 1999.
18. E. Omey, S. Van Gulk, R. Vesilo, Semi-heavy tails, *Lith. Math. J.*, **58**:480–499, 2018.
19. E. Omey, R. Vesilo, Random sums of random variables and vectors: Including infinite means and unequal length sums, *Lith. Math. J.*, **55**:433–450, 2015.
20. V. Paulauskas, A. Skučaitė, Some asymptotic results for one-sided large deviation probabilities, *Lith. Math. J.*, **43**:318–326, 2003.
21. O. Ragulina, J. Šiaulys, Randomly stopped minima and maxima with exponential-type distributions, *Nonlinear Anal. Model. Control*, **24**:297–313, 2018.
22. G. Samorodnitsky, *Stochastic Processes and Long Range Dependence*, Springer, New York, 2016.
23. T. Shimura, Decomposition problem of probability measures related to monotone regularly varying functions, *Nagoya Math. J.*, **135**:87–111, 1994.
24. A. Skučaitė, Large deviations for sums of independent heavy-tailed random variables, *Lith. Math. J.*, **44**:198–208, 2004.
25. D. Wang, Q. Tang, Tail probabilities of randomly weighted sums of random variables with dominated variation, *Stoch. Models*, **22**:253–272, 2006.
26. S. Wang, D. Guo, W. Wang, Closure property of consistently varying random variables based on precise large deviation principles, *Commun. Stat., Theory Methods*, **48**:2218–2228, 2019.