Front Dynamics with Delays in a Spatially Extended Bistable System: Computer Simulation

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Abstract. Front dynamics with delays in a spatially extended bistable system of the reaction-diffusion type is studied by the use of nonlinear partial differential equation (PDE) of the parabolic type. The response of the self-ordered front, joining two steady states of the different stability in the system, to the multi-harmonic (step-like) force is examined. The relaxation rate of the system, that characterizes the delayed response of the front to the alternating current (ac) drive, is found to be sensitive to the peculiarities (shape) of the rate function (nonlinearity) of the governing PDE. By using computer simulations of the drift motion of the ac driven bistable front (BF) we are able to show that the characteristic relaxation time of the system decreases with the increasing outer slope parameters of the rate function and is not sensitive to the inner one.

Keywords: dissipative systems, self-organization, spontaneously formed front-structures, nonlinear partial differential equations, mathematical physics.

1 Introduction

Continuous bistable systems driven far beyond their thermal equilibrium have been widely studied as the simplest examples of self-organization. The bistable fronts, i.e. the spontaneously formed front-structures, joining two states of different stability in a spatially extended system out of thermal equilibrium, have been widely known in physically diverse systems and have attracted increasing attention in many branches of physics, chemistry, biophysics, etc [1,2]. The prototype evolution equation that describes the self-ordered BF propagating in the ac driven system reads,

$$u_t - u_{zz} - cu_z + R(u) = f(t), \tag{1}$$

where the function $u(z,t)$ denotes the step-like field of the front propagating at the moment velocity $c(t)$, $z = x - ct$ is the traveling coordinate (t – time). The disturbing force $f(t)$ describes the action of the external fields on the system. The rate (reaction) function $R(u)$, which characterizes the rate of the transient processes in the system, is given by the N-shaped R-u dependence with three zero-points at $u = u_1$, u_2, u_3, witch are found from $R(u) = 0$. In addition, the free parameters u_i satisfy the relation $u_1 < u_2 < u_3$. In the considered case of the bistable system one has that $R'(u_{1,3}) > 0$ and $R'(u_2) < 0$, where the prime denotes the derivative. The front-solution of the free, undisturbed BF $u_0(z,t)$ is found from equation:

$$u_{0t} - u_{0zz} - cu_{0z} + R(u_0) = 0, \tag{2}$$

It should be noted that general methods for the analytical solution of the governing equation of BF are presently lacking; an analytic solution of the governing equation with an arbitrary rate function is not feasible even in the case of the free, unperturbed ($f(t)=0$) system. Free front-solutions of BFs are presently known only in few special cases of the rate function approximated by the cubic polynomial, sinus-type and piecewise linear dependencies (e.g., see [3–9]). On the other hand, the analytic treatment of the ac driven BFs requires the use of the approximate approaches. The analytic techniques that are frequently used in the studies of the ac driven BFs are of limited usefulness, namely, they involve two very special cases of the forcing function $f(t)$ that describes the slightly disturbed and the slowly (quasi-stationary) driven fronts.

Elementary self-ordered structures in bistable dissipative media simulated by both versions of the stochastic (noisy driver) and deterministic (regular driver) external forces relevant issues when researching self-organizantion phenomenon in physically diverse systems [3–8, 10, 11]. Previous results showed [7, 8] that the unforced dc motion (the ratchet-like transport) of BFs was sensitive to the symmetry properties of the considered R-u dependence. In common case of bistable dissipative media there are two types of rate functions with different symmetry properties: symmetric functions and asymmetric ones [7,8]. Control of the front-structures by symmetric rate function $R(u)$, described by the linear pieces (see Fig. 1), has been studied in the extensive literature analytically and by numerical simulations [7, 8, 12, 13].

$$R(u) = R_0(u) + C, \quad R_0(u) = \begin{cases} a_1(u+1), & u < u_M, \\ -a_2 u, & u_M < u < u_m, \\ a_3(u-1), & u > u_m, \end{cases} \tag{3}$$

where C is the free constant, a_i – the slope coefficient of the rate function, and the parameters u_M and u_m – the extremes of the rate function $R(u)$. When $a_1 = a_3$, then we have type of $symmetric$ functions, and with $asymmetric$ functions $a_1 \neq a_3$. The analytical front solutions of this pseudolinear model have been presented in [9]. In addition, in the case of symmetric rate functions, numerical methods to obtain the solution of equation (1) have been used. It has been shown [12, 13], that parameters a_1, a_3 of the rate function have influence on the dynamic properties of BF stimulated by zero-mean forces $f(t)$. The response of front-solution on the zero-mean force always delays: the response delay time depends on the parameters a_1 and a_3, which describe the rate of relaxation processes in the bistable system – the delay phenomenon reduces the effect of the soliton ratchet and at the same time, decreases the ability to control fronts with the help of zero-mean forces.

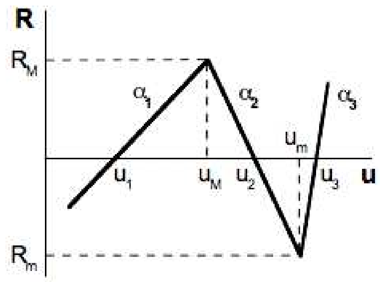

Fig. 1. The piecewise-linear rate function.

The present study investigates the dependence of the characteristic relaxation rates on the strength (magnitude) of the driving force and the parameters a_1, a_3 of the asymmetric rate function, using numerical methods to obtain the solution of equation (1) and pseudolinear “flexible” rate function (3). In this case, we approximate the driving force by the step-like forcing function

$$f(t) = F_0\Theta(t - t_0). \tag{4}$$

The retardation effects in dynamics of the ac driven BFs generated by the asymmetrical rate functions breaking the rigorous symmetry relation $a_1 = a_3$ have not been studied as yet.

2 The mathematical model

The solution of the non-linear parabolic type partial differential equation (1) is obtained in two steps. Firstly, homogeneous problem (2) is solved, when ($f(t) = 0$). The initial conditions of the homogeneous equation ($t = 0$)

$$u_0(z, t = 0) = a\ \tanh(qz) + b, \tag{5}$$

when $a = q = 0.5(u_3 - u_1), b = 0.5(u_3 + u_1)$. Consequently, the boundary conditions are $u_0(t; z \to -\infty) \to u_1$ and $u_0(t; z \to +\infty) \to u_3$.

In this step the front-solution of the free, undisturbed BF $u_0(z, t)$ are found. Secondly, the solution of homogeneous problem (2) is sustained as the initial conditions of the non-homogenous problem (1). Boundary conditions are obtained from the solution of $u_t + R(u) = f(t)$ problem with the initial conditions $V_1(t = t_0) = u_1$, $V_3(t = t_0) = u_3$, where $u_{1,3}$ are the boundary conditions of the homogeneous ($f(t) = 0$) problem.

The problem (1) with the initial condition (2) can be solved by two methods: analytical and numerical ones. The perturbation theory can be applied to the analytical solution but this method can be used only for very narrow set of the variables, therefore it can not be applied to the common case. One of the methods to solve (1) with the initial condition (2) is using numerical methods.

Seeking to obtain the results, the numerical method of the finite differences [14] was employed. This method has been chosen due to its simplicity, high calculation speed, and software unexceptionality. We use the discrete lattice having the steps Δt and Δx in the area Q, where:

$$\Delta x = X/N_x, \quad x_i = i\Delta x, \quad i = 0 \dots N_x,$$
$$\Delta t = X/N_t, \quad x_j = j\Delta t, \quad i = 0 \dots N_t.$$

The definitions of the finite differences: $u_{x_i}^{t_j} = u(x_i, t_j)$. Therefore the equation using this definition takes the form:

$$\frac{u_{x_i}^{t_{j+1}} - u_{x_i}^{t_j}}{\Delta t} - \frac{u_{x_{i+1}}^{t_j} - 2u_{x_i}^{t_j} + u_{x_{i-1}}^{t_j}}{\Delta x^2} - c\frac{u_{x_{i+1}}^{t_j} - u_{x_i}^{t_j}}{\Delta x} + R(u_{x_i}^{t_j}) = f(t_j). \tag{6}$$

3 Results and discusion

The occurrence of delays (lag time) between the driving force and the propagation velocity on the spurious drift of BF, applying the asymmetric rate function, has not been considered as yet. We will use the introduced [7,8] auxiliary parameter τ_S, which characterizes the rate of the temporal relaxation of the speed function $s(t)$ of BF being under the action of the step-like force (4). More specifically, its inverse τ_S^{-1}, indicates the steepness of the step-like s-t dependence. Using the following expression,

$$s_\tau(t) = \begin{cases} s_0, & t < t_0 \\ s_0 + \Delta_{S_\infty}\big[1 - \exp\big(-(t - t_0)/\tau_s\big)\big], & t > t_0, \end{cases} \tag{7}$$

we were able to describe with good accuracy the numerically found s-t dependencies. Both discussed s-t and s_τ-t dependencies show that the approximation (7) is good enough (see Fig. 2). This implies that the numerically found speed function $s(t)$ follows the exponential law in the case of the $asymmetric$ piecewise-linear rate function $R(u)$.

Let us turn to the "speed relaxation" and investigate it in greater detail. Namely, the response of BF to extremely fast driving force (4) was considered. The typical τ_S-a_i dependencies that have been derived by the direct (numerical) solution of the governing equation (1) are presented in Fig. 3. One can see that the presented $\tau_S - a_2$ characteristics shown by curves a, b and c demonstrate that the speed relaxation time τ_S is *independent* of the inner slope coefficient a_2. Thus, the relaxation time derived within the perturbative approach does not depend on a_2. However, the τ_S-a_3 dependencies obtained by changing slope coefficients $a_{1,2}$ are presented by dashed curves A, B, C shows that the speed relaxation time τ_S *dependents* on the outer slope coefficient a_3 (or a_1) when the *asymmetric* rate function $R(u)$ is used.

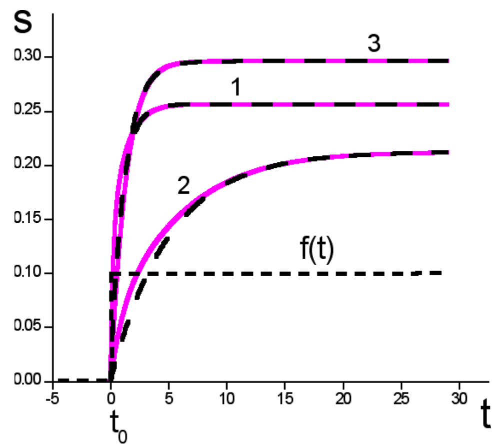

Fig. 2. The dependence of propagation velocity of the dragged front versus time. Solid curves show the actual motion described by governing equation (1), and dashed curves show s_τ-t dependence described by expression (7). The parameter values are: $a_3 = 5$, $s_\infty = 0.256,\ \tau_S = 0.93,\ \tau_R = 1$ (curves 1); $a_3 = 0.2,\ s_\infty = 0.212,\ \tau_S = 5.04,\ \tau_R = 5$ (curves 2); $a_3 = 1,\ s_\infty = 0.296,\ \tau_S = 1.23,\ \tau_R = 1$ (curves 3). The remaining parameters are as follows: $a_{1,2} = 1,\ F_0 = -0.5 f_{Mx},\ t_0 = 0,\ D_R = 1,\ g_H = 1,\ s_0 = 0.$

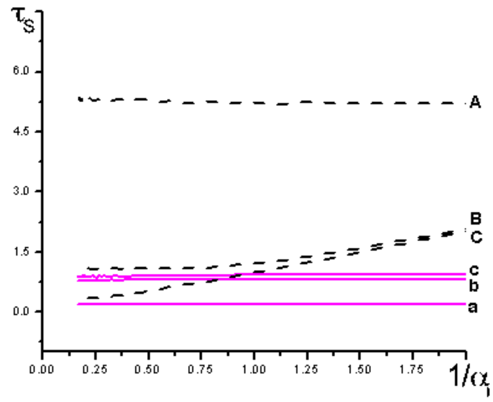

Fig. 3. The dependence of the relaxation time τ_S versus the slope coefficients of the rate function a_i. 1) The solid curves a, b, c show τ_S-a_2 dependencies derived for the different slope coefficients $a_{1,3}$. The parameter values are: $g_H = 1.2$, $F_0 = -2f_{Mx}/3$, $a_1 = 1.0$. The remaining parameters are as follows: $a_3 = 0.2$ (curve a); $a_3 = 1.0$ (curve b); $a_3 = 5.0$ (curve c). 2) The dashed curves A, B, C show τ_S-a_3 dependencies derived for the different slope coefficients $a_{1,2}$.The parameter values are: $g_H = 1.2$, $F_0 = -2f_{Mx}/3$, $a_2 = 1.0$. The remaining parameters are as follows: $a_1 = 0.2$ (curve A); $a_1 = 1.0$ (curve B); $a_1 = 5.0$ (curve C).

4 Conclusions

Front dynamics with time delays, namely, the propagation of the BF, joining two states of different stability in a bistable reaction-diffusion system under the action of fast driving was considered within the asymmetric piecewise linear model of reaction kinetics. The results presented show that (i) the numerically found front speed function follows the exponential law in the case of the $asymmetric$ piecewise rate function. (ii) The lag time that describes the size of the retardation depends on the outer slope coefficients of the $asymmetric$ piecewise linear rate function. The lag time is found to be independent of the inner slope coefficient of the rate function.

References

1. M. C. Cross, P. S. Hohenberg, Pattern formation outside the equilibrium, *Rev. Mod. Phys.*, **65**, pp. 851–1112, 1993.
2. C. Bowman, A. C. Newell, Natural patterns and wavelets, *Rev. Mod. Phys.*, **70**, pp. 289–302, 1998.
3. A. S. Mikhailov, L. Schimansky-Geier, W. Ebeling, Stochastic motion of the propagating front in bistable media, *Phys. Lett. A*, **96**, p. 453–456, 1983.
4. L. Schimansky-Geier, Ch. Zulicke, Kink propagation induced by multiplicative noise, *Z. Phys. B Con. Mat.*, **82**, pp. 157–162, 1991.
5. F. Marchesoni, Thermal ratchets in $1 + 1$ dimensions, *Phys. Rev. Lett.*, **77**, pp. 2364–2357, 1996.
6. M. G. Clerc, C. Falconi, E. Tirapegui, Additive noise induces front propagation, *Phys. Rev. Lett.*, **94**, pp. 148302-(1–4), 2005.
7. R. Bakanas, Unidirectional drift of fronts under zero-mean force and broken symmetries of the rate function, *Phys. Rev. E*, **69**, pp. 016103-(1–10), 2004.
8. R. Bakanas, Unidirectional drift of bistable front under asymmetrically oscillating zero-mean force, *Phys. Rev. E*, **71**, pp. 026201-(1–12), 2005.
9. R. Bakanas, Traveling fronts in a piecewise-linear bistable system, *Nonlinearity*, **16**, pp. 313–325, 2003.
10. P. Reimann, Brownian motors: noisy transport far from equilibrium, *Phys. Rep.*, **361**, pp. 57–265, 2002.
11. M. Salerno, Y. Zolotaryuk, Soliton ratchets, *Phys. Rev. E*, **65**, pp. 025602-(1–4), 2002.
12. A. Raguotis, F. Ivanauskas, R. Bakanas, Retarded accelerations of the self-organized front: propagation of the bistable front under step-like force, *Lith. J. Phys.*, **45**, pp. 153–160, 2005.
13. A. Raguotis, F. Ivanauskas, R. Bakanas, Retarding accelerations and ratchet-like transport of the self-ordered front in a bistable system of reaction-diffusion type, *Phys. Scripta*, **76**, pp. 629–637, 2006.
14. A. A. Samarskii, *The Theory of Difference Schemes*, Marcel Dekker, New York, 2001.