

On the Green's formula for a Stokes type problem

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Abstract. A time-periodic Stokes problem is studied in the domain with cylindrical outlets to infinity. Using the Fourier series the problem is reduced to a sequence of elliptic problem. For each of these elliptic boundary value problems a generalized Green's formula is constructed. The analogous Green's formula for the steady Stokes problem was obtained in [1].

Keywords: cylindrical outlets to infinity, time-periodic Stokes problem, generalized Green's formula.

1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be a domain with cylindrical outlets to infinity, i.e., outside the ball $B_R = \{x \in \mathbb{R}^3: |x| \leq R\}$ the domain Ω coincides with a system of J semi-infinite cylinders Π_+^j of a constant cross section ω^j . Let $\Pi_+^j \cap \Pi_+^k = \emptyset$, $j \neq k$ and let the boundary $\partial\Omega$ be smooth. We consider in Ω the time-periodic Stokes problem

$$\mathbf{v}_t - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \quad (x, t) \in \Omega \times (0, 2\pi), \quad (1)$$

$$-\nabla \cdot \mathbf{v} = \mathbf{0}, \quad (x, t) \in \Omega \times (0, 2\pi), \quad (2)$$

$$\mathbf{v} = \mathbf{0}, \quad (x, t) \in \partial\Omega \times (0, 2\pi), \quad (3)$$

$$\mathbf{v}(x, 0) = \mathbf{v}(x, 2\pi), \quad x \in \Omega. \quad (4)$$

We assume that the external force $\mathbf{f} = (f_1, f_2, f_3)^t$ is 2π -time-periodic function. Problem (1)–(4) could be decomposed into a sequence of elliptic problems. Indeed, we can look for the solution to problem (1)–(4) in the form

$$\mathbf{v}(x, t) = \frac{\mathbf{v}_{c0}}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \{\mathbf{v}_{ck}(x) \cos kt + \mathbf{v}_{sk}(x) \sin kt\}, \quad (5)$$

$$p(x, t) = \frac{p_{c0}}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \{p_{ck}(x) \cos kt + p_{sk}(x) \sin kt\}. \quad (6)$$

*This work was partly supported by the Lithuanian Science Council Student Research Fellowship Award.

Inserting series (5), (6) into equations and boundary conditions we get for coefficients $\mathbf{v}_{ck}, \mathbf{v}_{sk}, p_{ck}, p_{sk}$ series of the systems of elliptic problems

$$\begin{cases} k\mathbf{v}_{sk} - \nu\Delta\mathbf{v}_{ck} + \nabla p_{ck} = \mathbf{f}_{ck}, & x \in \Omega, \\ -k\mathbf{v}_{ck} - \nu\Delta\mathbf{v}_{sk} + \nabla p_{sk} = \mathbf{f}_{sk}, & x \in \Omega, \\ -\nabla \cdot \mathbf{v}_{ck} = 0, \quad -\nabla \cdot \mathbf{v}_{sk} = 0, & x \in \Omega, \\ \mathbf{v}_{ck} = \mathbf{0}, \quad \mathbf{v}_{sk} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \quad (7)$$

Here $\mathbf{f}_{c0}/(2\pi), \mathbf{f}_{ck}/\pi, \mathbf{f}_{sk}/\pi, k = 0, 2, \dots$, are Fourier coefficients of the function $\mathbf{f} = \mathbf{f}(x, t)$.

In this paper we derive so-called generalized Green's formula for problem (7). The analogous Green's formula for the steady Stokes problem was obtained in [1]. The obtained below results are important for the construction of correct asymptotic conditions at infinity which describe real time-periodic physical processes (for example bloodstream).

2. The asymptotics of the solution to problem (7)

Let $x^j = (x_1^j, x_2^j, x_3^j)$ be the local coordinate system related to the cylinder Π_+^j such that the axis x_3^j is directed along cylinder axis. We consider problem (7) in a weighted Sobolev space $W_\beta^l(\Omega)$ which is a closure of $C_0^\infty(\overline{\Omega})$ (a class of infinitely differentiable functions with compact supports in $\overline{\Omega}$) with respect to the norm

$$\|u; W_\beta^l(\Omega)\|^2 = \sum_{|\alpha| \leq l} \int_\Omega \rho_\beta(x) |D_x^\alpha u(x)|^2 dx,$$

where ρ_β is a smooth positive function on $\overline{\Omega}$ such that $\rho_\beta(x) = \exp(\beta x_3^j)$ on $\Pi_+^j \setminus B_R$, $j = 1, \dots, J$. If $\beta > 0$, elements of this space exponentially vanish as x_3^j tends to infinity, and they may exponentially grow, if $\beta < 0$.

Consider problem (7) in the cylinder Π_+^j . Using the methods of the book [2] and arguing in the same way as in [1] we obtain four special solutions of the homogeneous problem (7):

$$\mathbf{u}_{ck}^{j0} = (0, 0, 0, 1, 0, 0, 0, 0)^t, \quad \mathbf{u}_{ck}^{j1} = (0, 0, \varphi_k^j, x_3^j, 0, 0, -\psi_k^j, 0)^t, \quad (8)$$

$$\mathbf{u}_{sk}^{j0} = (0, 0, 0, 0, 0, 0, 0, 1)^t, \quad \mathbf{u}_{sk}^{j1} = (0, 0, \psi_k^j, 0, 0, 0, \varphi_k^j, x_3^j)^t, \quad (9)$$

where the pair of functions (φ_k^j, ψ_k^j) is the unique solution of the problem

$$\begin{cases} k\psi_k^j + \nu\Delta\varphi_k^j = 1, & x^{j'} = (x_1^j, x_2^j) \in \omega^j, \\ k\varphi_k^j - \nu\Delta\psi_k^j = 0, & x^{j'} \in \omega^j, \\ \varphi_k^j = \psi_k^j = 0, & x^{j'} \in \partial\omega^j. \end{cases} \quad (10)$$

According to Theorem 3.1.4 in [2] the sum of linear combinations of these solutions gives the main term (up to an exponentially vanishing term) of the asymptotic decom-

position of the "growing" at infinity solution. Let $\chi_j(x)$ be a smooth cut-off function such that $\text{supp}(\chi_j) \subseteq \Pi_+^j$ and $\chi_j(x) = 1$ if $x_3^j > L$ for $j = 1, \dots, J$.

THEOREM 1. *Let $\beta > 0$. If $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \mathcal{D}_{-\beta}^l W(\Omega)$ is the solution to problem (7) with the right-hand side $\mathbf{f}_k = (\mathbf{f}_{ck}, \mathbf{f}_{sk}) \in W_\beta^{l-1}(\Omega)^6$, then*

$$\mathbf{u}_k(x) = \sum_{j=1}^J \chi_j(x) \left\{ a_{ck}^j \mathbf{u}_{ck}^{j0} + a_{sk}^j \mathbf{u}_{sk}^{j0} + b_{ck}^j \mathbf{u}_{ck}^{j1} + b_{sk}^j \mathbf{u}_{sk}^{j0} \right\} + \tilde{\mathbf{u}}_k, \quad (11)$$

where $\tilde{\mathbf{u}}_k \in \mathcal{D}_\beta^l W(\Omega)$, $a_{ck}^j, a_{sk}^j, b_{ck}^j, b_{sk}^j \in \mathbb{C}$. Here $\mathcal{D}_\beta^l W(\Omega) = W_\beta^{l+1}(\Omega)^6 \times W_\beta^l(\Omega)^2$.

3. Generalized Green's formula

Let $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})$, $\mathbf{U}_k = (\mathbf{V}_{ck}, P_{ck}, \mathbf{V}_{sk}, P_{sk}) \in C_0^\infty(\overline{\Omega})$. Integrating twice by parts in Ω one gets the standard Green's formula (see [3])

$$\begin{aligned} & (-\nu \Delta \mathbf{v}_{ck} + \nabla p_{ck} + k \mathbf{v}_{sk}, \mathbf{V}_{ck})_\Omega + (-\nabla \cdot \mathbf{v}_{ck}, P_{ck})_\Omega \\ & + (-\nu \Delta \mathbf{v}_{sk} + \nabla p_{sk} - k \mathbf{v}_{ck}, \mathbf{V}_{sk})_\Omega + (-\nabla \cdot \mathbf{v}_{sk}, P_{sk})_\Omega \\ & + (\mathbf{v}_{ck}, \mathbf{n} P_{ck} - \nu \partial_{\mathbf{n}} \mathbf{V}_{ck})_{\partial\Omega} + (\mathbf{v}_{sk}, \mathbf{n} P_{sk} - \nu \partial_{\mathbf{n}} \mathbf{V}_{sk})_{\partial\Omega} \\ & - (\mathbf{v}_{ck}, -\nu \Delta \mathbf{V}_{ck} + \nabla P_{ck} - k \mathbf{V}_{sk})_\Omega - (p_{ck}, -\nabla \cdot \mathbf{V}_{ck})_\Omega \\ & - (\mathbf{v}_{sk}, -\nu \Delta \mathbf{V}_{sk} + \nabla P_{sk} + k \mathbf{V}_{ck})_\Omega - (p_{sk}, -\nabla \cdot \mathbf{V}_{sk})_\Omega \\ & - (\mathbf{n} p_{ck} - \nu \partial_{\mathbf{n}} \mathbf{v}_{ck}, \mathbf{V}_{ck})_{\partial\Omega} - (\mathbf{n} p_{sk} - \nu \partial_{\mathbf{n}} \mathbf{v}_{sk}, \mathbf{V}_{sk})_{\partial\Omega} = 0, \end{aligned} \quad (12)$$

here $(\cdot)_\Omega$ stands for a scalar product in $L_2(\Omega)$. Denoting by $q(\mathbf{u}_k, \mathbf{U}_k)$ the left-hand side of the above formula we get

$$q(\mathbf{u}, \mathbf{U}) = q(\mathbf{U}, \mathbf{u}) = 0$$

for any $\mathbf{u} \in \mathcal{D}_\beta^l W(\Omega)$ and $\mathbf{U} \in \mathcal{D}_{-\beta}^l W(\Omega)$. Let S be an operator of problem (7) and S^* be an operator of the problem

$$\begin{cases} -k \mathbf{V}_{sk} - \nu \Delta \mathbf{V}_{ck} + \nabla P_{ck} = \mathbf{F}_{ck}, & x \in \Omega, \\ k \mathbf{V}_{ck} - \nu \Delta \mathbf{V}_{sk} + \nabla P_{sk} = \mathbf{F}_{sk}, & x \in \Omega, \\ -\nabla \cdot \mathbf{V}_{ck} = 0, \quad -\nabla \cdot \mathbf{V}_{sk} = 0, & x \in \Omega, \\ \mathbf{V}_{ck} = \mathbf{0}, \quad \mathbf{V}_{sk} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \quad (13)$$

It is clear that S^* is an adjoint operator to S with respect to the Green's formula (12). Note that S is not self-adjoint operator. Homogeneous problem (13) in the cylinder Π_+^j has four special solutions

$$\mathbf{U}_{ck}^{j0} = (0, 0, 0, 1, 0, 0, 0, 0)^t, \quad \mathbf{U}_{ck}^{j1} = (0, 0, \varphi_k^j, x_3^j, 0, 0, \psi_k^j, 0)^t, \quad (14)$$

$$\mathbf{U}_{sk}^{j0} = (0, 0, 0, 0, 0, 0, 0, 1)^t, \quad \mathbf{U}_{sk}^{j1} = (0, 0, -\psi_k^j, 0, 0, 0, \varphi_k^j, x_3^j)^t, \quad (15)$$

where functions φ_k^j and ψ_k^j are defined by formula (10). We denote by $\mathbb{D}_{\pm\beta}^l W(\Omega)$ the subset of functions $\mathbf{u}_k \in \mathcal{D}_{-\beta}^l W(\Omega)$ having expansion (11) and by $\mathbb{D}_{\pm\beta}^l W(\Omega)^*$ the

subset of $\mathcal{D}_{-\beta}^l W(\Omega)$ consisting of functions having an expansions

$$\mathbf{U}_k = \sum_{j=1}^J \chi_j \left\{ A_{ck}^j \mathbf{U}_{ck}^{j0} + A_{sk}^j \mathbf{U}_{sk}^{j0} + B_{ck}^j \mathbf{U}_{ck}^{j1} + B_{sk}^j \mathbf{U}_{sk}^{j1} \right\} + \tilde{\mathbf{U}}_k, \quad (16)$$

where $\mathbf{U}_{\diamond,k}^{jh}$, $h \in \{0, 1\}$, $\diamond \in \{c, s\}$, are defined by (14) and (15), $\tilde{\mathbf{U}}_k \in \mathcal{D}_{-\beta}^l W(\Omega)$, $A_{ck}^j, A_{sk}^j, B_{ck}^j, B_{sk}^j \in \mathbb{C}$.

Since $\text{supp}(\chi_j) \cap \text{supp}(\chi_l) = \emptyset$, $j \neq l$, we have

$$q(\chi_j \mathbf{u}_{\diamond,k}^{jh}, \chi_l \mathbf{U}_{\diamond\circ,k}^{lm}) = 0, \quad h, m \in \{0, 1\}, \quad \diamond, \diamond\circ \in \{c, s\}.$$

Using the fact that functions (8), (9) and (14), (15) are exact solutions to homogeneous problems (7) and (13), respectively, we get, after cumbersome computation, that

$$q(\chi_j \mathbf{u}_{\diamond,k}^{jh}, \chi_j \mathbf{U}_{\diamond\circ,k}^{jh}) = 0, \quad h = 0, 1, \quad \diamond, \diamond\circ \in \{c, s\}.$$

Inserting representations (11) and (16) into $q(\mathbf{u}_k, \mathbf{U}_k)$ we get that a number of terms in $q(\mathbf{u}_k, \mathbf{U}_k)$ vanishes and, finally, we find

$$\begin{aligned} q(\mathbf{u}_k, \mathbf{U}_k) = & \sum_{j=1}^J \left\{ a_{ck}^j \bar{B}_{ck}^j q(\chi_j \mathbf{u}_{ck}^{j0}, \chi_j \mathbf{U}_{ck}^{j1}) + a_{sk}^j \bar{B}_{sk}^j q(\chi_j \mathbf{u}_{sk}^{j0}, \chi_j \mathbf{U}_{sk}^{j1}) \right. \\ & + a_{sk}^j \bar{B}_{ck}^j q(\chi_j \mathbf{u}_{sk}^{j0}, \chi_j \mathbf{U}_{ck}^{j1}) + a_{sk}^j \bar{B}_{sk}^j q(\chi_j \mathbf{u}_{sk}^{j0}, \chi_j \mathbf{U}_{sk}^{j1}) \\ & + b_{ck}^j \bar{A}_{ck}^j q(\chi_j \mathbf{u}_{ck}^{j1}, \chi_j \mathbf{U}_{ck}^{j0}) + b_{ck}^j \bar{A}_{sk}^j q(\chi_j \mathbf{u}_{ck}^{j1}, \chi_j \mathbf{U}_{sk}^{j0}) \\ & \left. + b_{sk}^j \bar{A}_{ck}^j q(\chi_j \mathbf{u}_{sk}^{j1}, \chi_j \mathbf{U}_{ck}^{j0}) + b_{sk}^j \bar{A}_{sk}^j q(\chi_j \mathbf{u}_{sk}^{j1}, \chi_j \mathbf{U}_{sk}^{j0}) \right\}. \end{aligned}$$

Let us calculate the term $q(\chi_j \mathbf{u}_{ck}^{j0}, \chi_j \mathbf{U}_{ck}^{j1})$. We note, firstly, that the cut-off function χ_j restricts all considerations to the cylinder Π_+^j , secondly, that $S(\chi_j \mathbf{u}_{\diamond,k}^{jh})$ and $S^*(\chi_j \mathbf{u}_{\diamond,k}^{jh})$ have compact supports. Applying the Green's formula (12) in the domain $\Omega_L = \{x \in \Omega: \text{if } x \in \Pi_+^j \text{ then } x_3^j < L, j = 1, \dots, J\}$ we get an additional integral over the cross-section ω^j . Let $\mathbf{n} = (0, 0, 1)^t$ be the outward normal to $\partial\Omega_L$ on ω^j and $\partial_3 = \partial \setminus \partial x_3^j$. Taking into account (8), (9) and (14), (15) we get

$$\begin{aligned} q(\chi_j \mathbf{u}_{ck}^{j0}, \chi_j \mathbf{U}_{ck}^{j1}) = & (\mathbf{v}_{ck}^{j0}, \mathbf{n} P_{ck}^{j1} - \nu \partial_3 \mathbf{V}_{ck}^{j1})_{\omega^j} + (\mathbf{v}_{sk}^{j0}, \mathbf{n} P_{sk}^{j1} - \nu \partial_3 \mathbf{V}_{sk}^{j1})_{\omega^j} \\ & - (\mathbf{n} p_{ck}^{j0} - \nu \partial_3 \mathbf{v}_{ck}^{j0}, \mathbf{V}_{ck}^{j1})_{\omega^j} - (\mathbf{n} p_{sk}^{j0} - \nu \partial_3 \mathbf{v}_{sk}^{j0}, \mathbf{V}_{sk}^{j1})_{\omega^j} \\ = & -(1, \varphi_k^j)_{\omega^j}. \end{aligned}$$

The rest terms in the Green's formula could be computed in the same way. Finally, we arrive at

$$q(\mathbf{u}_k, \mathbf{U}_k) = \sum_{j=1}^J \left\{ \left(b_{ck}^j \bar{A}_{ck}^j + b_{sk}^j \bar{A}_{sk}^j - a_{ck}^j \bar{B}_{ck}^j - a_{sk}^j \bar{B}_{sk}^j \right) (\varphi_k^j, 1)_{\omega^j} \right. \\ \left. + \left(a_{ck}^j \bar{B}_{sk}^j + b_{sk}^j \bar{A}_{ck}^j - b_{ck}^j \bar{A}_{sk}^j - a_{sk}^j \bar{B}_{ck}^j \right) (\psi_k^j, 1)_{\omega^j} \right\}.$$

Now we define operators $\pi_c^0, \pi_s^0, \pi_c^1, \pi_s^1: \mathbb{D}_{\pm\beta}^j W(\Omega) \rightarrow \mathbb{C}^J$ (operators $\pi_c^0, \pi_s^0, \pi_c^1, \pi_s^1: \mathbb{D}_{\pm\beta}^j W(\Omega)^* \rightarrow \mathbb{C}^J$ are defined in the same way) as follows:

$$\pi_c^0 \mathbf{u} = (a_c^1, a_c^2, \dots, a_c^J)^t, \quad \pi_s^0 \mathbf{u} = (a_s^1, a_s^2, \dots, a_s^J)^t, \\ \pi_c^1 \mathbf{u} = (b_c^1, b_c^2, \dots, b_c^J)^t, \quad \pi_s^1 \mathbf{u} = (b_s^1, b_s^2, \dots, b_s^J)^t,$$

where the numbers $a_c^j, a_s^j, b_c^j, b_s^j$ are the coefficients in expansion (11) of the function $\mathbf{u} \in \mathbb{D}_{\pm\beta}^j W(\Omega)$ (in expansion (16) for $\mathbf{U} \in \mathbb{D}_{\pm\beta}^j W(\Omega)^*$). Let

$$c_k^j = \int_{\omega^j} \varphi_k^j dx^{j'}, \quad d_k^j = - \int_{\omega^j} \psi_k^j dx^{j'}, \quad x^{j'} = (x_1^j, x_2^j),$$

and

$$\mathcal{C}_k = \text{diag}\{c_k^1, c_k^2, \dots, c_k^J\}, \quad \mathcal{D}_k = \text{diag}\{d_k^1, d_k^2, \dots, d_k^J\}$$

be the $J \times J$ matrices. Taking into account previous results and notations we get the following formula

$$q(\mathbf{u}_k, \mathbf{U}_k) = \langle \mathcal{C}_k \pi_c^1 \mathbf{u}_k - \mathcal{D}_k \pi_s^1 \mathbf{u}_k, \pi_c^0 \mathbf{U}_k \rangle_J + \langle \mathcal{C}_k \pi_s^1 \mathbf{u}_k + \mathcal{D}_k \pi_c^1 \mathbf{u}_k, \pi_s^0 \mathbf{U}_k \rangle_J \\ - \langle \pi_c^0 \mathbf{u}_k, \mathcal{C}_k \pi_c^1 \mathbf{U}_k + \mathcal{D}_k \pi_s^1 \mathbf{U}_k \rangle_J - \langle \pi_s^0 \mathbf{u}_k, \mathcal{C}_k \pi_s^1 \mathbf{U}_k - \mathcal{D}_k \pi_c^1 \mathbf{U}_k \rangle_J, \quad (17)$$

where $\langle \cdot, \cdot \rangle_J$ stands for a scalar product in \mathbb{C}^J . We call (17) the *generalized Green's formula*.

References

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REZIUMĖ

M. Skujus. Apie Gryno formulę vienam Stokso tipo uždaviniui

Laiko atžvilgiu periodinis Stokso uždavinys begalinių cilindrių sistemoje Furjė eilučių pagalba suvedamas į elipsinių uždavinių seką. Šiems Stokso tipo kraštiniais uždaviniams įvedama apibendrintoji Gryno formulė.

Raktiniai žodžiai: begalinių cilindrių sistema, laiko atžvilgiu periodinis Stokso uždavinys, apibendrintoji Gryno formulė.