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Cut free sequent calculus for logic $\mathrm{S5}_n$(ED)

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Abstract. Hilbert style, Gentzen style sequent and Kanger style sequent calculi for logic $\mathrm{S5}_n$(ED) are considered in this paper. Gentzen style sequent calculus is constructed and its equivalence with Hilbert style system is proved, getting soundness and completeness of Gentzen style system. Kanger style indexed sequent calculus is defined for cut elimination.

Keywords: epistemic logic, distributed knowledge, knowledge of everybody, sequent calculus, cut elimination.

Introduction

$\mathrm{S5}_n$(ED) is one of the epistemic logics. Epistemic logic is the logic of knowledge and belief. It provides insight into the properties of individual agents and means to model complicated scenarios involving groups of agents. In addition to its relevance for traditional philosophical problems, epistemic logic has many applications in computer science and artificial intelligence. It can be used for distributed systems, knowledge base merging, robotics or network security.

Applying logic to computer science, it is important to have mechanical proof system of the logic. Suggestions of proof theory of epistemic logic are usually limited to Hilbert style axiomatizations [2]. Several works were done by Regimantas Pliuškevičius and Aida Pliuškevičienė in [7], and Raul Hakli and Sara Negri in [2], proposing cut free Gentzen style sequent calculi for $\mathrm{S4}_n$(D) and $\mathrm{S5}_n$(D). Aida Pliuškevičienė in [6] presented indexed sequent calculus, which is known as Kanger style, for logic $\mathrm{S5}_2$. Our work is strongly related to [6] and describes Kanger style sequent calculus for logic $\mathrm{S5}_n$(ED).

Firstly, we define syntax of $\mathrm{S5}_n$(ED) in Section 1, then prove equivalence between Hilbert style calculus HS-$\mathrm{S5}_n$(ED) and Gentzen style sequent calculus GS-$\mathrm{S5}_n$(ED) in Sections 2 and 3, to get soundness and completeness of GS-$\mathrm{S5}_n$(ED). Finally, cut free Kanger style sequent calculus is presented in Section 4.

1 Syntax of logic $\mathrm{S5}_n$(ED)

The language of $\mathrm{S5}_n$(ED) contains:

- Symbols of atomic propositions: $p_1, p_2, p_3, q_1, q_2, q_3, \dots;$
- Symbols of formulas: $A_1, A_2, A_3, B_1, B_2, \dots;$
- Logical connectives: $\wedge, \vee, \neg, \rightarrow;$

- Constants of agents: $i_1, i_2, i_3, \ldots, i_n$;
- Operators: $\mathrm{E}, \mathrm{D}, \mathrm{K}_1, \mathrm{K}_2, \ldots, \mathrm{K}_n$.

Formula $\mathrm{K}_i A$ with knowledge operator K_i means "agent i knows A". Operator E signifies "everybody knows" and $\mathrm{E}A$ is defined as the conjuction $\mathrm{K}_1 A \wedge \ldots \wedge \mathrm{K}_n A$, expressing "everybody knows that A". Distributed knowledge A of within a group of agents is signed $\mathrm{D}A$ and means that A follows from what the members of group of agents individually know. For instance, having the group of three agents of which the first one knows B, the second one knows $B \to C$, and the third one knows $B \wedge C \to A$, A is distributed knowledge in this group.

2 Hilbert style calculus for logic $S5_n$(ED)

W. van der Hoek and J.J.-Ch. Meyer defined Hilbert style calculus for logic $S5_n$(EDC) in [8]. We'll use the subsystem of this calculus, selecting axioms and rules related to operators K_i, E and D:

- Any axiomatization for propositional logic;
- Axioms for operator K_i:

  K1. $(\mathrm{K}_i A \wedge \mathrm{K}_i(A \to B)) \to \mathrm{K}_i B$,

  K2. $\mathrm{K}_i A \to A$,

  K3. $\mathrm{K}_i A \to \mathrm{K}_i \mathrm{K}_i A$,

  K4. $\neg \mathrm{K}_i A \to \mathrm{K}_i \neg \mathrm{K}_i A$;

- Axioms for operator E:

  E1. $(\mathrm{K}_1 A \wedge ... \wedge \mathrm{K}_n A) \to \mathrm{E}A$,

  E2. $\mathrm{E}A \to (\mathrm{K}_1 A \wedge ... \wedge \mathrm{K}_n A)$;

- Axioms for operator D:

  D1. $\mathrm{K}_i A \to \mathrm{D}A$,

  D2. $(\mathrm{D}A \wedge \mathrm{D}(A \to B)) \to \mathrm{D}B$,

  D3. $\mathrm{D}A \to A$,

  D4. $\mathrm{D}A \to \mathrm{DD}A$,

  D5. $\neg \mathrm{D}A \to \mathrm{D}\neg \mathrm{D}A$;

- Rules:

$$\frac{A, A \to B}{B}\ (\mathrm{R1}) \qquad \frac{A}{\mathrm{K}_i A}\ (\mathrm{R2}) \qquad \frac{A}{\mathrm{D}A}\ (\mathrm{R3}).$$

W. van der Hoek and J.J.-Ch. Meyer have proved that HS-$S5_n$(EDC) system is sound and complete. Therefore we get soundness and completeness for HS-$S5_n$(ED).

3 Gentzen style sequent calculus for logic $S5_n$(ED)

Regimantas Pliuškevičius and Aida Pliuškevičienė considered Gentzen style sequent calculus for $S4_n$(D) in [7]. Masao Ohnishi and Kazuo Matsumoto suggested Gentzen style sequent system for $S5_n$ in [4] and [5]. We'll use these ideas to construct Gentzen style sequent system GS-$S5_n$(ED) for logic $S5_n$(ED):

- Axiom: $\Gamma, A \vdash \Delta, A$;
- Inference rules of the sequent calculus G for propositional logic [1] (Jean H. Gallier, 2003);
- Inference rules for operators K_i, E, D:

$$\frac{\Gamma, A, K_i A \vdash \Delta}{\Gamma, K_i A \vdash \Delta}\ (K_i \vdash), \qquad \frac{K_i \Gamma_2 \vdash K_i \Delta_2, A}{\Gamma_1, K_i \Gamma_2 \vdash K_i \Delta_2, K_i A, \Delta_1}\ (\vdash K_i),$$

$$\frac{\Gamma, K_1 A \wedge \ldots \wedge K_n A \vdash \Delta}{\Gamma, EA \vdash \Delta}\ (E \vdash), \qquad \frac{\Gamma \vdash K_1 A \wedge \ldots \wedge K_n A, \Delta}{\Gamma \vdash EA, \Delta}\ (\vdash E),$$

$$\frac{\Gamma, A, DA \vdash \Delta}{\Gamma, DA \vdash \Delta}\ (D \vdash), \qquad \frac{D\Gamma_2 \vdash D\Delta_2, A}{\Gamma_1, D\Gamma_2 \vdash D\Delta_2, DA, \Delta_1}\ (\vdash D),$$

$$\frac{\Gamma_2 \vdash A}{\Gamma_1, K\Gamma_2 \vdash \Delta, DA}\ (\text{Interaction});$$

- Structural rules and cut rule:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\text{Weakening} \vdash), \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (\vdash \text{Weakening}),$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\text{Contraction} \vdash), \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (\vdash \text{Contraction}),$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\text{Cut})$$

where Γ and Δ are multisets of formulas. $K_i\Gamma_2$ and $K_i\Delta_2$ are empty sets or consists of formulas $K_iA_1, \ldots, K_iA_n$ ($n \geqslant 1$). Analogously, $D\Gamma_2$ and $D\Delta_2$ are empty sets or $DA_1, \ldots, DA_n$ ($n \geqslant 1$). $K\Gamma$ means empty set or multiset of formulas $K_1A_{1,1}, \ldots, K_1A_{1,l_1}, \ldots, K_nA_{n,1}, \ldots, K_nA_{n,l_n}$ ($n \geqslant 1$).

To obtain soundness and completeness for system GS-$S5_n$(ED), the equivalence between GS-$S5_n$(ED) and HS-$S5_n$(ED) must be proved.

Definition 1 [Formula of a sequent]. If S is a sequent $A_1, \ldots, A_l \vdash B_1, \ldots, B_k$ then *formula of a sequent* is $F(S) = (A_1 \wedge \ldots \wedge A_l) \rightarrow (B_1 \vee \ldots \vee B_k)$.

Lemma 1 [Equivalence between GS-$S5_n$(ED) and HS-$S5_n$(ED)]. *Sequent S is provable in* GS-$S5_n$(ED) *if and only if $F(S)$ is provable in* HS-$S5_n$(ED).

Proof. "⇒". We'll use induction on number of steps N of proof of $F(S)$ in HS-$S5_n$(ED). Step is defined as usage of axiom or rule.

- Case $N = 1$: One of the axiom was used.
  - Let axiom K3 was used. Then its proof in GS-S5$_n$(ED) is:

$$\dfrac{\dfrac{\mathrm{K}_i A \vdash \mathrm{K}_i A}{\mathrm{K}_i A \vdash \mathrm{K}_i \mathrm{K}_i A}\ (\vdash \mathrm{K}_i)}{\vdash \mathrm{K}_i A \to \mathrm{K}_i \mathrm{K}_i A}\ (\vdash\to)$$

  - Other axioms are proved analogously.

- Case $N > 1$: One of the rule R1, R2 or R3 was applied on the last step.
  - Let rule R2 was used. By induction hypothesis sequence $\vdash A$ is provable in GS-S5$_n$(ED). Then conclusion $\mathrm{K}_i A$ of the rule R2 is provable in such a way:

$$\dfrac{\vdash A}{\vdash \mathrm{K}_i A}\ (\vdash \mathrm{K}_i)$$

  - R1 and R3 rules are considered similarly.

"$\Leftarrow$". Analogously as in [7] for logic S4$_n$(D).

4 Kanger style sequent calculus for logic S5$_n$(ED)

Stig Kanger proposed to use indexes to get cut free sequent calculus for S5 [3]. Kanger style index is defined as natural number. In [6] Aida Pliuškevičienė suggested to use ordered pair $\langle i, l\rangle$, where $i \in \{1, 2, \ldots, n\}$ and l is either zero or an arbitrary natural number, to get cut free sequent calculus for S5$_2$.

Let p be a propositional symbol, then p^{Index} is an indexed propositional symbol. If *Index* is any index, then indexation procedure of formula in S5$_n$(ED) is as follows:

- $(p^{\langle i,r\rangle})^{Index} = p^{\langle i,r\rangle, Index}$
- $(A \wedge B)^{Index} = A^{Index} \wedge B^{Index}$, $(A \vee B)^{Index} = A^{Index} \vee B^{Index}$, $(A \to B)^{Index} = A^{Index} \to B^{Index}$, $(\neg A)^{Index} = \neg(A)^{Index}$
- $(\mathrm{K}_i A)^{Index} = \mathrm{K}_i(A)^{Index}$, $(\mathrm{E}A)^{Index} = \mathrm{E}(A)^{Index}$, $(\mathrm{D}A)^{Index} = \mathrm{D}(A)^{Index}$

It is assumed that $p^{Index_1, \langle i,0\rangle, Index_2} = p^{Index_1, Index_2}$ and $p^{Index_1, \langle i,l\rangle, \langle i,r\rangle, Index_2} = p^{Index_1, \langle i,r\rangle, Index_2}$.

Kanger style calculus KS-S5$_n$(ED) for logic S5$_n$(ED) is as follows:

- Axiom:
  - $\Gamma, A \vdash \Delta, A$,
  - $\Gamma, A^{Index} \vdash \Delta, A^{Index}$,
  - $\Gamma, A^{\langle i,r\rangle} \vdash \Delta, A^{\langle G,r\rangle}$;
- Inference rules of the sequent calculus G for propositional logic [1];

- Inference rules for operators K_i, E, D:

$$\frac{\Gamma, A^{\langle i,\alpha\rangle}, \mathrm{K}_i A \vdash \Delta}{\Gamma, \mathrm{K}_i A \vdash \Delta}\ (\mathrm{K}_i \vdash), \qquad \frac{\Gamma \vdash \Delta, A^{\langle i,r\rangle}}{\Gamma \vdash \Delta, \mathrm{K}_i A}\ (\vdash \mathrm{K}_i),$$

$$\frac{\Gamma, \mathrm{K}_1 A \wedge \ldots \wedge \mathrm{K}_n A \vdash \Delta}{\Gamma, \mathrm{E}A \vdash \Delta}\ (\mathrm{E} \vdash), \qquad \frac{\Gamma \vdash \mathrm{K}_1 A \wedge \ldots \wedge \mathrm{K}_n A, \Delta}{\Gamma \vdash \mathrm{E}A, \Delta}\ (\vdash \mathrm{E}),$$

$$\frac{\Gamma, A^{\langle G,\alpha\rangle}, \mathrm{D}A \vdash \Delta}{\Gamma, \mathrm{D}A \vdash \Delta}\ (\mathrm{D} \vdash), \qquad \frac{\Gamma \vdash \Delta, A^{\langle G,r\rangle}}{\Gamma \vdash \Delta, \mathrm{D}A}\ (\vdash \mathrm{D}),$$

$$\frac{\Gamma_2 \vdash A^{\langle G,r\rangle}}{\Gamma_1, \mathrm{K}\Gamma_2 \vdash \Delta, \mathrm{D}A}\ (\text{Interaction}).$$

A natural number $r \in \mathrm{N}$ is such that any index in the conclusion does not contain a pair $\langle i, r\rangle$. A value of the metavariable α is either zero or natural number defined as follows: if any index in the conclusion does not contain a pair of the shape $\langle i, b\rangle$ where b is an arbitrary natural number, then α is zero, otherwise value of metavariable α is natural number $l \in \mathrm{N}$ such that $\langle i, l\rangle$ enters in some index of the conclusion. Index G means all indexes of agents in a group. We consider only one group of agents, so there is only one special index G.

As in [6] the following theorem can be proved.

Theorem 1. *The cut rule is admissible in* KS-S5$_n$(ED)*.*

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REZIUMĖ

Sekvencinis skaičiavimas be pjūvio taisyklės logikai $S5_n$(ED)

H. Giedra

Darbe nagrinėjami Hilbert'o tipo, Gentzen'o tipo sekvencinis ir Kanger'io tipo sekvencinis skaičiavimai žinių logikai $S5_n$(ED). Pateikiamas Gentzen'o tipo sekvencinis skaičiavimas ir įrodomas jo ekvivalentumas su Hilbert'o tipo skaičiavimu, gaunant Gentzen'o tipo skaičiavimo pagrįstumą ir pilnumą. Suformuojamas indeksinis Kanger'io tipo sekvencinis skaičiavimas be pjūvio taisyklės.

Raktiniai žodžiai: žinių logika, paskirstytas žinojimas, visuotinis žinojimas, sekvencinis skaičiavimas, pjūvio pašalinamumas.