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Some estimates of the normal approximation for mixture of Poisson and gamma random variables*

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Abstract. In the paper, we present the upper bound of L_p norms Δ_p of the order $(a_1 + a_2)/(\mathbb{D}Z)^{-1/2}$ for all $1 \leqslant p \leqslant \infty$, of the normal approximation for a standardized random variable $(Z - \mathbb{E}Z)/\sqrt{\mathbb{D}Z}$, where the random variable $Z = a_1X + a_2Y$, $a_1 + a_2 = 1$, $a_i \geqslant 0$, $i = 1, 2$, the random variable X is distributed by the Poisson distribution with the parameter $\lambda > 0$, and the random variable Y by the standard gamma distribution $\Gamma(\alpha, 0, 1)$ with the parameter $\alpha > 0$.

Keywords: normal approximation, L_p norms, Poisson distribution, gamma distribution, mixture of Poisson and gamma r.v.

1 Introduction

Let the random variable (r.v.) X be distributed by the Poisson distribution with the parameter $\lambda > 0$ (for short, $X \sim \mathcal{P}(\lambda)$),

$$\mathbb{P}\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots,$$

and the r.v. Y by the standard gamma distribution with the parameter $\alpha > 0$ (for short, $Y \sim \Gamma(\alpha, 0, 1)$), i.e., its probability density function has the form [1, p. 180]

$$f_Y(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \cdot 1_{(0,\infty)}(x),$$

where $\Gamma(\alpha)$ is the gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, and 1_A is the indicator of event A.

Assume that the r.v.'s X and Y are independent and consider a mixture of r.v.

$$Z = a_1X + a_2Y, \quad \text{where } a_1 + a_2 = 1,\ a_i \geqslant 0,\ i = 1, 2.$$

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Denote

$$\Delta(x) = \mathbb{P}\{\xi < x\} - \Phi(x), \qquad \xi = \frac{Z - \mathbb{E}Z}{\sqrt{\mathbb{D}Z}}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

$$\Delta_p = \begin{cases} (\int_{-\infty}^{\infty} |\Delta(x)|^p\, dx)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup_{x\in\mathbb{R}} |\Delta(x)| & \text{if } p = \infty. \end{cases}$$

Here and in what follows $\mathbb{R}$ is the real line.

It is easy to proove that the distribution function of the standardized Poisson r.v. $\frac{X-\mathbb{E}X}{\sqrt{\mathbb{D}X}}$, where $X \sim \mathcal{P}(\lambda)$, and the standardized gamma r.v. $\frac{Y-\mathbb{E}Y}{\sqrt{\mathbb{D}Y}}$, where $Y \sim \Gamma(\alpha, 0, 1)$, as $\mathbb{D}X \to \infty$ and $\mathbb{D}Y \to \infty$ respectively, converges to the standard normal distribution function $\Phi(x)$, i.e.,

$$\lim_{\mathbb{D}X\to\infty} \mathbb{P}\left\{\frac{X - \mathbb{E}X}{\sqrt{\mathbb{D}X}} < x\right\} = \lim_{\mathbb{D}Y\to\infty} \mathbb{P}\left\{\frac{Y - \mathbb{E}Y}{\sqrt{\mathbb{D}Y}} < x\right\} = \Phi(x), \quad x \in \mathbb{R}. \tag{1}$$

In this paper we are interested in the rate of convergence of the L_p norm Δ_p for all $1 \leqslant p \leqslant \infty$. However, in this case, the author has not found any published results on the rates of convergence of the norms Δ_p for all $1 \leqslant p \leqslant \infty$. We have obtained here the upper bound of the norms Δ_p of the order $(a_1 + a_2)/\sqrt{a_1^2\lambda + a_2^2\alpha}$ for all $1 \leqslant p \leqslant \infty$ with explicit constants (see Theorem 1). Obviously, these constants are not the best possible, but that was not the main author's aim.

To obtain the upper estimates of the norm Δ_∞ (for uniform metric) and the norm Δ_1 (for L_1), we formed linear differential equation from the characteristic function of the standardized r.v. $\xi = \frac{Z-\mathbb{E}Z}{\sqrt{\mathbb{D}Z}} = \frac{a_1(X-\lambda)+a_2(Y-\alpha)}{\sqrt{a_1^2\lambda+a_2^2\alpha}}$ by virtue of which we succeeded in getting proper estimates of differences: between this characteristic function and the normal one, and between their derivatives as well. The chosen proofs of estimates for the L_p norms are elementary.

Particular cases $a_1 = 0$ (for a standardized gamma r.v. $\xi = \frac{Y-\alpha)}{\sqrt{\alpha}}$) and $a_2 = 0$ (for a standardized Poisson r.v. $\xi = \frac{X-\lambda}{\sqrt{\lambda}}$) are investigated in the paper [9].

2 Main and auxiliary results

Now we formulate the main result.

Theorem 1. *Let the r.v. X be distributed by the Poisson distribution with the parameter $\lambda > 0$, the r.v. Y by the standard gamma distribution with the parameter $\alpha > 0$, and r.v.'s X and Y be independent. Let*

$$Z = a_1X + a_2Y, \quad \text{where } a_1 + a_2 = 1,\ a_i \geqslant 0,\ i = 1, 2.$$

Then, for all $1 \leqslant p \leqslant \infty$,

$$\Delta_\infty \leqslant \frac{7a_1 + 18a_2}{\sqrt{a_1^2\lambda + a_2^2\alpha}}, \tag{2}$$

$$\Delta_p \leqslant \frac{71a_1 + 189a_2}{\sqrt{a_1^2\lambda + a_2^2\alpha}}. \tag{3}$$

Recall that $\mathbb{E}X = \mathbb{D}X = \lambda$ for the r.v. $X \sim \mathcal{P}(\lambda)$ and $\mathbb{E}Y = \mathbb{D}Y = \alpha$ for the r.v. $Y \sim \mathit{\Gamma}(\alpha, 0, 1)$.

Denote the characteristic function of the standardized r.v. $\xi = \frac{Z-\mathbb{E}Z}{\sqrt{\mathbb{D}Z}}$ by $f(t) = \mathbb{E}e^{it\xi}$, and the derivative of the characteristic function $f(t)$ with respect to t by $f'(t)$.

To prove Theorem 1, we use an auxiliary result, Lemma 2, on the behaviour of the functions $f(t)$ and $f'(t)$.

Denote by θ_1, θ_2, θ_3, θ_4 complex functions such that all $|\theta_i| \leqslant 1$.

The following statement is valid.

Lemma 1. *Let the r.v. X be distributed by the Poisson distribution with the parameter $\lambda > 0$, the r.v. Y by the standard gamma distribution with the parameter $\alpha > 0$, and r.v.'s X and Y be independent. Let*

$$Z = a_1X + a_2Y, \quad \text{where } a_1 + a_2 = 1,\ a_i \geqslant 0,\ i = 1, 2.$$

Denote

$$b_1 = \frac{a_1}{\sqrt{a_1^2\lambda + a_2^2\alpha}}, \qquad b_2 = \frac{a_2}{\sqrt{a_1^2\lambda + a_2^2\alpha}}, \qquad c = 1.5b_1 + 4b_2.$$

Then the characteristic function $f(t)$ of the standardized r.v. $\frac{Z-\mathbb{E}Z}{\sqrt{\mathbb{D}Z}}$ satisfies the following homogeneous linear differential equation for all $|t| \leqslant \frac{1}{2b_2}$:

$$f'(t) = \bigl(-t + \theta_1 ct^2\bigr) f(t). \tag{4}$$

Moreover, for all $|t| \leqslant \frac{1}{c}$

$$\bigl|f(t) - e^{-t^2/2}\bigr| \leqslant \frac{1}{3}c|t|^3 e^{-t^2/6}, \tag{5}$$

$$\bigl|f'(t) - \bigl(e^{-t^2/2}\bigr)'\bigr| \leqslant ct^2 e^{-t^2/2} + \frac{1}{3}c\bigl(1 + c|t|\bigr)t^4 e^{-t^2/6}. \tag{6}$$

Proof. The characteristic functions of independent r.v.'s $X - \mathbb{E}X$ and $Y - \mathbb{E}Y$ are as follows:

$$\mathbb{E}e^{it(X-\mathbb{E}X)} = \exp\bigl\{\lambda(e^{it} - 1 - it)\bigr\}, \qquad \mathbb{E}e^{it(Y-\mathbb{E}Y)} = \frac{e^{-it\alpha}}{(1-it)^\alpha}.$$

Therefore

$$f(t) = \mathbb{E}e^{i(tb_1)(X-\lambda)} \cdot \mathbb{E}e^{i(tb_2)(Y-\alpha)} = \frac{\exp\{\lambda(e^{itb_1} - 1 - itb_1) - itb_2\alpha\}}{(1 - itb_2)^\alpha}.$$

Taking the derivatives with respect to t on both sides of this expression, we get that for all $t \in \mathbb{R}$

$$f'(t) = \frac{(\lambda b_1 - it\lambda b_1 b_2)(1 - e^{itb_1}) - it\alpha b_2^2}{tb_2 + i} \cdot f(t) = fr \cdot f(t), \tag{7}$$

where fr denotes the fraction in (7). Since $|e^{ix} - 1 - ix| \leqslant \frac{1}{2}x^2$ for all $x \in \mathbb{R}$, and $\lambda b_1^2 + \alpha b_2^2 = 1$, we can rewrite the fraction in (7) in the form

$$fr = -t + \frac{t^2 b_2 - \lambda t^2 b_1^2(b_2 + \theta_2 \frac{1}{2} b_1) + \theta_3 \frac{1}{2}\lambda t^3 b_1^3 b_2}{tb_2 + i} = -t + K, \tag{8}$$

where K denotes the fraction in (8). Using the fact that $\lambda b_1^2 \leqslant 1$ and $|tb_2 + i| \geqslant \frac{1}{2}$ for all $|t| \leqslant \frac{1}{2b_2}$, we have that

$$|K| \leqslant \left(\frac{3}{2}b_1 + 4b_2\right)t^2. \tag{9}$$

Substituting (9) into (8), and afterwards substituting (8) into (7), we get (4).

Now, solving the linear differential equation (4) with the boundary condition $f(0) = 1$, we get that the characteristic function $f(t)$ may be written in the form

$$f(t) = \exp\left\{ -\frac{t^2}{2} + \theta_4 \frac{1}{3} c|t|^3 \right\} \tag{10}$$

for all $|t| \leqslant \frac{1}{2b_2}$.

To estimate the difference $|f(t) - e^{-t^2/2}|$, we use the well-known fact that $|e^z - 1| \leqslant |z|e^{|z|}$ for all complex numbers z. We obtain that for all $|t| \leqslant \frac{1}{c}$

$$\left|f(t) - e^{-t^2/2}\right| \leqslant \frac{1}{3} c|t|^3 e^{-t^2/6},$$

i.e., (5) is proved.

Substituting (5) into (4), we get (6).

Lemma 1 is proved.

3 Proof of Theorem 1

Estimation of $\varDelta_\infty$. To estimate the uniform metric $\varDelta_\infty$, we use the smoothing inequality of Esséen [5, p. 297] with $T = \frac{1}{c} > 0$ and (5), and obtain that

$$\varDelta_\infty \leqslant \frac{2}{\pi} \int_0^T \left| \frac{f(t) - e^{-t^2/2}}{t} \right| dt + \frac{24}{\pi\sqrt{2\pi}} \frac{1}{T} \leqslant \left(\frac{12}{\pi} \sqrt{\frac{2}{\pi}} + \sqrt{\frac{6}{\pi}} \right) c. \tag{11}$$

Estimation of $\varDelta_1$. To estimate the L_1 norm $\varDelta_1$, we use the following inequality with $T = \frac{1}{c} \geqslant 1$ ([4, p. 25] and [6, p. 395]):

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\mathbb{P}\{\xi < x\} - \Phi(x)\right| dx &\leqslant 3 \left(\int_0^T \left| \frac{f(t) - e^{-t^2/2}}{t} \right|^2 dt \right)^{1/2} \\
&\quad + \sqrt{2} \left(\int_0^T \left| \frac{d}{dt} \left(\frac{f(t) - e^{-t^2/2}}{t} \right) \right|^2 dt \right)^{1/2} + \frac{8\pi}{T} \\
&\leqslant 3I_1 + 2(I_2 + I_3) + \frac{8\pi}{T},
\end{aligned} \tag{12}$$

where

$$I_1^2 = \int_0^T \left| \frac{f(t) - e^{-t^2/2}}{t} \right|^2 dt, \qquad I_2^2 = \int_0^T \left| \frac{f'(t) - (e^{-t^2/2})'}{t} \right|^2 dt,$$
$$I_3^2 = \int_0^T \left| \frac{f(t) - e^{-t^2/2}}{t^2} \right|^2 dt.$$

Using inequalities (5) and (6), we estimate the quantities I_1, I_2, and I_3 from (12) with $T = \frac{1}{c} \geqslant (0.03)^{-1}$, and obtain that

$$I_1 \leqslant \frac{1}{2}\sqrt{\frac{3}{2}\sqrt{3\pi}} \cdot c, \qquad I_2 \leqslant \sqrt{25.551\sqrt{3\pi} + \frac{3}{4}\sqrt{\pi}} \cdot c, \qquad I_3 \leqslant \frac{1}{2}\sqrt{\frac{1}{3}\sqrt{3\pi}} \cdot c.$$

Substituting these estimates into (12), we have that, for $T = \frac{1}{c} \geqslant (0.03)^{-1}$,

$$\Delta_1 \leqslant 47.226c. \tag{13}$$

The proof of Theorem 1 for $T = \frac{1}{c} \geqslant (0.03)^{-1}$ now follows from (11) and (13), because

$$\Delta_p \leqslant \Delta_\infty^{(p-1)/p} \Delta_1^{1/p}$$

for all $1 \leqslant p < \infty$. The proof as $T = \frac{1}{c} < (0.03)^{-1}$ is trivial, since $\Delta_p \leqslant \sqrt{2}$ for all $1 \leqslant p \leqslant \infty$ (for $\Delta_1 \leqslant \sqrt{2}$, see [**3**, p. 528]).

Theorem 1 is proved.

References

[1] N. Balakrishnan and V.B. Nevzorov. *A Primer on Statistical Distributions*. Wiley, New Jersey, 2003.

[2] R.V. Erickson. On an l_p version of the berry–esséen theorem for independent and m-dependent random variables. *Ann. Probab.*, **1**(3):497–503, 1973.

[3] R.V. Erickson. L_1 bounds for asymptotic normality of m-dependent sums using Stein's technique. *Ann. Probab.*, **2**:522–529, 1974.

[4] I.A. Ibragimov and Yu.V. Linnik. *Independent and Stationary Connected Variables*. Nauka, Moscow, 1965 (in russian).

[5] M. Loève. *Probability Theory*. I. Springer-Verlag, New York, Berlin, Heidelberg, 4^{th} edition, 1977.

[6] E. Lukacs. *Characteristic Functions*. Nauka, Moscow, 1979 (in russian).

[7] Ch. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proc. Math. Statist. and Probab., Vol.* 2, pp. 583–602, Berkeley, CA, 1972. Univ. Calif. Press.

[8] J. Sunklodas. On the rate of convergence of L_p norms in the central limit theorem for independent random variables. *Lithuanian Math. J.*, **42**(3):296–307, 2002.

[9] J. Sunklodas. On the rate of convergence of L_p norms in the CLT for Poisson and gamma random variables. *Lithuanian Math. J.*, **49**(2):216–221, 2009.

[10] A.N. Tikhomirov. On the rate of convergence in the central limit theorem for weakly dependent variables. *Theory Probab. Appl.*, **25**:790–809, 1980.

REZIUMĖ

Normaliosios aproksimacijos įverčiai mišriajam Puasono ir gama atsitiktiniam dydžiui

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Darbe gautas standartizuoto atsitiktinio dydžio $(Z - \mathbb{E}Z)/\sqrt{\mathbb{D}Z}$, kur $Z = a_1X + a_2Y$, $a_1 + a_2 = 1$, $a_i \geqslant 0$, $i = 1, 2$, X yra pasiskirstęs pagal Puasono skirstinį su parametru $\lambda > 0$, o Y – pagal standartinį gama skirstinį su parametru $\alpha > 0$, normos Δ_p viršutinis įvertis metrikoje L_p su visais $1 \leqslant p \leqslant \infty$.

Raktiniai žodžiai: normalioji aproksimacija, L_p norma, Puasono skirstinys, standartinis gama skirstinys, Puasono ir gama a.d. mišinys.