Approximation by Shifts of Compositions of Dirichlet L-Functions with the Gram Function

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Abstract: In this paper, a joint approximation of analytic functions by shifts of Dirichlet L-functions \( L(s + ia_1t_1, \chi_1), \ldots, L(s + ia_t, \chi_t) \), where \( a_1, \ldots, a_t \) are non-zero real algebraic numbers linearly independent over the field \( \mathbb{Q} \) and \( t_1 \) is the Gram function, is considered. It is proved that the set of their shifts has a positive lower density.

Keywords: Dirichlet L-function; Gram function; joint universality

1. Introduction

Let \( \chi : \mathbb{N} \to \mathbb{C} \) be a Dirichlet character modulo \( q \in \mathbb{N} \). Note that \( \chi(m) \) is periodic with period \( q \), completely multiplicative (i.e., \( \chi(mn) = \chi(m)\chi(n) \) for all \( m, n \in \mathbb{N} \) and \( \chi(1) = 1 \)), \( \chi(m) = 0 \) for \( (m, q) = 1 \) and \( \chi(m) \neq 0 \) for \( (m, q) = 1 \). Let \( s = \sigma + it \). In [1], L. Dirichlet introduced a function

\[
L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad (\sigma > 1),
\]

which is now called the Dirichlet L-function. In virtue of the complete multiplicativity of \( \chi(m) \), the function (1) can be written as an Euler product

\[
L(s, \chi) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1},
\]

where \( \mathbb{P} \) is the set of all prime numbers and has a meromorphic continuation to the whole complex plane with a unique simple pole at the point \( s = 1 \) (if \( \chi \) is the principal character modulo \( q \)) with residue \( \prod_{p \mid q}(1 - 1/p) \). Since then, the function (1) has become a subject of intensive investigation. See, for instance, References [2–4] for some very recent papers on its zeros and moments. For \( q = 1 \), the function \( L(s, \chi) \) becomes the Riemann zeta-function \( \zeta(s) \).

In Reference [5], S. M. Voronin established the universality of Dirichlet L-functions. He proved that if \( f(s) \) is a continuous non-vanishing function on the disc \( |s| \leq r \) with any fixed \( r, 0 < r < 1/4 \), and analytic in the interior of that disc, then, for every \( \varepsilon > 0 \), there exists a real number \( \tau = \tau(\varepsilon) \) such that

\[
\max_{|s| \leq r} |L(s + 3/4 + it_1, \chi) - f(s)| < \varepsilon.
\]

The Voronin theorem was extended to more general compact sets independently in References [6–8]. Denote by \( \mathcal{K} \) the class of compact subsets of the strip \( D = \{ s \in \mathbb{C} : 1/2 < \sigma < 1 \} \) with connected complements, and by \( H_0(\mathcal{K}) \), where \( \mathcal{K} \in \mathcal{K} \), the class of continuous non-vanishing functions on \( \mathcal{K} \) that
are analytic in the interior of $K$. Then the modern version of the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \meas \left\{ \tau \in [0, T] : \sup_{s \in K} |L(s + i\tau, \chi) - f(s)| < \varepsilon \right\} > 0,$$

where $\meas A$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ (see, for example, Reference [9]). The latter inequality shows that there are infinitely many shifts $L(s + i\tau, \chi)$ approximating a given function from the class $H_0(K)$.

In Reference [10], Voronin considered the joint functional independence of Dirichlet $L$-functions using the joint universality. We recall that two Dirichlet characters are called non-equivalent if they are not generated by the same primitive character. Thus, the following statement is valid [10,11]; see also References [9,12,13].

**Theorem 1.** Let $\chi_1, \ldots, \chi_r$ be pairwise non-equivalent Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \meas \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r, s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

The non-equivalence of the characters $\chi_1, \ldots, \chi_r$ ensures a certain independence of the functions $L(s, \chi_1), \ldots, L(s, \chi_r)$ which is necessary for a simultaneous approximation of the collection $f_1(s), \ldots, f_r(s)$. Later, it turned out that, in place of non-equivalent characters, different shifts can be used. This was observed by Nakamura [14]. More precisely, he proved the following theorem.

**Theorem 2.** Let $a_1 = 1, a_2, \ldots, a_r$ be real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$ and $\chi_1, \ldots, \chi_r$ be arbitrary Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $a \in \mathbb{R} \setminus \{0\}$,

$$\liminf_{T \to \infty} \frac{1}{T} \meas \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r, s \in K_j} |L(s + iaa_j\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

In Reference [15], Pańkowski obtained the joint universality of Dirichlet $L$-functions using the shifts $L(s + iaa_j\tau \log^b(s), \chi_j)$, $j = 1, \ldots, r$, where $a_1, \ldots, a_r \in \mathbb{R}$, $a_1, \ldots, a_r \in \mathbb{R}^+$ are distinct, $b_1, \ldots, b_r$ are distinct and satisfy

$$b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \in \mathbb{N}, \\ (-\infty, 0) \cup (1 + \infty) & \text{if } a_j \not\in \mathbb{N}. \end{cases}$$

The aim of this paper is to introduce new shifts of Dirichlet $L$-functions that approximate collections of analytic functions from the class $H_0(K)$. Let, as usual, $\Gamma(s)$ be the Euler gamma-function. For $t > 0$, denote the increment $\theta(t)$ of the argument of the function $\pi^{-s/2} \Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. Then it is known (see, for example, Reference [16] [Lemma 1.1]) that, for $\tau \geq 0$, the equation

$$\theta(t) = (\tau - 1)\pi$$

has the unique solution $t_\tau$ satisfying $\theta'(t_\tau) > 0$. For $n \in \mathbb{N}$, the numbers $t_n$ are called the Gram points. They were introduced and studied in Reference [17]. Therefore, we call $t_n$ the Gram function. A very interesting property of the Gram points is the relation $t_n \sim \gamma_n$ as $n \to \infty$, where $\gamma_n > 0$ are imaginary parts of non-trivial zeros of the Riemann zeta-function. In the paper, we will consider the
joint approximation of analytic functions by shifts of Dirichlet $L$-functions involving the Gram function. More precisely, we will prove the following joint universality theorem.

**Theorem 3.** Suppose that $a_1, \ldots, a_r$ are real non-zero algebraic numbers linearly independent over $\mathbb{Q}$, and $\chi_1, \ldots, \chi_r$ are arbitrary Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T - 2} \text{meas}\left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia_j t, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$ 

Moreover, the limit

$$\lim_{T \to \infty} \frac{1}{T - 2} \text{meas}\left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia_j t, \chi_j) - f_j(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of Theorem 3, we will use the probabilistic approach based on weakly convergent probability measures in the space of analytic functions.

2. Lemmas

We start with a lemma on the functional properties of the function $t_\tau$. (Its proof can be found in Reference [16] [Lemma 1.1].)

**Lemma 1.** Suppose that $\tau \to \infty$. Then

$$t_\tau = \frac{2\pi \tau}{\log \tau} \left(1 + \frac{\log \log \tau}{\log \tau}(1 + o(1))\right),$$

$$t'_\tau = \frac{2\pi}{\log \tau} \left(1 + \frac{\log \log \tau}{\log \tau}(1 + o(1))\right)$$

and

$$t''_\tau = -\frac{\pi}{\tau(\log \tau)^2} \left(1 + \frac{\log \log \tau}{\log \tau}(2 + o(1))\right).$$

The next lemma provides an estimate for certain trigonometric integral.

**Lemma 2.** Suppose that $F(x)$ is a real differentiable function, the derivative $F'(x)$ is monotonic and $F'(x) \geq \lambda > 0$ or $F'(x) \leq -\lambda < 0$ on the interval $(a, b)$. Then

$$\left| \int_a^b \exp\{iF(x)\} \, dx \right| \leq \frac{4}{\lambda}.$$

The proof of the lemma is given, for example, in Reference [11].

We will also use Baker’s theorem on linear forms in logarithms of algebraic numbers (see, for example, Reference [18]).

**Lemma 3.** Suppose that $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}$ are such that their logarithms $\log \lambda_1, \ldots, \log \lambda_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \ldots, \beta_r$, not all zero, we have

$$|\beta_0 + \beta_1 \log \lambda_1 + \cdots + \beta_r \log \lambda_r| > H^{-C},$$

where $H$ is the maximum of the heights of $\beta_0, \beta_1, \ldots, \beta_r$, and $C$ is an effectively computable constant depending on $r, \lambda_1, \ldots, \lambda_r$ and the maximum of the degrees of $\beta_0, \beta_1, \ldots, \beta_r$. 
Let \( \gamma = \{ s \in \mathbb{C} : |s| = 1 \} \), and 
\[
\Omega = \prod_{p \in \mathbb{P}} \gamma_p,
\]
where \( \gamma_p = \gamma \) for all \( p \in \mathbb{P} \). With the product topology and pointwise multiplication, the infinite-dimensional torus \( \Omega \) is a compact topological Abelian group. Define 
\[
\Omega' = \Omega_1 \times \cdots \times \Omega_r,
\]
where \( \Omega_j = \Omega \) for \( j = 1, \ldots, r \). Then \( \Omega' \) is also a compact topological Abelian group. Therefore, denoting by \( \mathcal{B}(X) \) the Borel \( \sigma \)-field of the space \( X \), we see that, on \((\Omega', \mathcal{B}(\Omega'))\), the probability Haar measure \( m'_H \) exists. This gives the probability space \((\Omega', \mathcal{B}(\Omega'), m'_H)\).

For \( A \in \mathcal{B}(\Omega') \), define
\[
Q_T(A) = \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \left( \left( p^{-i_1 \tau} : p \in \mathbb{P} \right), \ldots, \left( p^{-i_r \tau} : p \in \mathbb{P} \right) \right) \in A \right\}.
\]
Then the following limit theorem holds.

**Lemma 4.** Under hypotheses of Theorem 2 on the numbers \( a_1, \ldots, a_r \), \( Q_T \) converges weakly to the Haar measure \( m'_H \) as \( T \to \infty \).

**Proof.** We apply the Fourier transform method. It is well known that the dual group of \( \Omega' \) is isomorphic to the group
\[
\bigoplus_{j=1}^r \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{jp},
\]
where \( \mathbb{Z}_{jp} = \mathbb{Z} \) for all \( j = 1, \ldots, r \), \( p \in \mathbb{P} \). Hence it follows that characters of the group \( \Omega' \) are of the form
\[
\prod_{j=1}^r \prod_{p \in \mathbb{P}} \omega_{jp}^k(p),
\]
where \( \omega_{jp}(p) \) is the \( p \)th component of an element \( \omega_j \in \Omega_j \), \( j = 1, \ldots, r \), and the sign "\( * \)" means that only a finite number of integers \( k_{jp} \) are distinct from zero. Therefore
\[
\int_{\Omega'} \left( \prod_{j=1}^r \prod_{p \in \mathbb{P}} \omega_{jp}^k(p) \right) d\mu \tag{2}
\]
is the Fourier transform of a measure \( \mu \) on \((\Omega', \mathcal{B}(\Omega'))\).

Let \( s_{Q_T}(\vec{k}) \), \( \vec{k} = (k_1, \ldots, k_r) \), \( k_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P}), j = 1, \ldots, r \), be the Fourier transform of \( Q_T \). In view of (2) we have
\[
s_{Q_T}(\vec{k}) = \int_{\Omega'} \left( \prod_{j=1}^r \prod_{p \in \mathbb{P}} \omega_{jp}^k(p) \right) dQ_T.
\]
Thus, by the definition of \( Q_T \),
\[
s_{Q_T}(\vec{k}) = \frac{1}{T-2} \int_2^T \prod_{j=1}^r \prod_{p \in \mathbb{P}} p^{-ik_{jp} \tau} d\tau
\]
\[= \frac{1}{T-2} \int_2^T \exp \left\{ -it \sum_{j=1}^r \sum_{p \in \mathbb{P}} a_j k_{jp} \log p \right\} d\tau. \tag{3}
\]
Obviously, if $k = (0, \ldots, 0)$, then
\[ g_{Q_T}(k) = 1. \] (4)

Now suppose that $k = (k_1, \ldots, k_r) \neq (0, \ldots, 0)$. Note that
\[ A_k \stackrel{\text{def}}{=} \sum_{j=1}^{r} \sum_{p \in \mathbb{P}} a_j k_j \log p = \sum_{p \in \mathbb{P}} \log p \sum_{j=1}^{r} a_j k_j p. \]

Since $k_j \neq 0$ for some $j \in \{1, 2, \ldots, r\}$, there is a prime number $p$ such that $k_j p \neq 0$. For this $p$, the sum $\beta_p \stackrel{\text{def}}{=} \sum_{j=1}^{r} a_j k_j p$ is non-zero, because the numbers $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, in view of Lemma 3,
\[ A_k = \sum_{p \in \mathbb{P}} \beta_p \log p \neq 0. \] (5)

Now, (3) and Lemmas 1 and 2 show that, in the case $k \neq (0, \ldots, 0)$,
\[ g_{Q_T}(k) \ll \log T \frac{T A_k}{k}. \]

This together with (4) and (5) give
\[ \lim_{T \to \infty} g_{Q_T}(k) = \begin{cases} 1 & \text{if } k = (0, \ldots, 0), \\ 0 & \text{if } k \neq (0, \ldots, 0). \end{cases} \]

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure $m_{H}'$, the lemma follows by a continuity theorem for probability measures on compact groups.

$H(D)$ denotes the space of analytic functions on the strip $D$ endowed with the topology of uniform convergence on compacta. Lemma 4 implies a limit theorem for probability measures on $(H(D), B(H(D)))$ defined by means of absolutely convergent Dirichlet series.

For a fixed number $\theta > 1/2$ and $m, n \in \mathbb{N}$, set
\[ v_n(m) = \exp \left\{ -\left( \frac{m}{n} \right)^\theta \right\}. \] (6)

Then we define the series
\[ L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)v_n(m)}{m^s}, \]
and
\[ L_n(s, \omega_j, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega_j(m)v_n(m)}{m^s}, \]
\[ j = 1, \ldots, r, \]
where the functions $\omega_j(p)$ are extended to the set $\mathbb{N}$ by the formula
\[ \omega_j(m) = \prod_{p | m} \omega_j(p), \quad m \in \mathbb{N}. \]

Denote the elements of $\Omega'$ by $\omega = (\omega_1, \ldots, \omega_r)$. Put $\chi = (\chi_1, \ldots, \chi_r)$, and set
\[ L_n(s, \chi) = (L_n(s, \chi_1), \ldots, L_n(s, \chi_r)) \] (7)
and
\[ L_n(s, \omega, \chi) = (L_n(s, \omega_1, \chi_1), \ldots, L_n(s, \omega_r, \chi_r)). \]
Moreover, let \( u_n : \Omega' \to H'(D) \) be given by the formula
\[
  u_n(\omega) = L_n(s, \omega, \chi).
\]
The absolute convergence of the series for \( L_n(s, \omega, \chi) \) implies the continuity of the mapping \( u_n \).

Let \( V_n = m'H u_n^{-1} \), where, for \( A \in \mathcal{B}(H'(D)) \),
\[
  V_n(A) = m'H u_n^{-1}(A) = m'_H (u_n^{-1} A).
\]

In view of (7) and (8) we conclude that Lemma 4, the continuity of \( u_n \) and the well-known property on preservation of weak convergence under mapping lead to the following statement.

**Lemma 5.** Under hypothesis of Theorem 3 on the numbers \( a = (a_1, \ldots, a_r) \), we have
\[
  P_{\tau, n}(A) \overset{\text{def}}{=} \frac{1}{T - 2} \text{meas} \left\{ \tau \in [2, T] : L_n(s + i\tau t, \chi) \in A \right\}, \quad A \in \mathcal{B}(H'(D)),
\]
converges weakly to the measure \( V_n \) as \( T \to \infty \).

The probability measure \( V_n \) is very important for the proof of Theorem 3. Let
\[
  L_s = (L(s, \omega_1, \chi_1), \ldots, L(s, \omega_r, \chi_r)),
\]
where
\[
  L_s = \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega_j(p) \chi(p)}{p^s} \right)^{-1}, \quad j = 1, \ldots, r. \tag{9}
\]

Note that the latter products are uniformly convergent on compact subsets of the strip \( D \) for almost all \( \omega_j \in \Omega_j \), and define the \( H(D) \)-valued random elements on the probability space \( (\Omega_j, \mathcal{B}(\Omega_j), m_{ijH}) \), where \( m_{ijH} \) is the probability Haar measure on \( (\Omega_j, \mathcal{B}(\Omega_j)) \). Therefore, \( L_s(s, \omega, \chi) \) is an \( H(D) \)-valued random element on \( (\Omega', \mathcal{B}(\Omega'), m'_{H}) \). Denote by \( P_L \) the distribution of the random element \( L(s, \omega, \chi) \), that is,
\[
  P_L(A) = m'_{H} \left\{ \omega \in \Omega' : L(s, \omega, \chi) \in A \right\}, \quad A \in \mathcal{B}(H'(D)).
\]

We recall that the support of a probability measure \( P \) on \( (\mathbb{X}, \mathcal{B}(\mathbb{X})) \), where the space \( \mathbb{X} \) is separable, is a minimal closed set \( S_P \subset \mathbb{X} \) such that \( P(S_P) = 1 \). The set \( S_P \) consists of all elements \( x \in \mathbb{X} \) such that, for every open neighbourhood \( G \) of \( x \), the inequality \( P(G) > 0 \) is satisfied.

The measure \( V_n \) is independent on any hypothesis. Therefore, from Reference [19] it follows that:

**Lemma 6.** The measure \( V_n \) converges weakly to \( P_L \) as \( n \to \infty \). Moreover, the support of \( P_L \) is the set \( S' \), where
\[
  S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.
\]

**Proof.** To be precise, in Reference [19] it was proved that a certain measure \( P_N \) converges weakly to a certain probability measure \( P \) on \( (H'(D), \mathcal{B}(H'(D))) \) (as \( N \to \infty \)), and the measure \( P \) is the limit measure of \( V_n \) as \( n \to \infty \). Moreover, it was proved that \( P = P_L \).

It remains to prove that the support of \( P_L \) is the set \( S' \). It is well known that the support of the random element
\[
  \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega(p) \chi(p)}{p^s} \right)^{-1}, \quad \omega \in \Omega,
\]

Under hypothesis of Theorem 3 on the numbers \( a = (a_1, \ldots, a_r) \), we have
\[
  P_{\tau, n}(A) \overset{\text{def}}{=} \frac{1}{T - 2} \text{meas} \left\{ \tau \in [2, T] : L_n(s + i\tau t, \chi) \in A \right\}, \quad A \in \mathcal{B}(H'(D)),
\]
converges weakly to the measure \( V_n \) as \( T \to \infty \).

The probability measure \( V_n \) is very important for the proof of Theorem 3. Let
\[
  L(s, \omega, \chi) = (L(s, \omega_1, \chi_1), \ldots, L(s, \omega_r, \chi_r)),
\]
where
\[
  L(s, \omega, \chi) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega_j(p) \chi(p)}{p^s} \right)^{-1}, \quad j = 1, \ldots, r. \tag{9}
\]

Note that the latter products are uniformly convergent on compact subsets of the strip \( D \) for almost all \( \omega_j \in \Omega_j \), and define the \( H(D) \)-valued random elements on the probability space \( (\Omega_j, \mathcal{B}(\Omega_j), m_{ijH}) \), where \( m_{ijH} \) is the probability Haar measure on \( (\Omega_j, \mathcal{B}(\Omega_j)) \). Therefore, \( L_s(s, \omega, \chi) \) is an \( H(D) \)-valued random element on \( (\Omega', \mathcal{B}(\Omega'), m'_{H}) \). Denote by \( P_L \) the distribution of the random element \( L(s, \omega, \chi) \), that is,
\[
  P_L(A) = m'_{H} \left\{ \omega \in \Omega' : L(s, \omega, \chi) \in A \right\}, \quad A \in \mathcal{B}(H'(D)).
\]

We recall that the support of a probability measure \( P \) on \( (\mathbb{X}, \mathcal{B}(\mathbb{X})) \), where the space \( \mathbb{X} \) is separable, is a minimal closed set \( S_P \subset \mathbb{X} \) such that \( P(S_P) = 1 \). The set \( S_P \) consists of all elements \( x \in \mathbb{X} \) such that, for every open neighbourhood \( G \) of \( x \), the inequality \( P(G) > 0 \) is satisfied.

The measure \( V_n \) is independent on any hypothesis. Therefore, from Reference [19] it follows that:

**Lemma 6.** The measure \( V_n \) converges weakly to \( P_L \) as \( n \to \infty \). Moreover, the support of \( P_L \) is the set \( S' \), where
\[
  S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.
\]

**Proof.** To be precise, in Reference [19] it was proved that a certain measure \( P_N \) converges weakly to a certain probability measure \( P \) on \( (H'(D), \mathcal{B}(H'(D))) \) (as \( N \to \infty \)), and the measure \( P \) is the limit measure of \( V_n \) as \( n \to \infty \). Moreover, it was proved that \( P = P_L \).

It remains to prove that the support of \( P_L \) is the set \( S' \). It is well known that the support of the random element
\[
  \prod_{p \in \mathbb{P}} \left( 1 - \frac{\omega(p) \chi(p)}{p^s} \right)^{-1}, \quad \omega \in \Omega,
\]
is the set $S$ for every Dirichlet character $\chi$. Since the space $H^r(D)$ is separable, we have
\[ B(H^r(D)) = \bigotimes_{i=1}^r B(H(D)) \]
(see [20]). Therefore, it suffices to consider the measure $P_L$ on the sets
\[ A = A_1 \times \cdots \times A_r, \quad A_1, \ldots, A_r \in B(H(D)). \]
Since the Haar measure $m'H$ is the product of the Haar measures $m_{jH}$ on $(\Omega_j, B(\Omega_j))$, $j = 1, \ldots, r$, we deduce that
\[ m'_H\{\omega \in \Omega' : L(s, \omega, \chi) \in A}\} = \prod_{j=1}^r m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j, \chi_j) \in A_j\}. \]
This equality and the minimality of the support together with remark on the support of the element (10) show that the support of $P_L$ is the set $S^r$. 

3. Mean Square Estimates

Define
\[ L(s, \chi) = (L(s, \chi_1), \ldots, L(s, \chi_r)). \quad (11) \]
To pass from $L\nu(s + iat, \chi)$ (defined by (7)) to $L(s + iat, \chi)$, certain mean square estimates for Dirichlet $L$-functions are necessary. Let $\chi$ be an arbitrary character modulo $q$.

**Lemma 7.** Suppose that $\sigma$, $1/2 < \sigma < 1$, and $a \in \mathbb{R} \setminus \{0\}$ are fixed. Then, for $t \in \mathbb{R}$,
\[ \int_2^T |L(\sigma + it + iat, \chi)|^2 d\tau \ll T(1 + |t|). \]

**Proof.** It is well known that, for fixed $\sigma > 1/2$,
\[ \int_2^T |L(\sigma + it, \chi)|^2 dt \ll_{\sigma} T. \]
Therefore, in view of Lemma 1, for $1/2 < \sigma < 1$,
\[ \int_2^T |L(\sigma + it + iat, \chi)|^2 d\tau = \frac{1}{a} \int_2^T \frac{1}{\tau} |L(\sigma + it + iat, \chi)|^2 d(ait) \ll_{\sigma,a} T(1 + |t|), \]
which is the required estimate. 

For $g_1, g_2 \in H(D)$, define
\[ \rho(g_1, g_2) = \sum_{l=1}^\infty 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}. \quad (12) \]
where \( \{K_l\} \subset D \) is a sequence of compact subsets such that
\[
D = \bigcup_{l=1}^{\infty} K_l.
\]

\( K_l \subset K_{l+1} \) for all \( l \in \mathbb{N} \), and if \( K \subset D \) is a compact set, then \( K \subset K_l \) for some \( l \in \mathbb{N} \). Then \( \rho \) is a metric in the space \( H(D) \) inducing the topology of uniform convergence on compacta. Now, putting, for \( \mathcal{g}_1 = (g_{11}, \ldots, g_{1r}), \mathcal{g}_2 = (g_{21}, \ldots, g_{2r}) \in H'(D), \)
\[
\rho(\mathcal{g}_1, \mathcal{g}_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j})
\]
gives a metric in \( H'(D) \) inducing the product topology. The next lemma provides a certain approximation of \( L(s, \chi) \) (see definition (11)) by \( L_n(s, \chi) \).

**Lemma 8.** Suppose that \( a \neq (0, \ldots, 0) \). Then
\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T - 2} \int_{T/2}^{T} \rho \left( L(s + iat, \chi), L_n(s + iat, \chi) \right) \, d\tau = 0.
\]

**Proof.** From the definition (13) of the metric \( \rho \), it follows that it suffices to prove that, for \( a \neq 0, \)
\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T - 2} \int_{T/2}^{T} \rho \left( L(s + iat, \chi), L_n(s + iat, \chi) \right) \, d\tau = 0
\]
for every \( j = 1, \ldots, r \). We will prove the above equality for the character \( \chi \) modulo \( q \).

Let \( \theta \) be from the definition (6) of \( v_n(m) \), and
\[
l_n(s) = \frac{s}{\theta} \Gamma \left( \frac{s}{\theta} \right) n^s.
\]
Then the representation
\[
L_n(s, \chi) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} L(s + z, \chi) l_n(z) \frac{dz}{z},
\]
is true. Its proof is the same as in Section 5.4 of [21] for the Riemann zeta-function. Hence, taking \( \theta_1 > 0 \), by the residue theorem, we obtain
\[
L_n(s, \chi) - L(s, \chi) = \frac{1}{2\pi i} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} L(s + z, \chi) l_n(z) \frac{dz}{z} + R_n(s, \chi),
\]
where
\[
R_n(s, \chi) = \begin{cases} 0 & \text{if } \chi \text{ is a non-principal character}, \\ \prod_{p \nmid \theta} \left( 1 - \frac{1}{p} \right) \frac{l_n(1-s)}{1-s} & \text{otherwise}. \end{cases}
\]
Let \( K \subset D \) be an arbitrary compact set. Denote by \( s = \sigma + iv \) the points of \( K \), and suppose that \( 1/2 + 2\epsilon \leq \sigma \leq 1 - \epsilon \) with fixed \( \epsilon > 0 \) for \( s \in K \). More precisely, we select \( \theta_1 = \sigma - \epsilon - 1/2 \geq \epsilon > 0 \). Then, in view of (16),
\[
\left| L_n(s + iat, \chi) - L(s + iat, \chi) \right| \ll \int_{-\infty}^{\infty} |L(s + it, \chi) - \theta_1 + it, \chi)| l_n(-\theta_1 + it) \frac{dz}{z} + |R_n(s + iat, \chi)|.}

Now, taking $t$ in place of $t + \nu$, we get that, for $s \in K$,
\[
|L_n(s + iat\tau, \chi) - L(s + iat\tau, \chi)|
\ll \int_{-\infty}^{\infty} |L(1/2 + \epsilon + i(t + at\tau), \chi)| \frac{|l_n(1/2 + \epsilon - s + it)|}{|1/2 + \epsilon - s + it|} dt
+ |R_n(s + iat\tau, \chi)|.
\]
This implies the estimate
\[
\frac{1}{T - 2} \int_2^T \sup_{s \in K} |L(s + iat\tau, \chi) - L_n(s + iat\tau, \chi)| d\tau
\ll \frac{1}{T - 2} \int_2^T \int_{-\infty}^{\infty} |L(1/2 + \epsilon + i(t + at\tau), \chi)| \sup_{s \in K} \frac{|l_n(1/2 + \epsilon - s + it)|}{|1/2 + \epsilon - s + it|} dt d\tau
+ \frac{1}{T - 2} \int_2^T \sup_{s \in K} |R_n(s + iat\tau, \chi)| d\tau
\ll J_1 + J_2,
\]
where
\[
J_1 = \int_{-\infty}^{\infty} \frac{1}{T - 2} \int_2^T (|L(1/2 + \epsilon + i(t + at\tau), \chi)| d\tau) \sup_{s \in K} \frac{|l_n(1/2 + \epsilon - s + it)|}{|1/2 + \epsilon - s + it|} dt
\]
and
\[
J_2 = \frac{1}{T - 2} \int_2^T \sup_{s \in K} |R_n(s + iat\tau, \chi)| d\tau.
\]
It is well known that uniformly in $\sigma, \sigma_1 \leq \sigma \leq \sigma_2$, with arbitrary $\sigma_1 < \sigma_2$,
\[
\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0.
\]
Therefore, by the definition (15) of the function $l_n(s)$, we find that, for $s \in K$,
\[
\left|\frac{l_n(1/2 + \epsilon - s + it)}{1/2 + \epsilon - s + it}\right| = \frac{n^{1/2 + \epsilon - \sigma}}{\Gamma\left(\frac{1/2 + \epsilon - \sigma}{\theta} + \frac{i(t - \nu)}{\theta}\right)}
\ll_{\theta, K} n^{-\epsilon} \exp\left\{-\frac{c_1}{\theta}|t|\right\}, \quad c_1 > 0.
\]
In the same way, for $s \in K$, we obtain
\[
R_n(s + iat\tau, \chi) \ll_{\theta, K} n^{1-\epsilon} \exp\left\{-\frac{c_2}{\theta}|a|t\tau\right\}.
\]
Suppose that $\theta = 1/2 + \epsilon$. Then (17), (19) and Lemma 7 lead to the bound
\[
J_1 \ll_{\epsilon, K} n^{-\epsilon} \int_{-\infty}^{\infty} (1 + |t|) \exp\{-c_3|t|\} dt \ll_{\epsilon, K, \theta} n^{-\epsilon}, \quad c_3 > 0.
\]
Moreover, by (18), Lemma 1 and (20),
Thus, in view of (17) and (21),

\[
\frac{1}{T-2} \int_2^T \sup_{s \in K} |L(s + iat, \chi) - L_n(s + iat, \chi)| \, d\tau \ll_{\epsilon, K, q} n^{-\epsilon} + n^{1/2 - 2\epsilon} \log T \frac{T}{T-2}.
\]

From this, it follows that

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T-2} \int_2^T \sup_{s \in K} |L(s + iat, \chi) - L_n(s + iat, \chi)| \, d\tau = 0.
\]

Now, the definition (12) of the metric \(\rho\) implies (14), which completes the proof of Lemma 8. \(\square\)

4. A Limit Theorem

For \(A \in B(H'(D))\), define

\[
P_T(A) = \frac{1}{T-2} \text{meas}\left\{ \tau \in [2, T] : L(s + iat, \chi) \in A \right\}.
\]

In this section, we will prove the following statement.

**Theorem 4.** Suppose that \(a_1, \ldots, a_r\) are non-zero real algebraic numbers linearly independent over \(\mathbb{Q}\), and \(\chi_1, \ldots, \chi_r\) are arbitrary Dirichlet characters. Then \(P_T\) converges weakly to \(P_L\) as \(T \to \infty\). The support of \(P_L\) is the set \(S'\).

First we recall a useful property of convergence in distribution \((\overset{D}{\to})\) (see Theorem 4.2 in Reference [20]).

**Lemma 9.** Suppose that the space \((\mathcal{X}, d)\) is separable, the random elements \(X_{kn}\) and \(Y_n\), \(k \in \mathbb{N}, n \in \mathbb{N}\), are defined on the same probability space with measure \(\mu\),

\[
X_{kn} \overset{D}{\to} X_k,
\]

for every \(k \in \mathbb{N}\),

\[
X_k \overset{D}{\to} X,
\]

and, for every \(\epsilon > 0\),

\[
\lim_{k \to \infty} \limsup_{n \to \infty} \mu \left\{ d(X_{kn}, Y_n) \geq \epsilon \right\} = 0.
\]

Then \(Y_n \overset{D}{\to} X\).

In the theory of weak convergence of probability measures, the notions of relative compactness and tightness of families of probability measures are very useful. We recall that the family \(\{P\}\) of probability measures on \((\mathcal{X}, B(\mathcal{X}))\) is called relatively compact if every sequence \(\{P_n\} \subset \{P\}\) contains a weakly convergent subsequence to a certain measure on \((\mathcal{X}, B(\mathcal{X}))\), and this family is called tight, if for every \(\epsilon > 0\), there exists a compact set \(K = K(\epsilon) \subset \mathcal{X}\) such that

\[
P(K) > 1 - \epsilon.
\]
for all \( P \in \{ P \} \). By the direct Prokhorov theorem (see Theorem 5.1 in Billingsley [20]), every tight family \( \{ P \} \) is relatively compact. We apply the above remarks to the sequence \( \{ V_n : n \in \mathbb{N} \} \), where \( V_n \) (defined by (8)) is the limit measure in Lemma 5.

**Lemma 10.** The sequence \( \{ V_n \} \) is relatively compact.

**Proof.** By the above mentioned Prokhorov theorem, it suffices to prove that the sequence \( \{ V_n \} \) is tight.

Suppose \( \theta_T \) is a random variable defined on a certain probability space with measure \( \mu \) and uniformly distributed on \([2, T]\). Define the \( H'(D) \)-valued random element

\[
X_{T,n} = X_{T,n}(s) = (X_{T,n,1}(s), \ldots, X_{T,n,r}(s)) = L_n(s + i\theta_T, \chi).
\]

Moreover, let

\[
X_n = X_n(s) = (X_{n,1}(s), \ldots, X_{n,r}(s))
\]

be the \( H'(D) \)-valued random element with the distribution \( V_n \). Then Lemma 5 implies the relation

\[
\frac{X_{T,n}}{T \to \infty} \to X_n.
\]

By Lemma 7 with \( t = 0 \), we have, for \( 1/2 < \sigma < 1 \),

\[
\int_2^T |L(\sigma + ia_j t, \chi_j)|^2 \, d\tau \ll_{\sigma, a_j} T, \quad j = 1, \ldots, r.
\]

Let \( K_j \) be a compact set from the definition of the metric \( \rho \). Then (26) together with the Cauchy integral formula show that

\[
\int_2^T \sup_{s \in K_j} |L(s + ia_j t, \chi_j)| \, d\tau \ll_{a_j} T, \quad j = 1, \ldots, r.
\]

This combined with (22) implies the inequality

\[
\sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T - 2} \int_2^T \sup_{s \in K_j} |L_n(s + ia_j t, \chi_j)| \, d\tau \ll R_{ij}, \quad j = 1, \ldots, r.
\]  

Fix \( \varepsilon > 0 \), and define \( M_{ij} = M_{ij}(s) = 2^j r R_{ij} \varepsilon^{-1} \). Then, in view of (27), we find that, for each \( n \in \mathbb{N} \),

\[
\limsup_{T \to \infty} \mu \left\{ \exists j : \sup_{s \in K_j} |X_{T,n,j}(s)| > M_{ij} \right\} \leq \sum_{j=1}^r \limsup_{T \to \infty} \mu \left\{ \sup_{s \in K_j} |X_{T,n,j}(s)| > M_{ij} \right\} \leq \sum_{j=1}^r \frac{R_{ij}}{M_{ij}} = \frac{\varepsilon}{2^j}.
\]

This together with (25) shows that, for all \( l, n \in \mathbb{N} \),

\[
\mu \left\{ \exists j : \sup_{s \in K_l} |X_{n,j}(s)| > M_{ij} \right\} \leq \frac{\varepsilon}{2^j}.
\]
Define the set
\[ K_j = K_j(s) = \left\{ g \in H(D) : \sup_{s \in K_j} |g(s)| \leq M_{lj}, \ l \in \mathbb{N} \right\}. \]

Then \( K_j \) is a compact set in \( H(D) \), and, in virtue of (24) and (28),
\[ \mu \{ X_n \in K \} \geq 1 - \varepsilon \]
for all \( n \in \mathbb{N} \). In other words, we have
\[ V_n(K) \geq 1 - \varepsilon \]
for all \( n \in \mathbb{N} \). Thus, the sequence \( \{ V_n : n \in \mathbb{N} \} \) is tight.

**Proof of Theorem 4.** By Lemma 10, there exists a subsequence \( \{ V_{n_k} \} \) of the sequence \( \{ V_n \} \) that is weakly convergent to a certain probability measure \( P \) on \((H'(D), B(H'(D)))\) as \( k \to \infty \). This can be written as
\[ X_{n_k} \xrightarrow{D} k \to \infty P. \tag{29} \]

Define one more \( H'(D) \)-valued random element
\[ X_T = X_T(s) = L(s + i\theta_T, \chi). \]

Then Lemma 8 implies that, for every \( \varepsilon > 0 \),
\[ \lim_{n \to \infty} \limsup_{T \to \infty} \mu \left\{ \|X_T - X_{T,n}\| \geq \varepsilon \right\} \leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{(T-2)\varepsilon} \int_2^T \frac{1}{p} \left( L(s + i\theta_T, \chi), L_n(s + i\theta_T, \chi) \right) \, d\tau = 0. \]

The latter equality together with (25), (29), and Lemma 9 shows that
\[ X_T \xrightarrow{D} T \to \infty P, \tag{30} \]
or, in other words, \( P_T \) converges weakly to \( P \) as \( T \to \infty \). Moreover, by the relation (30), the measure \( P \) is independent of the subsequence \( \{ V_{n_k} \} \). Thus, we deduce that
\[ X_n \xrightarrow{D} n \to \infty P, \]
or \( V_n \) converges weakly to \( P \) as \( n \to \infty \). Therefore, the theorem follows by Lemma 6. \( \square \)

5. **Proof of Universality**

The proof of Theorem 3 is based on Mergelyan’s theorem on the approximation of analytic functions by polynomials [22], Theorem 4, and the properties of weak convergence. For convenience, we state them as lemmas.

**Lemma 11 (Mergelyan theorem).** Suppose that \( K \subset \mathbb{C} \) is a compact set with connected complement, and \( f(s) \) be a continuous function on \( K \) and analytic in the interior of \( K \). Then, for every \( \varepsilon > 0 \), there exists a polynomial \( p(s) \) such that
\[ \sup_{s \in K} |f(s) - p(s)| < \varepsilon. \]

We recall that \( A \in B(\mathbb{X}) \) is called a continuity set of the measure \( P \) on \((\mathbb{X}, B(\mathbb{X}))\) if \( P(\partial A) = 0 \), where \( \partial A \) is a boundary of \( A \).
Lemma 12. Let $P_n$, $n \in \mathbb{N}$, and $P$ be probability measures on $(X, B(X))$. Then the following statements are equivalent:

1° $P_n$ converges weakly to $P$ as $n \to \infty$;
2° For every open set $G \subset X$,
   \[ \liminf_{n \to \infty} P_n(G) \geq P(G); \]
3° For every continuity set $A$ of $P$,
   \[ \lim_{n \to \infty} P_n(A) = P(A). \]

The above lemma is a part of Theorem 2.1 from Reference [20]. Now, we can give the proof of Theorem 3.

Proof of Theorem 3. First part. In view of Lemma 11, there exist polynomials $p_1(s), \ldots, p_r(s)$ such that

\[ \sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2}. \tag{31} \]

The set

\[ G'_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \right\} \tag{32} \]

is an open neighbourhood of the element $(e^{p_1(s)}, \ldots, e^{p_r(s)}) \in S'$. Thus, by Theorem 4, $P_L(G'_\varepsilon) > 0$, where the distribution $P_L$ is defined by (9). Hence, from Theorem 4 again and Lemma 12,

\[ \liminf_{T \to \infty} P_T(G'_\varepsilon) \geq P_L(G'_\varepsilon) > 0, \]

and the definitions (23) and (32) of $P_T$ and $G'_\varepsilon$ together with (31) prove the first part of the theorem.

Second part. Introduce one more set

\[ A_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}. \tag{33} \]

Then the boundary of $A_\varepsilon$ lies in the set

\[ \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\}, \]

thus, $\partial A_{\varepsilon_1} \cap \partial A_{\varepsilon_2} = \emptyset$ for different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. This shows that the set $A_\varepsilon$ is a continuity set of the measure $P_L$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemma 12,

\[ \lim_{T \to \infty} P_T(A_\varepsilon) = P_L(A_\varepsilon) \tag{34} \]

for all but at most countably many $\varepsilon > 0$. Moreover, (31) shows the inclusion $G'_\varepsilon \subset A_\varepsilon$. This, (34) and the definitions (23) and (33) of $P_T$ and $A_\varepsilon$ prove the second assertion of the theorem. \(\square\)

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