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Approximation by Shifts of Compositions of Dirichlet L-Functions with the Gram Function

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Abstract: In this paper, a joint approximation of analytic functions by shifts of Dirichlet L-functions $L(s+ia_1t_\tau,\chi_1),\ldots,L(s+ia_rt_\tau,\chi_r)$, where $a_1,\ldots,a_r$ are non-zero real algebraic numbers linearly independent over the field $\mathbb{Q}$ and t_τ is the Gram function, is considered. It is proved that the set of their shifts has a positive lower density.

Keywords: Dirichlet L-function; Gram function; joint universality

1. Introduction

Let $\chi:\mathbb{N}\to\mathbb{C}$ be a Dirichlet character modulo $q\in\mathbb{N}$. Note that $\chi(m)$ is periodic with period q, completely multiplicative (i.e., $\chi(mn)=\chi(m)\chi(n)$ for all $m,n\in\mathbb{N}$ and $\chi(1)=1$), $\chi(m)=0$ for $(m,q)=1$ and $\chi(m)\neq 0$ for $(m,q)=1$. Let $s=\sigma+it$. In [1], L. Dirichlet introduced a function

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},\qquad(\sigma>1),\tag{1}$$

which is now called the Dirichlet L-function. In virtue of the complete multiplicativity of $\chi(m)$, the function (1) can be written as an Euler product

$$L(s,\chi)=\prod_{p\in\mathbb{P}}\left(1-\frac{\chi(p)}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers and has a meromorphic continuation to the whole complex plane with a unique simple pole at the point $s=1$ (if χ is the principal character modulo q) with residue $\prod_{p|q}(1-1/p)$. Since then, the function (1) has become a subject of intensive investigation. See, for instance, References [2–4] for some very recent papers on its zeros and moments. For $q=1$, the function $L(s,\chi)$ becomes the Riemann zeta-function $\zeta(s)$.

In Reference [5], S. M. Voronin established the universality of Dirichlet L-functions. He proved that if $f(s)$ is a continuous non-vanishing function on the disc $|s|\le r$ with any fixed r, $0<r<1/4$, and analytic in the interior of that disc, then, for every $\varepsilon>0$, there exists a real number $\tau=\tau(\varepsilon)$ such that

$$\max_{|s|\le r}|L(s+3/4+i\tau,\chi)-f(s)|<\varepsilon.$$

The Voronin theorem was extended to more general compact sets independently in References [6–8]. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$ with connected complements, and by $H_0(K)$, where $K\in\mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the modern version of the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(s+i\tau,\chi) - f(s)| < \varepsilon\right\} > 0,$$

where meas$A$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ (see, for example, Reference [9]). The latter inequality shows that there are infinitely many shifts $L(s+i\tau,\chi)$ approximating a given function from the class $H_0(K)$.

In Reference [10], Voronin considered the joint functional independence of Dirichlet L-functions using the joint universality. We recall that two Dirichlet characters are called non-equivalent if they are not generated by the same primitive character. Thus, the following statement is valid [10,11]; see also References [9,12,13].

Theorem 1. *Let $\chi_1,\dots,\chi_r$ be pairwise non-equivalent Dirichlet characters. For $j = 1,\dots,r$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\le j\le r}\sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon\right\} > 0.$$

The non-equivalence of the characters $\chi_1,\dots,\chi_r$ ensures a certain independence of the functions $L(s,\chi_1),\dots,L(s,\chi_r)$ which is necessary for a simultaneous approximation of the collection $f_1(s),\dots,f_r(s)$. Later, it turned out that, in place of non-equivalent characters, different shifts can be used. This was observed by Nakamura [14]. More precisely, he proved the following theorem.

Theorem 2. *Let $a_1 = 1, a_2,\dots,a_r$ be real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$ and $\chi_1,\dots,\chi_r$ be arbitrary Dirichlet characters. For $j = 1,\dots,r$, let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $a \in \mathbb{R}\setminus\{0\}$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\le j\le r}\sup_{s\in K_j} |L(s+iaa_j\tau,\chi_j) - f_j(s)| < \varepsilon\right\} > 0.$$

In Reference [15], Pańkowski obtained the joint universality of Dirichlet L-functions using the shifts $L(s+i\alpha_j\tau^{a_j}\log^{b_j}\tau,\chi_j)$, $j = 1,\dots,r$, where $\alpha_1,\dots,\alpha_r \in \mathbb{R}$, $a_1,\dots,a_r \in \mathbb{R}^+$ are distinct, $b_1,\dots,b_r$ are distinct and satisfy

$$b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \notin \mathbb{N}, \\ (-\infty,0]\cup(1+\infty) & \text{if } a_j \in \mathbb{N}. \end{cases}$$

The aim of this paper is to introduce new shifts of Dirichlet L-functions that approximate collections of analytic functions from the class $H_0(K)$. Let, as usual, $\Gamma(s)$ be the Euler gamma-function. For $t > 0$, denote the increment $\theta(t)$ of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2+it$. Then it is known (see, for example, Reference [16] [Lemma 1.1]) that, for $\tau \ge 0$, the equation

$$\theta(t) = (\tau-1)\pi$$

has the unique solution t_τ satisfying $\theta'(t_\tau) > 0$. For $n \in \mathbb{N}$, the numbers t_n are called the Gram points. They were introduced and studied in Reference [17]. Therefore, we call t_τ the Gram function. A very interesting property of the Gram points is the relation $t_n \sim \gamma_n$ as $n\to\infty$, where $\gamma_n > 0$ are imaginary parts of non-trivial zeros of the Riemann zeta-function. In the paper, we will consider the joint approximation of analytic functions by shifts of Dirichlet L-functions involving the Gram function. More precisely, we will prove the following joint universality theorem.

Theorem 3. *Suppose that $a_1, \dots, a_r$ are real non-zero algebraic numbers linearly independent over $\mathbb{Q}$, and $\chi_1, \dots, \chi_r$ are arbitrary Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + ia_j t_\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + ia_j t_\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of Theorem 3, we will use the probabilistic approach based on weakly convergent probability measures in the space of analytic functions.

2. Lemmas

We start with a lemma on the functional properties of the function t_τ. (Its proof can be found in Reference [16] [Lemma 1.1].)

Lemma 1. *Suppose that $\tau \to \infty$. Then*

$$t_\tau = \frac{2\pi\tau}{\log\tau} \left(1 + \frac{\log\log\tau}{\log\tau} (1 + o(1)) \right),$$
$$t'_\tau = \frac{2\pi}{\log\tau} \left(1 + \frac{\log\log\tau}{\log\tau} (1 + o(1)) \right)$$

and

$$t''_\tau = -\frac{\pi}{\tau(\log\tau)^2} \left(1 + \frac{\log\log\tau}{\log\tau} (2 + o(1)) \right).$$

The next lemma provides an estimate for certain trigonometric integral.

Lemma 2. *Suppose that $F(x)$ is a real differentiable function, the derivative $F'(x)$ is monotonic and $F'(x) \ge \lambda > 0$ or $F'(x) \le -\lambda < 0$ on the interval (a, b). Then*

$$\left| \int_a^b \exp\{iF(x)\}\, dx \right| \le \frac{4}{\lambda}.$$

The proof of the lemma is given, for example, in Reference [11].

We will also use Baker's theorem on linear forms in logarithms of algebraic numbers (see, for example, Reference [18]).

Lemma 3. *Suppose that $\lambda_1, \dots, \lambda_r \in \overline{\mathbb{Q}}$ are such that their logarithms $\log\lambda_1, \dots, \log\lambda_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \dots, \beta_r$, not all zero, we have*

$$|\beta_0 + \beta_1 \log\lambda_1 + \cdots + \beta_r \log\lambda_r| > H^{-C},$$

where H is the maximum of the heights of $\beta_0, \beta_1, \dots, \beta_r$, and C is an effectively computable constant depending on r, $\lambda_1, \dots, \lambda_r$ and the maximum of the degrees of $\beta_0, \beta_1, \dots, \beta_r$.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Define

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then Ω^r is also a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, we see that, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r exists. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$.

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_T(A) = \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \left(\left(p^{-ia_1 t_\tau} : p \in \mathbb{P} \right), \dots, \left(p^{-ia_r t_\tau} : p \in \mathbb{P} \right) \right) \in A \right\}.$$

Then the following limit theorem holds.

Lemma 4. *Under hypotheses of Theorem 2 on the numbers $a_1, \dots, a_r$, Q_T converges weakly to the Haar measure m_H^r as $T \to \infty$.*

Proof. We apply the Fourier transform method. It is well known that the dual group of Ω^r is isomorphic to the group

$$\bigoplus_{j=1}^{r} \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for all $j = 1, \dots, r$, $p \in \mathbb{P}$. Hence it follows that characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p),$$

where $\omega_j(p)$ is the pth component of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$, and the sign "$*$" means that only a finite number of integers k_{jp} are distinct from zero. Therefore

$$\int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p) \right) \mathrm{d}\mu \tag{2}$$

is the Fourier transform of a measure μ on $(\Omega^r, \mathcal{B}(\Omega^r))$.

Let $g_{Q_T}(\underline{k})$, $\underline{k} = (\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z},\ p \in \mathbb{P})$, $j = 1, \dots, r$, be the Fourier transform of Q_T. In view of (2) we have

$$g_{Q_T}(\underline{k})) = \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p) \right) \mathrm{d}Q_T.$$

Thus, by the definition of Q_T,

$$\begin{aligned} g_{Q_T}(\underline{k}) &= \frac{1}{T-2} \int_2^T \prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} p^{-ik_{jp} a_j t_\tau} \, \mathrm{d}\tau \\ &= \frac{1}{T-2} \int_2^T \exp \left\{ -it_\tau \sum_{j=1}^{r} \sum_{p \in \mathbb{P}}{}^{*} a_j k_{jp} \log p \right\} \mathrm{d}\tau. \end{aligned} \tag{3}$$

Obviously, if $\underline{k} = (\underline{0}, \dots, \underline{0})$, then

$$g_{Q_T}(\underline{k}) = 1. \tag{4}$$

Now suppose that $\underline{k} = (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Note that

$$A_{\underline{k}} \overset{def}{=} \sum_{j=1}^{r} \sideset{}{^*}\sum_{p \in \mathbb{P}} a_j k_{jp} \log p = \sideset{}{^*}\sum_{p \in \mathbb{P}} \log p \sum_{j=1}^{r} a_j k_{jp}.$$

Since $\underline{k}_j \neq \underline{0}$ for some $j \in \{1, 2, \dots, r\}$, there is a prime number p such that $k_{jp} \neq 0$. For this p, the sum $\beta_p \overset{def}{=} \sum_{j=1}^{r} a_j k_{jp}$ is non-zero, because the numbers $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, in view of Lemma 3,

$$A_{\underline{k}} = \sideset{}{^*}\sum_{p \in \mathbb{P}} \beta_p \log p \neq 0. \tag{5}$$

Now, (3) and Lemmas 1 and 2 show that, in the case $\underline{k} \neq (\underline{0}, \dots, \underline{0})$,

$$g_{Q_T}(\underline{k}) \ll \frac{\log T}{T A_{\underline{k}}}.$$

This together with (4) and (5) give

$$\lim_{T \to \infty} g_{Q_T}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } \underline{k} \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r, the lemma follows by a continuity theorem for probability measures on compact groups. □

$H(D)$ denotes the space of analytic functions on the strip D endowed with the topology of uniform convergence on compacta. Lemma 4 implies a limit theorem for probability measures on $(H(D), \mathcal{B}(H(D)))$ defined by means of absolutely convergent Dirichlet series.

For a fixed number $\theta > 1/2$ and $m, n \in \mathbb{N}$, set

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}. \tag{6}$$

Then we define the series

$$L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) v_n(m)}{m^s}$$

and

$$L_n(s, \omega_j, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) \omega_j(m) v_n(m)}{m^s},$$

$j = 1, \dots, r$, where the functions $\omega_j(p)$ are extended to the set $\mathbb{N}$ by the formula

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N}.$$

Denote the elements of Ω^r by $\omega = (\omega_1, \dots, \omega_r)$. Put $\underline{\chi} = (\chi_1, \dots, \chi_r)$, and set

$$\underline{L}_n(s, \underline{\chi}) = (L_n(s, \chi_1), \dots, L_n(s, \chi_r)) \tag{7}$$

and

$$\underline{L}_n(s, \omega, \underline{\chi}) = (L_n(s, \omega_1, \chi_1), \dots, L_n(s, \omega_r, \chi_r)).$$

Moreover, let $u_n : \Omega^r \to H^r(D)$ be given by the formula

$$u_n(\omega) = \underline{L}_n(s, \omega, \underline{\chi}).$$

The absolute convergence of the series for $L_n(s, \omega_j, \chi_j)$ implies the continuity of the mapping u_n. Let $V_n = m^r_H u_n^{-1}$, where, for $A \in \mathcal{B}(H^r(D))$,

$$V_n(A) = m^r_H u_n^{-1}(A) = m^r_H(u_n^{-1}A). \tag{8}$$

In view of (7) and (8) we conclude that Lemma 4, the continuity of u_n and the well-known property on preservation of weak convergence under mapping lead to the following statement.

Lemma 5. *Under hypothesis of Theorem 3 on the numbers $\underline{a} = (a_1, \dots, a_r)$, we have*

$$P_{T,n}(A) \stackrel{def}{=} \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \underline{L}_n(s + i\underline{a}t_\tau, \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)),$$

converges weakly to the measure V_n as $T \to \infty$.

The probability measure V_n is very important for the proof of Theorem 3. Let

$$\underline{L}(s, \omega, \underline{\chi}) = (L(s, \omega_1, \chi_1), \dots, L(s, \omega_r, \chi_r)),$$

where

$$\underline{L}(s, \omega, \underline{\chi}) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p)\chi_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r. \tag{9}$$

Note that the latter products are uniformly convergent on compact subsets of the strip D for almost all $\omega_j \in \Omega_j$, and define the $H(D)$-valued random elements on the probability space $(\Omega_j, \mathcal{B}(\Omega_j), m_{jH})$, where m_{jH} is the probability Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$. Therefore, $\underline{L}(s, \omega, \underline{\chi})$ is the $H^r(D)$-valued random element on $(\Omega^r, \mathcal{B}(\Omega^r), m^r_H)$. Denote by $P_{\underline{L}}$ the distribution of the random element $\underline{L}(s, \omega, \underline{\chi})$, that is,

$$P_{\underline{L}}(A) = m^r_H \left\{ \omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

We recall that the support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, where the space $\mathbb{X}$ is separable, is a minimal closed set $S_P \subset \mathbb{X}$ such that $P(S_P) = 1$. The set S_P consists of all elements $x \in \mathbb{X}$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied.

The measure V_n is independent on any hypothesis. Therefore, from Reference [19] it follows that:

Lemma 6. *The measure V_n converges weakly to $P_{\underline{L}}$ as $n \to \infty$. Moreover, the support of $P_{\underline{L}}$ is the set S^r, where*

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Proof. To be precise, in Reference [19] it was proved that a certain measure P_N converges weakly to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ (as $N \to \infty$), and the measure P is the limit measure of V_n as $n \to \infty$. Moreover, it was proved that $P = P_{\underline{L}}$.

It remains to prove that the support of $P_{\underline{L}}$ is the set S^r. It is well known that the support of the random element

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{\omega(p)\chi(p)}{p^s}\right)^{-1}, \quad \omega \in \Omega, \tag{10}$$

is the set S for every Dirichlet character χ. Since the space $H^r(D)$ is separable, we have

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_{r}$$

(see [20]). Therefore, it suffices to consider the measure $P_{\underline{L}}$ on the sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in \mathcal{B}(H(D)).$$

Since the Haar measure m_H^r is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$, we deduce that

$$m_H^r\{\omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A\} = \prod_{j=1}^{r} m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j, \chi_j) \in A_j\}.$$

This equality and the minimality of the support together with remark on the support of the element (10) show that the support of $P_{\underline{L}}$ is the set S^r. $\square$

3. Mean Square Estimates

Define

$$\underline{L}(s, \underline{\chi}) = (L(s, \chi_1), \dots, L(s, \chi_r)). \tag{11}$$

To pass from $\underline{L}_n(s + i\underline{a}t_\tau, \underline{\chi})$ (defined by (7)) to $\underline{L}(s + i\underline{a}t_\tau, \underline{\chi})$, certain mean square estimates for Dirichlet L-functions are necessary. Let χ be an arbitrary character modulo q.

Lemma 7. *Suppose that σ, $1/2 < \sigma < 1$, and $a \in \mathbb{R} \setminus \{0\}$ are fixed. Then, for $t \in \mathbb{R}$,*

$$\int_2^T |L(\sigma + it + iat_\tau, \chi)|^2 \, \mathrm{d}\tau \ll T(1 + |t|).$$

Proof. It is well known that, for fixed $\sigma > 1/2$,

$$\int_2^T |L(\sigma + it, \chi)|^2 \, \mathrm{d}t \ll_\sigma T.$$

Therefore, in view of Lemma 1, for $1/2 < \sigma < 1$,

$$\begin{aligned}
\int_2^T |L(\sigma + it + iat_\tau, \chi)|^2 \, \mathrm{d}\tau &= \frac{1}{a} \int_2^T \frac{1}{t'_\tau} |L(\sigma + it + iat_\tau, \chi)|^2 \, \mathrm{d}(at_\tau) \\
&= \frac{1}{a} \int_2^T \frac{1}{t'_\tau} \, \mathrm{d} \left(\int_2^{t + at_\tau} |L(\sigma + iu, \chi)|^2 \, \mathrm{d}u \right) \\
&\ll \frac{\log T}{a} \int_2^{|t| + |a| t_T} |L(\sigma + iu, \chi)|^2 \, \mathrm{d}u \\
&\ll_{\sigma,a} \log T \left(|t| + |a| \frac{T}{\log T} \right) \ll_{\sigma,a} T(1 + |t|),
\end{aligned}$$

which is the required estimate. $\square$

For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \tag{12}$$

where $\{K_l\} \subset D$ is a sequence of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is a metric in the space $H(D)$ inducing the topology of uniform convergence on compacta. Now, putting, for $\underline{g}_1 = (g_{11}, \ldots, g_{1r}), \underline{g}_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D)$,

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \le j \le r} \rho(g_{1j}, g_{2j}) \tag{13}$$

gives a metric in $H^r(D)$ inducing the product topology. The next lemma provides a certain approximation of $\underline{L}(s, \underline{\chi})$ (see definition (11)) by $\underline{L}_n(s, \underline{\chi})$.

Lemma 8. *Suppose that $\underline{a} \neq (0, \ldots, 0)$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-2} \int_2^T \underline{\rho}\left(\underline{L}(s + i\underline{a}t_\tau, \underline{\chi}), \underline{L}_n(s + i\underline{a}t_\tau, \underline{\chi})\right) \mathrm{d}\tau = 0.$$

Proof. From the definition (13) of the metric $\underline{\rho}$, it follows that it suffices to prove that, for $a \neq 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-2} \int_2^T \rho\left(L(s + iat_\tau, \chi_j), L_n(s + iat_\tau, \chi)\right) \mathrm{d}\tau = 0 \tag{14}$$

for every $j = 1, \ldots, r$. We will prove the above equality for the character χ modulo q.

Let θ be from the definition (6) of $v_n(m)$, and

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s. \tag{15}$$

Then the representation

$$L_n(s, \chi) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} L(s + z, \chi) l_n(z) \frac{\mathrm{d}z}{z},$$

is true. Its proof is the same as in Section 5.4 of [21] for the Riemann zeta-function. Hence, taking $\theta_1 > 0$, by the residue theorem, we obtain

$$L_n(s, \chi) - L(s, \chi) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} L(s + z, \chi) l_n(z) \frac{\mathrm{d}z}{z} + R_n(s, \chi), \tag{16}$$

where

$$R_n(s, \chi) = \begin{cases} 0 & \text{if } \chi \text{ is a non-principal character,} \\ \prod\limits_{p|q} \left(1 - \frac{1}{p}\right) \frac{l_n(1-s)}{1-s} & \text{otherwise.} \end{cases}$$

Let $K \subset D$ be an arbitrary compact set. Denote by $s = \sigma + iv$ the points of K, and suppose that $1/2 + 2\varepsilon \le \sigma \le 1 - \varepsilon$ with fixed $\varepsilon > 0$ for $s \in K$. More precisely, we select $\theta_1 = \sigma - \varepsilon - 1/2 \ge \varepsilon > 0$. Then, in view of (16),

$$\begin{aligned} &|L_n(s + iat_\tau, \chi) - L(s + iat_\tau, \chi)| \\ &\ll \int_{-\infty}^{\infty} |L(s + iat_\tau - \theta_1 + it, \chi)| \frac{|l_n(-\theta_1 + it)|}{|-\theta_1 + it|} \mathrm{d}t + |R_n(s + iat_\tau, \chi)|. \end{aligned}$$

Now, taking t in place of $t+v$, we get that, for $s \in K$,

$$\begin{aligned} &|L_n(s+iat_\tau,\chi) - L(s+iat_\tau,\chi)| \\ &\ll \int_{-\infty}^{\infty} |L(1/2+\varepsilon+i(t+at_\tau),\chi)| \frac{|l_n(1/2+\varepsilon-s+it)|}{|1/2+\varepsilon-s+it|}\,\mathrm{d}t \\ &\quad + |R_n(s+iat_\tau,\chi)|. \end{aligned}$$

This implies the estimate

$$\begin{aligned} &\frac{1}{T-2}\int_2^T \sup_{s\in K} |L(s+iat_\tau,\chi) - L_n(s+iat_\tau,\chi)|\,\mathrm{d}\tau \\ &\ll \frac{1}{T-2}\int_2^T \int_{-\infty}^{\infty} |L(1/2+\varepsilon+i(t+at_\tau),\chi)| \sup_{s\in K} \frac{|l_n(1/2+\varepsilon-s+it)|}{|1/2+\varepsilon-s+it|}\,\mathrm{d}t\,\mathrm{d}\tau \\ &\quad + \frac{1}{T-2}\int_2^T \sup_{s\in K} |R_n(s+iat_\tau,\chi)|\,\mathrm{d}\tau \\ &\ll J_1 + J_2, \end{aligned} \tag{17}$$

where

$$J_1 = \int_{-\infty}^{\infty} \frac{1}{T-2}\int_2^T \left(|L(1/2+\varepsilon+i(t+at_\tau),\chi)|\,\mathrm{d}\tau\right) \sup_{s\in K} \frac{|l_n(1/2+\varepsilon-s+it)|}{|1/2+\varepsilon-s+it|}\,\mathrm{d}t$$

and

$$J_2 = \frac{1}{T-2}\int_2^T \sup_{s\in K} |R_n(s+iat_\tau,\chi)|\,\mathrm{d}\tau. \tag{18}$$

It is well known that uniformly in σ, $\sigma_1 \le \sigma \le \sigma_2$, with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0.$$

Therefore, by the definition (15) of the function $l_n(s)$, we find that, for $s \in K$,

$$\begin{aligned} \left|\frac{l_n(1/2+\varepsilon-s+it)}{1/2+\varepsilon-s+it}\right| &= \frac{n^{1/2+\varepsilon-\sigma}}{\theta} \left|\Gamma\left(\frac{1/2+\varepsilon-\sigma}{\theta} + \frac{i(t-v)}{\theta}\right)\right| \\ &\ll_{\theta,K} n^{-\varepsilon} \exp\left\{-\frac{c_1}{\theta}|t|\right\}, \quad c_1 > 0. \end{aligned} \tag{19}$$

In the same way, for $s \in K$, we obtain

$$R_n(s+iat_\tau,\chi) \ll_{\theta,q,K} n^{1-\sigma} \exp\left\{-\frac{c_2}{\theta}|a|t_\tau\right\}. \tag{20}$$

Suppose that $\theta = 1/2+\varepsilon$. Then (17), (19) and Lemma 7 lead to the bound

$$J_1 \ll_{\varepsilon,K} n^{-\varepsilon} \int_{-\infty}^{\infty} (1+|t|)\exp\{-c_3|t|\}\,\mathrm{d}t \ll_{\varepsilon,K,a} n^{-\varepsilon}, \quad c_3 > 0. \tag{21}$$

Moreover, by (18), Lemma 1 and (20),

$$\begin{aligned}
J_2 &\ll_{\varepsilon,K,q} n^{1/2-2\varepsilon} \frac{1}{T-2} \int_2^T \exp\left\{ -c_4|a| \frac{\tau}{\log \tau} \right\} \mathrm{d}\tau \\
&\ll_{\varepsilon,K,q} n^{1/2-2\varepsilon} \frac{\log T}{T-2} + \frac{n^{1/2-2\varepsilon}}{T-2} \int_{\log T}^T \exp\left\{ -c_4|a| \frac{\tau}{\log \tau} \right\} \mathrm{d}\tau \\
&\ll_{\varepsilon,K,q,a} n^{1/2-2\varepsilon} \frac{\log T}{T-2}.
\end{aligned}$$

Thus, in view of (17) and (21),

$$\frac{1}{T-2} \int_2^T \sup_{s\in K} |L(s+iat_\tau,\chi) - L_n(s+iat_\tau,\chi)| \,\mathrm{d}\tau \ll_{\varepsilon,K,q,a} n^{-\varepsilon} + n^{1/2-2\varepsilon} \frac{\log T}{T-2}.$$

From this, it follows that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-2} \int_2^T \sup_{s\in K} |L(s+iat_\tau,\chi) - L_n(s+iat_\tau,\chi)| \,\mathrm{d}\tau = 0. \tag{22}$$

Now, the definition (12) of the metric ρ implies (14), which completes the proof of Lemma 8. □

4. A Limit Theorem

For $A \in \mathcal{B}(H^r(D))$, define

$$P_T(A) = \frac{1}{T-2} \mathrm{meas} \left\{ \tau \in [2,T] : \underline{L}(s+i\underline{a}t_\tau, \underline{\chi}) \in A \right\}. \tag{23}$$

In this section, we will prove the following statement.

Theorem 4. *Suppose that $a_1,\dots,a_r$ are non-zero real algebraic numbers linearly independent over $\mathbb{Q}$, and $\chi_1,\dots,\chi_r$ are arbitrary Dirichlet characters. Then P_T converges weakly to $P_{\underline{L}}$ as $T\to\infty$. The support of $P_{\underline{L}}$ is the set S^r.*

First we recall a useful property of convergence in distribution ($\xrightarrow{\mathcal{D}}$) (see Theorem 4.2 in Reference [20]).

Lemma 9. *Suppose that the space $(\mathbb{X}, d)$ is separable, the random elements X_{kn} and Y_n, $k\in\mathbb{N}$, $n\in\mathbb{N}$, are defined on the same probability space with measure μ,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$

for every $k\in\mathbb{N}$,

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\left\{ d(X_{kn}, Y_n) \geq \varepsilon \right\} = 0.$$

Then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

In the theory of weak convergence of probability measures, the notions of relative compactness and tightness of families of probability measures are very useful. We recall that the family $\{P\}$ of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a weakly convergent subsequence to a certain measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and this family is called tight, if for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1-\varepsilon$$

for all $P \in \{P\}$. By the direct Prokhorov theorem (see Theorem 5.1 in Billingsley [20]), every tight family $\{P\}$ is relatively compact. We apply the above remarks to the sequence $\{V_n : n \in \mathbb{N}\}$, where V_n (defined by (8)) is the limit measure in Lemma 5.

Lemma 10. *The sequence $\{V_n\}$ is relatively compact.*

Proof. By the above mentioned Prokhorov theorem, it suffices to prove that the sequence $\{V_n\}$ is tight.

Suppose θ_T is a random variable defined on a certain probability space with measure μ and uniformly distributed on $[2, T]$. Define the $H^r(D)$-valued random element

$$\underline{X}_{T,n} = \underline{X}_{T,n}(s) = (X_{T,n,1}(s), \ldots, X_{T,n,r}(s)) = \underline{L}_n(s + i\underline{a}t_{\theta_T}, \underline{\chi}).$$

Moreover, let

$$\underline{X}_n = \underline{X}_n(s) = (X_{n1}(s), \ldots, X_{nr}(s)) \tag{24}$$

be the $H^r(D)$-valued random element with the distribution V_n. Then Lemma 5 implies the relation

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n. \tag{25}$$

By Lemma 7 with $t = 0$, we have, for $1/2 < \sigma < 1$,

$$\int_2^T |L(\sigma + ia_j t_\tau, \chi_j)|^2 \, \mathrm{d}\tau \ll_{\sigma, a_j} T, \quad j = 1, \ldots, r. \tag{26}$$

Let K_l be a compact set from the definition of the metric ρ. Then (26) together with the Cauchy integral formula show that

$$\int_2^T \sup_{s\in K_l} |L(s + ia_j t_\tau, \chi_j)| \, \mathrm{d}\tau \ll_{l, a_j} T, \quad j = 1, \ldots, r.$$

This combined with (22) implies the inequality

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T-2} \int_2^T \sup_{s\in K_l} |L_n(s + ia_j t_\tau, \chi_j)| \, \mathrm{d}\tau \ll R_{lj}, \quad j = 1, \ldots, r. \tag{27}$$

Fix $\varepsilon > 0$, and define $M_{lj} = M_{lj}(s) = 2^l r R_{lj} \varepsilon^{-1}$. Then, in view of (27), we find that, for each $n \in \mathbb{N}$,

$$\begin{aligned} &\limsup_{T\to\infty} \mu \left\{ \exists j : \sup_{s\in K_l} |X_{T,n,j}(s)| > M_{lj} \right\} \\ &\le \sum_{j=1}^r \limsup_{T\to\infty} \mu \left\{ \sup_{s\in K_l} |X_{T,n,j}(s)| > M_{lj} \right\} \\ &\le \sum_{j=1}^r \limsup_{T\to\infty} \frac{1}{(T-2)M_{lj}} \int_2^T \sup_{s\in K_l} |L_n(s + ia_j t_\tau, \chi_j)| \, \mathrm{d}\tau \le \sum_{j=1}^r \frac{R_{lj}}{M_{lj}} = \frac{\varepsilon}{2^r}. \end{aligned}$$

This together with (25) shows that, for all $l, n \in \mathbb{N}$,

$$\mu \left\{ \exists j : \sup_{s\in K_l} |X_{n,j}(s)| > M_{lj} \right\} \le \frac{\varepsilon}{2^l}. \tag{28}$$

Define the set

$$K_j = K_j(s) = \left\{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leq M_{lj},\ l \in \mathbb{N} \right\}.$$

Then K_j is a compact set in $H(D)$, and, in virtue of (24) and (28),

$$\mu\{\underline{X}_n \in K\} \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. In other words, we have

$$V_n(K) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{V_n : n \in \mathbb{N}\}$ is tight. □

Proof of Theorem 4. By Lemma 10, there exists a subsequence $\{V_{n_k}\}$ of the sequence $\{V_n\}$ that is weakly convergent to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. This can be written as

$$\underline{X}_{n_k} \xrightarrow[k \to \infty]{\mathcal{D}} P. \tag{29}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}_T = \underline{X}_T(s) = \underline{L}(s + i\underline{a}t_{\theta_T}, \underline{\chi}).$$

Then Lemma 8 implies that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n \to \infty} \limsup_{T \to \infty} \mu \left\{ \underline{\rho}(X_T, X_{T,n}) \geq \varepsilon \right\} \\ &\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{(T-2)\varepsilon} \int_2^T \underline{\rho} \left(\underline{L}(s + i\underline{a}t_\tau, \underline{\chi}), \underline{L}_n(s + i\underline{a}t_\tau, \underline{\chi}) \right) \mathrm{d}\tau = 0. \end{aligned}$$

The latter equality together with (25), (29), and Lemma 9 shows that

$$\underline{X}_T \xrightarrow[T \to \infty]{\mathcal{D}} P, \tag{30}$$

or, in other words, P_T converges weakly to P as $T \to \infty$. Moreover, by the relation (30), the measure P is independent of the subsequence $\{V_{n_k}\}$. Thus, we deduce that

$$\underline{X}_n \xrightarrow[n \to \infty]{\mathcal{D}} P,$$

or V_n converges weakly to P as $n \to \infty$. Therefore, the theorem follows by Lemma 6. □

5. Proof of Universality

The proof of Theorem 3 is based on Mergelyan's theorem on the approximation of analytic functions by polynomials [22], Theorem 4, and the properties of weak convergence. For convenience, we state them as lemmas.

Lemma 11 (Mergelyan theorem)**.** *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $f(s)$ be a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

We recall that $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of the measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $P(\partial A) = 0$, where ∂A is a boundary of A.

Lemma 12. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following statements are equivalent:*

1° *P_n converges weakly to P as $n \to \infty$;*

2° *For every open set $G \subset \mathbb{X}$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G);$$

3° *For every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The above lemma is a part of Theorem 2.1 from Reference [20]. Now, we can give the proof of Theorem 3.

Proof of Theorem 3. First part. In view of Lemma 11, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2}. \tag{31}$$

The set

$$G_\varepsilon^r = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leq j\leq r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\} \tag{32}$$

is an open neighbourhood of the element $\left(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)}\right) \in S^r$. Thus, by Theorem 4, $P_{\underline{L}}(G_\varepsilon^r) > 0$, where the distribution $P_{\underline{L}}$ is defined by (9). Hence, from Theorem 4 again and Lemma 12,

$$\liminf_{T\to\infty} P_T(G_\varepsilon^r) \geq P_{\underline{L}}(G_\varepsilon^r) > 0,$$

and the definitions (23) and (32) of P_T and G_ε^r together with (31) prove the first part of the theorem.

Second part. Introduce one more set

$$A_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leq j\leq r} \sup_{s\in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}. \tag{33}$$

Then the boundary of A_ε lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leq j\leq r} \sup_{s\in K_j} \left| g_j(s) - f_j(s) \right| = \varepsilon \right\},$$

thus, $\partial A_{\varepsilon_1} \cap \partial A_{\varepsilon_2} = \varnothing$ for different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. This shows that the set A_ε is a continuity set of the measure $P_{\underline{L}}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemma 12,

$$\lim_{T\to\infty} P_T(A_\varepsilon) = P_{\underline{L}}(A_\varepsilon) \tag{34}$$

for all but at most countably many $\varepsilon > 0$. Moreover, (31) shows the inclusion $G_\varepsilon^r \subset A_\varepsilon$. This, (34) and the definitions (23) and (33) of P_T and A_ε prove the second assertion of the theorem. □

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