Nonlinear Analysis: Modelling and Control, Vol. 25, No. 2, 301–320 ISSN: 1392-5113
https://doi.org/10.15388/namc.2020.25.16519 eISSN: 2335-8963

Square root of a multivector in 3D Clifford algebras

Adolfas Dargys[a], **Artūras Acus**[b,1]

[a]Center for Physical Sciences and Technology,
Semiconductor Physics Institute,
Saulėtekio 3, LT-10257, Vilnius, Lithuania
adolfas.dargys@ftmc.lt

[b]Institute of Theoretical Physics and Astronomy,
Vilnius University,
Saulėtekio 3, LT-10257 Vilnius, Lithuania
arturas.acus@tfai.vu.lt

Received: April 8, 2019 / **Revised:** July 1, 2019 / **Published online:** March 2, 2020

Abstract. The problem of square root of multivector (MV) in real 3D ($n = 3$) Clifford algebras $Cl_{3,0}$, $Cl_{2,1}$, $Cl_{1,2}$ and $Cl_{0,3}$ is considered. It is shown that the square root of general 3D MV can be extracted in radicals. Also, the article presents basis-free roots of MV grades such as scalars, vectors, bivectors, pseudoscalars and their combinations, which may be useful in applied Clifford algebras. It is shown that in mentioned Clifford algebras, there appear isolated square roots and continuum of roots on hypersurfaces (infinitely many roots). Possible numerical methods to extract square root from the MV are discussed too. As an illustration, the Riccati equation formulated in terms of Clifford algebra is solved.

Keywords: geometric Clifford algebra, experimental mathematics, square root of multivector, infinitely many roots, Riccati equation.

1 Introduction

The square root and the power of a real number are the two simplest nonlinear operations with a long history [10]. Solution by radicals of a cubic equation was first published after very long period, in 1545 by G. Cardano. Simultaneously, a concept of square root of a negative number as well as of complex number have been developed [10]. A. Cayley was the first to carry over the notion of the square root to matrices [6]. In the recent book by Higham [11], where an extensive literature is presented on nonlinear functions of matrices, two sections are devoted to analysis of square and pth roots of general matrix. The existence of matrix square root here is related to matrix positive eigenvalues. In the context of noncommutative Clifford algebra (CA), the main attempts up till now were concentrated on the square roots of quaternions [19, 20] or their derivatives such as coquaternions (also called split quaternions) or nectarines [9, 20, 21]. The square root

[1]The author was supported by the European Social Fund under grant No. 09.3.3-LMT-K-712-01-0051.

of biquaternion (complex quaternion) was considered in [22]. All they are isomorphic to 2D Clifford algebras (CA) $Cl_{0,2}$, $Cl_{1,1}$, $Cl_{2,0}$, and therefore, the quaternionic square root analysis can be easily carried over to CA. In this paper, we shall be interested in higher 3D CAs, where the main object is the 8-component multivector (MV).

The understanding and investigation of square roots in noncommutative CAs is still in its infancy. The published formulas are not complete or even erroneous in case of a general MV [7]. The most akin to present results are the investigations of conditions for existence of square root of -1 [12, 13, 22] in the Clifford–Fourier transforms and CA-based wavelet theory [14]. In contrast to complex number Fourier transform (we shall remind that complex number algebra is isomorphic to $Cl_{0,1}$), the existence of variety of square roots of -1 in noncommutative CAs allows to create new geometric kernels for Fourier transforms and wavelets as well as for left-right and double-sided Fourier transforms. More important is the fact that the CA allows to extend analysis beyond complex numbers [5, 13]. In paper [7], along with functions for MVs, the authors provide several formulas for square roots in $Cl_{3,0}$. However, the general formula in [7] as shown in this article is incorrect.

Since the square root is a nonlinear operation, a nonstandard approach is needed. For this purpose, we have applied the Gröbner-basis algorithm to analyze the system of polynomial equations that ensue from MV equation $\mathsf{A}^2 = \mathsf{B}$, where A and B are the MVs. The Gröbner basis is accessible in symbolic mathematical packages such as *Mathematica* and *Maple*. In this paper, the formulas for practical calculations of the square root of MVs in all 3D Clifford algebra are presented. In case of MV, we have found that a characteristic property of roots in 3D Clifford algebras is that the MV may have no roots, a single or multiple isolated roots, or even an infinitely many roots in the parameter space.

In Section 2, the needed notation is introduced. In Section 3, the method of calculation of roots in described. Then, we prove that in 3D CAs, the square root of MV can be expressed in radicals. In Sections 4 and 5, formulas for roots of simple MVs are presented. Possible numerical methods of calculating the roots are discussed in Section 6, and finally, application of MV square root to solve Riccati equation in CA is demonstrated in Section 7.

2 Clifford algebra and notation

The Clifford algebra $Cl_{p,q}$ is an associative noncommutative algebra, where (p, q) indicates vector space metric. In 3D case, the MV consists of the following elements (basis blades) $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}$, where $\mathbf{e}_i$ are the orthogonal basis vectors, and $\mathbf{e}_{ij}$ are the bivectors (oriented planes). The last term is called the pseudoscalar. The shorthand notation, for example, $\mathbf{e}_{ij}$ means $\mathbf{e}_{ij} = \mathbf{e}_i \circ \mathbf{e}_j$, where the circle indicates the geometric or Clifford product. Usually, the multiplication symbol is omitted, and one writes $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j$. The number of subscripts indicates the grade of basis element, so that the scalar is a grade-0 element, the vectors are the grade-1 elements etc. In the orthonormalized basis, the products of vectors satisfies

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}.$$

For $Cl_{3,0}$ and $Cl_{0,3}$ algebras, correspondingly, the squares of basis vectors are $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_i^2 = -1$, where $i = 1, 2, 3$. For mixed signature algebras $Cl_{2,1}$ and $Cl_{1,2}$, we have, respectively, $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3^2 = -1$ and $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = -1$. As a result, the sign of squares of blades depends on considered algebra. For example, in $Cl_{3,0}$, we have $\mathbf{e}_{12}^2 = \mathbf{e}_{12}\mathbf{e}_{12} = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1(+1)\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_1 = -1$. However, in $Cl_{1,2}$, we have $\mathbf{e}_{12}^2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1(-1)\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_1 = +1$.

The MV is a sum of different blades multiplied by real coefficients a_A. In 3D algebras, the general MV expanded in coordinates (basis elements) reads

$$\begin{aligned}\mathsf{A} &= a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{31}\mathbf{e}_{31} + a_{123}I \\ &= a_0 + \mathbf{a} + \mathcal{B} + a_{123}I,\end{aligned} \tag{1}$$

where $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathcal{B} = a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{31}\mathbf{e}_{31}$ represents, respectively, the vector and bivector parts of MV. The geometric product of two MVs A and B gives new multivector $\mathsf{C} = \mathsf{A} \circ \mathsf{B}$, which has the same structure. In the following, the circle will be omitted, and AB will be interpreted as the geometric product of two MVs. Since we shall need the square of MV, below we present the full expression of A^2 in case of $Cl_{3,0}$,

$$\mathsf{C} = \mathsf{AA} \equiv \mathsf{A}^2 = c_0 + \mathbf{c} + \mathcal{C} + c_{123}I, \tag{2}$$

where the scalar c_0, vector $\mathbf{c}$, bivector $\mathcal{C}$ and pseudoscalar $c_{123}I$, $c_{123} \in \mathbb{R}$, expressed in components are

$$\begin{aligned}c_0 &= a_0^2 + (a_1^2 + a_2^2 + a_3^2) - (a_{12}^2 + a_{13}^2 + a_{23}^2) - a_{123}^2, \\ \mathbf{c} &= 2(a_0a_1 - a_{23}a_{123})\mathbf{e}_1 + 2(a_0a_2 + a_{13}a_{123})\mathbf{e}_2 + 2(a_0a_3 - a_{13}a_{123})\mathbf{e}_3, \\ \mathcal{C} &= 2(a_0a_{12} + a_3a_{123})\mathbf{e}_{12} + 2(a_0a_{13} - a_2a_{123})\mathbf{e}_{13} + 2(a_0a_{23} + a_1a_{123})\mathbf{e}_{23}, \\ c_{123} &= 2(a_0a_{123} + a_1a_{23} - a_2a_{13} + a_3a_{12}).\end{aligned} \tag{3}$$

Since we are considering real CAs, all coefficients a_A, where A is the multiindex, are real numbers. For remaining algebras $Cl_{2,1}$, $Cl_{1,2}$ and $Cl_{0,3}$, the product formulas are similar to (3) except that signs before scalar coefficients may be different. Two main operations in CAs are the addition of MVs and their geometric product. Also, two supplementary operations – dot (or inner) and wedge (or outer) products – are frequently used in CA since they provide geometric interpretation to CA formulas and allow to introduce simpler notation. For example, in case of vectors $\mathbf{a}$ and $\mathbf{b}$ (italic notation, a and b, will be used too), the vector dot and wedge products can be expressed via geometric product respectively as $\mathbf{a} \cdot \mathbf{b} \equiv a \cdot b = (ab + ba)/2$ and $\mathbf{a} \wedge \mathbf{b} \equiv a \wedge b = (ab - ba)/2$. To select the scalar a_0 in MV (1), the grade selector $\langle \mathsf{A} \rangle_0 = a_0$ is introduced. In general, $\langle \mathsf{A} \rangle_k$ selects kth grade elements from the MV A, where $k = 1, 2, 3$ designates vector, bivector, or pseudoscalar. More about Clifford algebras and MV properties can be found, for example, in books [8, 18].

3 Square root of MV in 3D Clifford algebras

To calculate the square root of MV, in this paper, we have applied the Gröbner-basis algorithm of *Mathematica* package (command **GroebnerBasis[..]**) as well as commands such as **Solve[..]**, **Eliminate[..]** and **Reduce[..]**, which are based on the Gröbner basis to solve polynomial systems of algebraic equations. The square roots of MV B are those multivectors A, which satisfy $\mathsf{A}^2 = \mathsf{B}$. The simplest considered multivector A is the elementary blade the square of which is equal to -1. For example, in $Cl_{3,0}$, the bivector $\mathbf{e}_{ij}$ and pseudoscalar $I = \mathbf{e}_{123}$ are of this type since $\mathbf{e}_{ij}^2 = -1$ and $I^2 = -1$. If B and C are two MVs, due to noncommutativity of CA, we have that, in general, $\sqrt{\mathsf{B}}\sqrt{\mathsf{C}} \neq \sqrt{\mathsf{BC}} \neq \sqrt{\mathsf{CB}} \neq \sqrt{\mathsf{C}}\sqrt{\mathsf{B}}$. However, we shall assume that $\sqrt{\mathsf{B}}\sqrt{\mathsf{B}} = \mathsf{B}$. This property implies that the square root may have two signs. However, as we shall see in Section 4, this property is not general enough. In cases when there is an infinite number of roots, instead of $\pm\sqrt{\mathsf{B}}$, we shall obtain more general (parameterized) expressions that also represent the square roots of B (see Sections 4 and 5).

3.1 Method of calculation

The method of calculation of MV square root is based on solution of equation $\mathsf{A}^2 = \mathsf{B}$, where $\mathsf{A} = a_0 + a_1\mathbf{e}_1 + \cdots + a_{12}\mathbf{e}_{12} + \ldots$ and $\mathsf{B} = b_0 + b_1\mathbf{e}_1 + \cdots + b_{12}\mathbf{e}_{12} + \cdots$ are the MVs in a fixed frame. Expanding A^2 (see Eqs. (2) and (3)) and equating scalar coefficients at the same grades of B, the system of real polynomial equations was constructed with respect to unknown coefficients $\{a_0, a_1, a_2, \ldots, a_{12}, \ldots, a_{123}\}$. In case of 3D CAs, we have eight coupled real nonlinear equations. Real solutions (coefficients a_A) were found and analyzed with *Mathematica* package. The computer result consisted of a set of possible solutions (from 2 up to 28 depending on a number of nonzero coefficients in B). Since we consider real CAs only, the complex solutions (coefficients a_A) were rejected. The remaining real coefficients were grouped in plus/minus pairs $\pm\sqrt{\mathsf{B}}$ in case of isolated roots, or in alike pairs in case of infinite many roots. Finally, the results were checked both symbolically and numerically whether $F^2(\mathsf{B}) = \mathsf{B}$ is satisfied indeed. The numerical check allowed to eliminate spurious roots due to *Mathematica* specific manipulations with roots.

3.1.1 Example

As an illustration of the method, we will calculate the square root of MV in 1D, namely, in algebra $Cl_{0,1}$, which is isomorphic to complex number algebra. The general MV in this algebra is $\mathsf{A} = a_0 + a_1\mathbf{e}_1$ the square of which gives $\mathsf{A}^2 = a_0^2 - a_1^2 + 2a_0a_1\mathbf{e}_1$. If $\mathsf{B} = b_0 + b_1\mathbf{e}_1 = \mathsf{A}^2$ is introduced, then MV equation $\mathsf{A}^2 = \mathsf{B}$ yields two scalar equations: $a_0 - a_1 = b_0$ and $2a_0a_1 = b_1$. Solution of this nonlinear system with respect to a_0 and a_1 gives four roots,

$$(a_0, a_1) = \pm\left(\frac{\sqrt{c_-}d_-}{(\sqrt{2}\,b_1)}, \frac{\sqrt{c_-}}{\sqrt{2}}\right), \qquad (a_0, a_1) = \pm\left(\frac{\sqrt{c_+}d_+}{(\sqrt{2}\,b_1)}, \frac{\sqrt{c_+}}{\sqrt{2}}\right), \tag{4}$$

where $c_\pm = -b_0 \pm (b_0^2 + b_1^2)^{1/2}$ and $d_\pm = b_0 \pm (b_0^2 + b_1^2)^{1/2}$. The first solution in (4) must be rejected since $\sqrt{c_-}$ brings in the imaginary coefficients into real CA. Thus, we have two square MV roots in $Cl_{0,1}$ algebra,

$$\sqrt{\mathsf{B}} = \pm\frac{\sqrt{-\frac{1}{2}(\mathsf{B}+\widehat{\mathsf{B}}) + \sqrt{\widehat{\mathsf{B}}\mathsf{B}}}\,(\sqrt{\widehat{\mathsf{B}}\mathsf{B}}+\mathsf{B})}{\sqrt{2}\sqrt{-\mathsf{B}\cdot\mathsf{B}}}, \tag{5}$$

where $\mathsf{B} = b_0 + b_1\mathbf{e}_1$, and $\widehat{\mathsf{B}} = b_0 - b_1\mathbf{e}_1$ is the grade inverse of B (an analogue of complex conjugate in the complex algebra). The grade inversion and inner product were used to represent the square root in the basis-free manner: $(\mathsf{B}+\widehat{\mathsf{B}})/2 = b_0$, $\widehat{\mathsf{B}}\mathsf{B} = b_0^2 + b_1^2$, $\mathsf{B}\cdot\mathsf{B} = -b_1^2$. If latter expressions are inserted back into (5), one finds the well-known formula for the square root of complex number. The role of imaginary unit in $Cl_{0,1}$ is played by basis vector $\mathbf{e}_1$ since $\mathbf{e}_1^2 = -1$. If $b_0 = 0$, then (5) reduces to $\sqrt{\mathsf{B}} = \pm\sqrt{b_1/2}(1+\mathbf{e}_1)$, which will remain in real CA if $b_1 > 0$. If MV is a scalar, i.e., $\mathsf{B} = b_0$, then Eq. (5) becomes singular, and the system $a_0^2 - a_1^2 = b_0$, $2a_0a_1 = 0$ is to be solved once more. Then the trivial root is $\sqrt{\mathsf{B}} = \pm\sqrt{b_0}$ if $a_1 = 0$, and $\sqrt{\mathsf{B}} = \pm\sqrt{|b_0|}\,\mathbf{e}_1$ if $a_0 = 0$. This simple example shows that for those MVs that consist of individual grades (scalar, vector, etc) the roots must be calculated anew. As we shall see in Section 4.1.1, in higher dimensional noncommutative algebras, in such cases, there may appear free parameters that are connected with a continuum of roots.

The described method of finding square root was also applied to 2D Clifford algebras $Cl_{0,2}$, $Cl_{1,1}$ and $Cl_{2,0}$, which are isomorphic to quaternion, coquaternion and conecterine algebras [20]. We have found that in these algebras, there are square roots with additional free parameters that bring in a continuum of roots (unpublished results).

3.2 Square roots of general MV in 3D

The paper is devoted to investigation of square roots of blades and their simple combinations. Nonetheless, below we shall outline a constructive proof that in 3D algebras, the problem of square root of general multivector (MV) can be solved in radicals. Numerical solutions (see Section 6.1) show that at least four roots exist in case of general MV. In case of individual blades, as we shall see in Sections 4 and 5, infinitely many roots may appear, in a sharp contrast to general case where only isolated roots are allowed. The situation where there are infinitely many roots of quaternionic scalar has been noted earlier [15] and explained by existence of equivalence class in such roots. Thus, in CA, we have the situation where partial solutions do not necessarily follow from general solution, and, therefore, they must be considered to be as important as the general one. This is one of reasons why the present paper is confined to simple MVs, mainly to blade roots using computer-aided mathematics. Let us return back to roots of general MV.

Mathematica was unable to find the square root of the most general MV symbolically by the method described in Section 3.1. Below we shall show that, nonetheless, such a root does exist and even can be expressed in radicals. At first, we shall rewrite the initial MV A in terms of two scalars, s and S, and two vectors, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and $\mathbf{V} = V_1\mathbf{e}_1 + V_2\mathbf{e}_2 + V_3\mathbf{e}_3$ in the following form: $\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I$. Equating $\mathsf{B} = b_0 + b_1\mathbf{e}_2 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{23} + b_{13}\mathbf{e}_{13} + b_{23}\mathbf{e}_{23} + b_{123}I$ to the square of A, one obtains the nonlinear system of scalar equations

$$b_0 = s^2 - S^2 + \mathbf{v}^2 - \mathbf{V}^2, \qquad b_{123} = 2(sS + \mathbf{v}\cdot\mathbf{V}), \tag{6}$$
$$b_1 = 2(sv_1 - SV_1), \qquad b_{23} = 2(sV_1 + Sv_1), \tag{7}$$
$$b_2 = 2(sv_2 - SV_2), \qquad b_{13} = 2(sV_2 + Sv_2), \tag{8}$$
$$b_3 = 2(sv_2 - SV_2), \qquad b_{12} = 2(sV_3 + Sv_3). \tag{9}$$

The above system can be divided into subsystems (6) and (7)–(9). Subsystem (7)–(9) of six equations can be solved with respect to vector components (v_1, v_2, v_3) and (V_1, V_2, V_3),

$$v_1 = \frac{b_1 s + b_{23} S}{2(s^2+S^2)}, \qquad V_1 = \frac{b_{23} s - b_1 S}{2(s^2+S^2)}, \tag{10}$$
$$v_2 = -\frac{b_{13} s + b_2 S}{2(s^2+S^2)}, \qquad V_2 = -\frac{b_{13} s + b_2 S}{2(s^2+S^2)}, \tag{11}$$
$$v_3 = \frac{b_3 s + b_{12} S}{2(s^2+S^2)}, \qquad V_3 = \frac{b_{12} s - b_3 S}{2(s^2+S^2)}. \tag{12}$$

It should be noted that the vectors $\mathbf{v}$ and $\mathbf{V}$ are functions of scalars s and S only. In this way obtained components of $\mathbf{v}$ and $\mathbf{V}$ are inserted back into scalar and pseudoscalar in Eqs. (6). This yields two coupled nonlinear algebraic equations for two unknowns s and S

$$\begin{aligned} &4\big(b_0 - s^2 + S^2\big)\big(s^2+S^2\big)^2 \\ &\quad = -(b_{23}s - b_1 S)^2 + (b_2 s - b_{13}S)^2 + (b_3 s + b_{12}S)^2 \\ &\qquad - (b_1 s + b_{23}S)^2 - (b_{13}s + b_2 S)^2 - (b_{12}s - b_3 S)^2, \\ &2(b_{123} - 2sS)\big(s^2+S^2\big)^2 \\ &\quad = -(b_2 s - b_{13}S)(b_{13}s + b_3 S) + (b_{23}s - b_1 S) \\ &\qquad \times (b_1 s + b_{23}S)(b_3 s + b_{12}S)(b_{12}s - b_3 S). \end{aligned} \tag{13}$$

Since the highest degree of s and S in (13) is $\geqslant 5$, the solution in radicals is not guaranteed to exist. However, the system can be easily solved numerically to give 4 isolated roots as demonstrated in Section 6.1. Fortunately, it appears that the change of variable $s\,S = t$ and $-s^2 + S^2 = T$ reduces system (13) to simpler one[2],

$$\begin{aligned} &2(b_0 + 2T)(4t - b_{123}) - b_I = 0, \\ &b_S - (b_0 - b_{123} + 2T + 4t)(b_0 + b_{123} + 2T - 4t) = 0, \end{aligned} \tag{14}$$

where the coordinate-free notation for the MV parts,

$$b_S = \langle \mathsf{B}\widetilde{\widehat{\mathsf{B}}}\rangle_0 = b_0^2 - b_1^2 - b_2^2 - b_3^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 - b_{123}^2$$

[2]The substitution $s^2 + S^2 = T$ also lowers the degree of the initial equation. However, the symmetry $s \leftrightarrow S$ has the side effect of producing parasite real solutions, which have to be discarded in the final answer. The symmetry $s \leftrightarrow S$ is absent in the multivector $\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I$, therefore, the asymmetric substitution better suits our purpose.

and

$$b_I = \langle \mathsf{B}\widetilde{\widehat{\mathsf{B}}} I\rangle_0 = 2b_3b_{12} - 2b_2b_{13} + 2b_1b_{23} - 2b_0b_{123},$$

has been introduced. The tilde is the index reversion operator, for example, $\widetilde{\mathbf{e}}_{12} = \mathbf{e}_{21}$. Since the substitution itself and the resulting system of equations with respect to t and T are both of degree $\leqslant 4$, we conclude that the initial system (13) can be solved in radicals, i.e., an explicit formula of square root of MV when $n \leqslant 3$ can be obtained. In particular, we can solve s and S from $s\,S = t$ and $-s^2 + S^2 = T$, which yields two real solutions:

$$s_{1,2} = \pm\frac{\sqrt{-T + \sqrt{T^2 + 4t^2}}}{\sqrt{2}}, \qquad S_{1,2} = \pm\frac{\sqrt{2}t}{\sqrt{-T + \sqrt{T^2 + 4t^2}}}. \tag{15}$$

In (15), the sign of both s_i and S_i should be the same. The other two solutions, which are obtained from (15) by the substitution $\sqrt{T^2 + 4t^2} \to -\sqrt{T^2 + 4t^2}$, are complex because of the inequality $\sqrt{T^2 + 4t^2} \geqslant T$.

Equation (14) can be solved in radicals in a compact way as well. The two of solutions

$$\begin{aligned} t_1 &= \frac{1}{4}\left(b_{123} - \frac{\sqrt{D - b_S}}{\sqrt{2}}\right), \qquad T_1 = \frac{-\sqrt{2}\sqrt{D - b_S}(b_S + D) - 2b_0b_I}{4b_I}, \\ t_2 &= \frac{1}{4}\left(b_{123} + \frac{\sqrt{D - b_S}}{\sqrt{2}}\right), \qquad T_2 = \frac{\sqrt{2}\sqrt{D - b_S}(b_S + D) - 2b_0b_I}{4b_I} \end{aligned} \tag{16}$$

are real-valued due to the inequality $D \geqslant b_S$, where $D = \sqrt{b_S^2 + b_I^2}$ denotes the square root of determinant norm of MV B [2]. The other two solutions, which can be obtained from (16) by substitution $D \to -D$, are complex valued.

To summarize, formulas (15), (16) and (6)–(12) yield four explicit real solutions and completely determine the square root $\mathsf{A} = s + \mathbf{v} + (S + \mathbf{V})I = \sqrt{\mathsf{B}}$ in $Cl_{3,0}$ algebra in terms of radicals. Using the suggested transformations and notations, similar explicit formulas can be easily obtained for $Cl_{2,1}$, $Cl_{1,2}$ and $Cl_{0,3}$ algebras too. More about square roots of a general MV can be found in [3].

4 Roots of simple MVs in $Cl_{3,0}$

Below the square roots of scalars, vectors, bivectors and pseudoscalars, and their combinations such as scalar-plus-vector and scalar-plus-pseudoscalar are calculated by the method described in Section 3.1 for algebras $Cl_{3,0}$, $Cl_{2,1}$, $Cl_{1,2}$ and $Cl_{0,3}$.

4.1 Root of plus/minus scalar

Let A be $Cl_{3,0}$ algebra MV the square of which is either positive or negative scalar, $\mathsf{A}^2 = \pm s$ and $s \in \mathbb{R}$. Since A^2 is assumed to consist of all grades, from $\mathsf{A}^2 = \pm s$ follows that the right-hand sides of resulting nonlinear system of equations are $\langle \mathsf{A}^2\rangle_k = 0$ if $k = 1, 2, 3$, and $\langle \mathsf{A}^2\rangle_0 = \pm s$ for scalar part. Such a system of eight equations is undetermined and allows free parameters in the solution. Using the method described in Section 3.1, below we will demonstrate that, apart from trivial roots, there may appear solutions that contain free parameters that represent infinitely many square roots.

4.1.1 Square root of scalar in $Cl_{3,0}$

When $\mathsf{A}^2 = \mathsf{B} = s > 0$, the trivial solution is $\mathsf{A} = \sqrt{\mathsf{B}} = \pm\sqrt{s}$. Since all three basis vectors in $Cl_{3,0}$ satisfy $\mathbf{e}_i^2 = 1$, the solution also can be written as a vector $\mathsf{A} = \pm\sqrt{s}\,\mathbf{e}_i$, where $i = 1, 2, 3$. When $\mathsf{A}^2 = \mathsf{B} = s < 0$, it is evident that the trivial solutions can be expressed by bivectors $\mathsf{A} = \sqrt{\mathsf{B}} = \pm\sqrt{s}\,\mathbf{e}_{ij}$, or pseudoscalar $\mathsf{A} = \sqrt{\mathsf{B}} = \pm\sqrt{s}\,I$ since $\mathbf{e}_{ij}^2 = -1$ and $I^2 = -1$.

If solution method based on Gröbner basis (Section 3.1) is addressed, apart from these trivial expressions, one can find a more general formulas for square roots of plus/minus scalar that include up to four real free parameters

$$\begin{aligned} \sqrt{s_{1,2}} &= \frac{-\alpha\beta\delta \pm w\gamma}{\gamma^2 + \delta^2}\mathbf{e}_1 + \frac{\alpha\beta\gamma \pm w\delta}{\gamma^2 + \delta^2}\mathbf{e}_2 + \alpha\mathbf{e}_3 + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13} + \delta\mathbf{e}_{23}, \\ w &= \sqrt{-\alpha^2\bigl(\beta^2 + \gamma^2 + \delta^2\bigr) + \bigl(\gamma^2 + \delta^2\bigr)\bigl(s + \beta^2 + \gamma^2 + \delta^2\bigr)}. \end{aligned} \tag{17}$$

The root $\sqrt{s_1}$ or $\sqrt{s_2}$, respectively, corresponds to upper and lower signs. The formula is valid when $s_{1,2} > 0$ and $s_{1,2} < 0$. The Greek letters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are optional real parameters, which, in fact, represent MV coefficients that appeared redundant in solving the system of polynomial equations: $a_3 \to \alpha$, $a_{12} \to \beta$, $a_{13} \to \gamma$, $a_{23} \to \delta$. It can be checked that after squaring of the right side of (17), one gets $(\sqrt{s_1})^2 = \sqrt{s_1} \times \sqrt{s_1} = s$ and $(\sqrt{s_2})^2 = \sqrt{s_2}\sqrt{s_2} = s$, i.e., after squaring, all free parameters will cancel out automatically. Thus, the appearance of free parameters allows Eq. (17) to have an infinite number of square roots. Also, one should note that the structure of the roots in (17) is different from simple expression $\pm\sqrt{\mathsf{B}}$ when parameters are absent. If all four parameters are equated to zero in an appropriate order to avoid the infinity, one gets simple expression, either $+\sqrt{s}\,\mathbf{e}_1$ and $+\sqrt{s}\,\mathbf{e}_2$, or $-\sqrt{s}\,\mathbf{e}_1$ and $-\sqrt{s}\,\mathbf{e}_2$, the squares of which give the initial scalar s. More interesting cases are those where either three, two or single of parameters are equated to zero.

Single real parameter

1. If $\alpha = \beta = \delta = 0$, then $\sqrt{s} = \pm\sqrt{s + \gamma^2}\,\mathbf{e}_1 + \gamma\mathbf{e}_{13}$.
2. If $\alpha = \beta = \gamma = 0$, then $\sqrt{s} = \pm\sqrt{s + \delta^2}\,\mathbf{e}_2 + \delta\mathbf{e}_{23}$.

Here the roots lie on hyperbolas in parameter space $(\pm\sqrt{s + \gamma^2}, \gamma)$ or $(\pm\sqrt{s + \delta^2}, \delta)$ with branches pointing to left and right.

Two real parameters

1. If $\alpha = \beta = 0$, then $\sqrt{s} = \pm\bigl(\sqrt{s + \gamma^2 + \delta^2}/\sqrt{\gamma^2 + \delta^2}\bigr)(\gamma\mathbf{e}_1 + \delta\mathbf{e}_2) + \gamma\mathbf{e}_{13} + \delta\mathbf{e}_{23}$.
2. If $\alpha = \gamma = 0$, then $\sqrt{s} = \pm\sqrt{s + \beta^2 + \delta^2}\,\mathbf{e}_2 + \beta\mathbf{e}_{12} + \delta\mathbf{e}_{23}$.
3. If $\alpha = \delta = 0$, then $\sqrt{s} = \pm\sqrt{s + \beta^2 + \gamma^2}\,\mathbf{e}_1 + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13}$.

4. If $\beta = \gamma = 0$, then $\sqrt{s} = \pm\sqrt{s - \alpha^2 + \delta^2}\,\mathbf{e}_2 + \alpha\mathbf{e}_3 + \delta\mathbf{e}_{23}$.
5. If $\beta = \delta = 0$, then $\sqrt{s} = \pm\sqrt{s - \alpha^2 + \gamma^2}\,\mathbf{e}_1 + \alpha\mathbf{e}_3 + \gamma\mathbf{e}_{13}$.

Here the roots lie on either one-sheeted, $(\pm\sqrt{s - \alpha^2 + \delta^2}, \alpha, \delta)$, or two-sheeted, $(\pm\sqrt{s + \beta^2 + \delta^2}, \beta, \delta)$, hyperboloids in the parameter space.

Three real parameters

1. If $\alpha = 0$, then
$$\sqrt{s} = \pm\frac{\sqrt{s + \beta^2 + \gamma^2 + \delta^2}}{\sqrt{\gamma^2 + \delta^2}}(\gamma\mathbf{e}_1 + \delta\mathbf{e}_2) + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13} + \delta\mathbf{e}_{23}.$$
2. If $\beta = 0$, then
$$\sqrt{s} = \pm\frac{\sqrt{s - \alpha^2 + \gamma^2 + \delta^2}}{\sqrt{\gamma^2 + \delta^2}}(\gamma\mathbf{e}_1 + \delta\mathbf{e}_2) + \alpha\mathbf{e}_3 + \gamma\mathbf{e}_{13} + \delta\mathbf{e}_{23}.$$
3. If $\gamma = 0$, then
$$\sqrt{s} = \pm\sqrt{s + \beta^2 + \delta^2 - \alpha^2\delta^{-2}(\beta^2 + \delta^2)}\,\mathbf{e}_2 - \alpha\beta\delta^{-1}\mathbf{e}_1 + \alpha\mathbf{e}_3 + \beta\mathbf{e}_{12} + \delta\mathbf{e}_{23}.$$
4. If $\delta = 0$, then
$$\sqrt{s} = \pm\sqrt{s + \beta^2 + \gamma^2 - \alpha^2\gamma^{-2}(\beta^2 + \gamma^2)}\,\mathbf{e}_1 + \alpha\beta\gamma^{-1}\mathbf{e}_2 + \alpha\mathbf{e}_3 + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13}.$$

Finally, we shall note that in case of quaternionic root of a scalar, there appear two free parameters [unpublished results], and the structure of root formula is very similar to $Cl_{3,0}$, namely, $\sqrt{s} = \pm\sqrt{s - \alpha^2 - \beta^2}\,\mathbf{e}_1 + \alpha\mathbf{e}_2 + \beta\mathbf{e}_{12}$. If basis vectors of $Cl_{0,2}$ algebra are replaced by quaternionic imaginary units ($\mathbf{e}_1 \to \mathrm{k}$, $\mathbf{e}_2 \to \mathrm{j}$, $\mathbf{e}_{12} \to \mathrm{i}$) and free parameters are replaced by angles $\beta = \sqrt{s}\sin\theta\cos\varphi$ and $\alpha = \sqrt{s}\sin\theta\sin\varphi$, the formula for quaternionic root of scalar can be reduced to $\sqrt{s}(\pm\cos\theta\,\mathrm{k} + \sin\theta\sin\varphi\,\mathrm{j} + \sin\theta\cos\varphi\,\mathrm{i})$, which represents the equation of sphere. Thus, in the quaternionic case, we have a continuum of roots on sphere of radius $\sqrt{s}$ [15].

4.1.2 *Square root of plus/minus scalar in* $Cl_{0,3}$

In contrast to Euclidean $Cl_{3,0}$ algebra, in case of anti-Euclidean $Cl_{0,3}$ algebra, the square root of scalar was found to have only trivial solutions. No free parameters have been detected. When scalar $s > 0$, the roots are
$$\sqrt{s} = \pm\sqrt{s}, \quad \sqrt{s} = \pm\sqrt{s}\,I, \quad s > 0.$$
The second root comes from pseudoscalar property $I^2 = 1$ in $Cl_{0,3}$.

More interesting is that in $Cl_{0,3}$, the square root of negative scalar, i.e., $\pm\sqrt{-s}$ when $s > 0$, does not exist at all, i.e., no MV solutions with real coefficients have been found by algorithm described in Section 3.1.

4.2 Roots of vector

Table 1 summarizes all possible square roots of vector $a = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ in 3D Clifford algebras. I is the pseudoscalar of a respective algebra. The value of a^2 depends on algebra: $a^2 = \pm(a_1^2 + a_2^2 + a_3^2)$ in $Cl_{3,0}$ (upper sign) and $Cl_{0,3}$ (lower sign); $a^2 = a_1^2 - a_2^2 - a_3^2$ in $Cl_{1,2}$ and $a^2 = a_1^2 + a_2^2 - a_3^2$ in $Cl_{2,1}$. One can see that, in general, there are more then two square roots of vector; the largest number (up to 8) is in $Cl_{1,2}$. The free parameters do not appear.

Table 1. Square roots of vector $a = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $a_i \in \mathbb{R}$, in 3D Clifford algebras. The numerical value and sign of a^2 depends on algebra and vector coefficients a_i. The root $\sqrt{a}$ belongs to real Clifford algebra if expression under the square root in formulas is positive.

Algebra	Square roots of vector a	Constraint
$Cl_{3,0}$	$\sqrt{a} = \begin{cases} \pm\dfrac{\sqrt{a^2}(1-I) + a(1+I)}{2(\sqrt{a^2})^{1/2}}, \\ \pm\dfrac{\sqrt{a^2}(1+I) + a(1-I)}{2(\sqrt{a^2})^{1/2}} \end{cases}$	$a^2 > 0$
$Cl_{0,3}$	$\sqrt{a} = \begin{cases} \pm\dfrac{\sqrt{-a^2} + a}{(2\sqrt{-a^2})^{1/2}}, \\ \pm\dfrac{(\sqrt{-a^2} + a)I}{(2\sqrt{-a^2})^{1/2}} \end{cases}$	$a^2 < 0$
$Cl_{1,2}$	$\sqrt{a} = \pm\dfrac{(a^2)^{1/4}}{2}\Big((-1+I) - \dfrac{a}{(\sqrt{a^2})^{1/2}}(1+I)\Big)$	$a^2 > 0$
	$\sqrt{a} = \pm\dfrac{(a - \sqrt{-a^2})I}{(2\sqrt{-a^2})^{1/2}}$	$a^2 < 0$
	$\sqrt{a} = \pm\dfrac{(a^2)^{1/4}}{2}\Big((-1-I) + \dfrac{a}{(\sqrt{a^2})^{1/2}}(-1+I)\Big)$	$a^2 > 0$
	$\sqrt{a} = \pm\dfrac{(a + \sqrt{-a^2})}{(2\sqrt{-a^2})^{1/2}}$	$a^2 < 0$
$Cl_{2,1}$	$\sqrt{a} = \begin{cases} \pm\dfrac{a + \sqrt{-a^2}}{(2\sqrt{-a^2})^{1/2}}, \\ \pm\dfrac{(a + \sqrt{-a^2})I}{(2\sqrt{-a^2})^{1/2}} \end{cases}$	$a^2 < 0$

4.2.1 Examples

In $Cl_{3,0}$, there are four square roots of vector $\mathbf{e}_1$ ($a^2 > 0$), namely, $\sqrt{\mathbf{e}_1} = \pm(1 + \mathbf{e}_1 + \mathbf{e}_{23} - I)/2$ and $\pm(-1 - \mathbf{e}_1 + \mathbf{e}_{23} - I)/2$. Similarly, in $Cl_{0,3}$, the roots are $\sqrt{\mathbf{e}_1} = \pm(1 + \mathbf{e}_1)I/\sqrt{2}$ and $\sqrt{\mathbf{e}_1} = \pm(1 + \mathbf{e}_1)/\sqrt{2}$.

In $Cl_{1,2}$, the square roots of basis vector $\mathbf{e}_2$ ($a^2 < 0$) are $\sqrt{\mathbf{e}_2} = \pm(1 + \mathbf{e}_2)/\sqrt{2}$, whereas the square roots of $\mathbf{e}_1$ ($a^2 > 0$) are $\sqrt{\mathbf{e}_1} = \pm(1 + \mathbf{e}_1 - \mathbf{e}_{23} + I)/2$.

In $Cl_{2,1}$, the roots of $\mathbf{e}_3$ ($a^2 < 0$) are $\sqrt{\mathbf{e}_3} = \pm(1+\mathbf{e}_3)/\sqrt{2}$ and $\sqrt{\mathbf{e}_3} = \pm(1+\mathbf{e}_3)I/\sqrt{2}$, and $\sqrt{-\mathbf{e}_3} = (1-\mathbf{e}_3)/\sqrt{2}$ and $\sqrt{-\mathbf{e}_3} = (1-\mathbf{e}_3)I/\sqrt{2}$. The square root of $\mathbf{e}_1$ ($a^2 > 0$) does not exist. Since in this algebra $I^2 = 1 > 0$, the square root of pseudoscalar I ($a^2 > 0$) does not exist too. In this algebra, the square roots of general vector $a = \mathbf{e}_1 + \mathbf{e}_2/2 + \sqrt{3}\,\mathbf{e}_3$ ($a^2 = -7/4 < 0$) are $\sqrt{a} = \pm(\sqrt{7} + 2(\mathbf{e}_1 + \mathbf{e}_2/2 + \sqrt{3}\,\mathbf{e}_3))/(2\cdot 7^{1/4})$ and $\sqrt{a} = \pm(\sqrt{7} + 2(\mathbf{e}_1 + \mathbf{e}_2/2 + \sqrt{3}\,\mathbf{e}_3))I/(2\cdot 7^{1/4})$.

4.3 Roots of bivector

Table 2 shows that in 3D algebras, the bivector can have two ($Cl_{3,0}$), four ($Cl_{0,3}$ and $Cl_{2,1}$) and eight ($Cl_{1,2}$) isolated square roots. In the table, $\mathcal{B} = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}$ is the bivector, and $\widetilde{\mathcal{B}} = -\mathcal{B}$ is the reverse bivector. I is the pseudoscalar of respective algebra. The product $\mathcal{B}\widetilde{\mathcal{B}}$ in different algebras is: $\mathcal{B}\widetilde{\mathcal{B}} = a_{12}^2 + a_{13}^2 + a_{23}^2$ in $Cl_{3,0}$; $\mathcal{B}\widetilde{\mathcal{B}} = -a_{12}^2 - a_{13}^2 - a_{23}^2$ in $Cl_{0,3}$; $\mathcal{B}\widetilde{\mathcal{B}} = -a_{12}^2 - a_{13}^2 + a_{23}^2$ in $Cl_{1,2}$; $\mathcal{B}\widetilde{\mathcal{B}} = a_{12}^2 - a_{13}^2 - a_{23}^2$ in $Cl_{2,1}$. The free parameters do not appear.

Table 2. Square roots of bivector $\mathcal{B} = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}$, $a_{ij} \in \mathbb{R}$ in 3D Clifford algebras.

Algebra	Square root of bivector $\mathcal{B}$	Constraint
$Cl_{3,0}$	$\sqrt{\mathcal{B}} = \pm\frac{\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}+\mathcal{B}}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}}$	$\mathcal{B}\widetilde{\mathcal{B}} > 0$
$Cl_{0,3}$	$\sqrt{\mathcal{B}} = \begin{cases} \pm\frac{\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}+\mathcal{B}}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}} \\ \pm\frac{(\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}+\mathcal{B})I}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}} \end{cases}$	$\mathcal{B}\widetilde{\mathcal{B}} > 0$
$Cl_{1,2}$	$\sqrt{\mathcal{B}} = \begin{cases} \pm\frac{\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}+\mathcal{B}}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}} \\ \pm\frac{(\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}-\mathcal{B})I}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}} \end{cases}$	$B\widetilde{B} > 0$
	$\sqrt{\mathcal{B}} = \pm\frac{\sqrt{-\mathcal{B}\widetilde{\mathcal{B}}}(1+I)+\mathcal{B}(1-I)}{(2\sqrt{-\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}}$	$B\widetilde{B} < 0$
	$\sqrt{\mathcal{B}} = \pm\frac{\sqrt{-\mathcal{B}\widetilde{\mathcal{B}}}(1+I)+\mathcal{B}(1-I)}{(2\sqrt{-\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}}$	$B\widetilde{B} < 0$
$Cl_{2,1}$	$\sqrt{\mathcal{B}} = \begin{cases} \pm\frac{\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}+\mathcal{B}}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}} \\ \pm\frac{(\sqrt{\mathcal{B}\widetilde{\mathcal{B}}}-\mathcal{B})I}{(2\sqrt{\mathcal{B}\widetilde{\mathcal{B}}})^{1/2}} \end{cases}$	$B\widetilde{B} > 0$

4.3.1 Examples

In $Cl_{3,0}$, the square roots of basis bivector $\mathbf{e}_{12}$ are $\sqrt{\mathbf{e}_{12}} = \pm(1+\mathbf{e}_{12})/\sqrt{2}$. For a full bivector, they are $\sqrt{\mathbf{e}_{12}-2\mathbf{e}_{13}+3\mathbf{e}_{23}} = (\sqrt{14}+\mathbf{e}_{12}-2\mathbf{e}_{13}+3\mathbf{e}_{23})/(2^3 7)^{1/4}$. In $Cl_{0,3}$, they are $\sqrt{\mathbf{e}_{12}} = \pm(\mathbf{e}_3 - I)/\sqrt{2} = (1-\mathbf{e}_{12})\mathbf{e}_3/\sqrt{2}$.

In $Cl_{1,2}$, $\sqrt{\mathbf{e}_{12}} = \pm(1+\mathbf{e}_3+\mathbf{e}_{12}-I)/2$ and $\sqrt{\mathbf{e}_{23}} = \pm\frac{1}{2}(1+\mathbf{e}_{23})/\sqrt{2}$. In $Cl_{2,1}$, $\sqrt{\mathbf{e}_{12}} = \pm(1+\mathbf{e}_{12})/\sqrt{2}$. For a full bivector $\mathcal{B} = 4\mathbf{e}_{12}+\mathbf{e}_{13}+3\mathbf{e}_{23}$ of $Cl_{2,1}$ algebra, we find $\sqrt{\mathcal{B}} = (\sqrt{6}+4\mathbf{e}_{12}+\mathbf{e}_{13}+3\mathbf{e}_{23})/(2^{3/4}3^{1/4})$.

4.4 Roots of pseudoscalar

Only square roots in Euclidean algebra $Cl_{3,0}$ is considered here. It should be remembered that in MV equation $\mathsf{A}^2 = \mathsf{B}$, the MV A takes into account all gradings, $\mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23} + a_{123}I$, whereas the root of B consists of pseudoscalar $\sqrt{\mathsf{B}} = \sqrt{pI}$ only, where $p \in \mathbb{R}$. *Mathematica* command **Solve[..]** finds 28 possible candidates for roots of $\mathsf{A}^2 = \mathsf{B}$, the most of which have complex coefficients and, thus, must be rejected. Only four of roots have real coefficients. Similarly as in the scalar case, those coefficients of A that survive in the final result will be treated as free parameters. All in all, we have found that five of coefficients in A may be considered as free parameters: $\varepsilon = a_2$, $\alpha = a_3$, $\beta = a_{12}$, $\beta = a_{12}$, $\gamma = a_{13}$, and $\delta = a_{23}$. The root

$$\sqrt{pI} = \pm\frac{\sqrt{p}(1+I)}{\sqrt{2}}, \quad p \in \mathbb{R},\ I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \tag{18}$$

has no free parameters. The second root has two free parameters β and ε,

$$\sqrt{pI} = \pm\sqrt{-(p')^2+\beta^2-\varepsilon^2}\,\mathbf{e}_1 + \varepsilon\mathbf{e}_2 + p'\mathbf{e}_3 + \beta\mathbf{e}_{12}, \quad p' = \frac{p}{2\beta}, \quad p \in \mathbb{R}. \tag{19}$$

The third root has three free parameters α, β, γ,

$$\begin{aligned}\sqrt{pI} = \pm\frac{1}{2\gamma}\Big(&\sqrt{-p^2+4p\alpha\beta-4\big(\alpha^2-\gamma^2\big)\big(\beta^2+\gamma^2\big)}\,\mathbf{e}_1 \\ &- p\mathbf{e}_2 + 2(\alpha+\gamma\mathbf{e}_1)(\beta\mathbf{e}_2+\gamma\mathbf{e}_3)\Big), \quad p \in \mathbb{R}.\end{aligned} \tag{20}$$

Finally, the fourth root has four free parameters α, β, δ, γ,

$$\begin{aligned}\sqrt{pI} = {}& \frac{\pm\gamma w + (p-2\alpha\beta)\delta}{2(\gamma^2+\delta^2)}\,\mathbf{e}_1 + \frac{\pm\delta w + (p-2\alpha\beta)\gamma}{2(\gamma^2+\delta^2)}\,\mathbf{e}_2 \\ &+ \alpha\mathbf{e}_3 + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13} + \delta\mathbf{e}_{23},\end{aligned} \tag{21}$$

where

$$w = \sqrt{-p^2+4p\alpha\beta-4\big(\alpha^2-\gamma^2-\delta^2\big)\big(\beta^2+\gamma^2+\delta^2\big)}.$$

Formulas (18)–(21) are valid when $p > 0$ and $p < 0$, i.e., for roots of positive and negative pseudoscalar. Depending on sign of p, it can be checked that the squares of right-hand sides of (18)–(21) simplify to either $+pI$ or $-pI$.

5 Square roots of two-element MVs

5.1 Square roots of paravector

The electromagnetism theory is inherently relativistic and requires 4D algebra $Cl_{1,3}$ for its full description [8]. Nonetheless, the covariant formulation of electromagnetism can be introduced if paravectors of smaller $Cl_{3,0}$ algebra are addressed [4]. The paravector is a sum of scalar and vector. The geometric product of paravector, $p = a_0 + \mathbf{a}$, and its grade conjugate, $\widehat{p} = a_0 - \mathbf{a}$, gives scalar $p\widehat{p} = a_0^2 - \mathbf{a}^2$, which is just the Minkowski space-time metric if a_0 is interpreted as time component.

We have found that in $Cl_{3,0}$, the paravector has 12 square roots as summarized in Table 3. The roots assume two different forms. In all formulas in Table 3, the expressions under square roots must be positive for MV coefficients to be real.

Table 3. Square roots of paravector $p = s + \mathbf{a}$, where s is the scalar and $\mathbf{a}$ is the vector. $\widehat{p} = s - \mathbf{a}$, $p\widehat{p} = s^2 - \mathbf{a}^2$, $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$, $|\mathbf{a}| = \sqrt{\mathbf{a}^2}$, $I \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

Algebra	Square root of paravector $p = s + \mathbf{a}$	Constraint
$Cl_{3,0}$	$\sqrt{p} = \pm\dfrac{(-s - \sqrt{p\widehat{p}})^{1/2}}{\sqrt{2\mathbf{a}^2}}\left(-p + \sqrt{p\widehat{p}}\right)\mathbf{a}I$	$p\widehat{p} > 0$ and
	$\sqrt{p} = \pm\dfrac{(-s + \sqrt{p\widehat{p}})^{1/2}}{\sqrt{2\mathbf{a}^2}}\left(-p - \sqrt{p\widehat{p}}\right)\mathbf{a}I$	$(\pm s \pm \sqrt{p\widehat{p}}) > 0$
	$\sqrt{p} = \pm\dfrac{(s - \sqrt{p\widehat{p}})^{1/2}}{\sqrt{2\mathbf{a}^2}}\left(p + \sqrt{p\widehat{p}}\right)\mathbf{a}$	
	$\sqrt{p} = \pm\dfrac{(s + \sqrt{p\widehat{p}})^{1/2}}{\sqrt{2\mathbf{a}^2}}\left(p - \sqrt{p\widehat{p}}\right)\mathbf{a}$	
	$\sqrt{p} = \pm\left(\sqrt{s + \lvert\mathbf{a}\rvert}\left(\dfrac{1+\hat{\mathbf{a}}}{2}\right) + \sqrt{-s + \lvert\mathbf{a}\rvert}\left(\dfrac{1-\hat{\mathbf{a}}}{2}\right)I\right)$	$\pm s + \lvert\mathbf{a}\rvert > 0$
	$\sqrt{p} = \pm\left(\sqrt{s - \lvert\mathbf{a}\rvert}\left(\dfrac{1-\hat{\mathbf{a}}}{2}\right) - \sqrt{-s - \lvert\mathbf{a}\rvert}\left(\dfrac{1+\hat{\mathbf{a}}}{2}\right)I\right)$	$\pm s - \lvert\mathbf{a}\rvert > 0$

5.2 Square root of scalar + pseudoscalar

In $Cl_{3,0}$, the square roots of $a = s + pI$, where $s, p \in \mathbb{R}$, in a simple (nonparameterized) form are

$$\begin{aligned}\sqrt{a} &= \sqrt{s + pI} \\ &= \pm\frac{p + (-s + \sqrt{s^2 + p^2})I}{\sqrt{2}\sqrt{-s + \sqrt{s^2 + p^2}}} = \pm\frac{\langle -aI\rangle + (-\langle a\rangle + (a\widetilde{a})^{1/2})I}{\sqrt{2}\sqrt{-\langle a\rangle + (a\widetilde{a})^{1/2}}},\end{aligned} \tag{22}$$

where $s = \langle a\rangle_0$ and $p = \langle -aI\rangle_0$. In the parameterized form, the expressions for square roots may have two, three or four parameters that represent the coefficients of A, namely, $\mu = a_2$, $\alpha = a_3$, $\beta = a_{12}$, $\gamma = a_{13}$, $\delta = a_{23}$.

The square root with two free parameters β and μ is

$$\sqrt{a} = \sqrt{s + pI} = \pm\sqrt{s - \left(\frac{p}{2\beta}\right)^2 + \beta^2 - \mu^2}\,\mathbf{e}_1 + \mu\mathbf{e}_2 + \frac{p}{2\beta}\mathbf{e}_3 + \beta\mathbf{e}_{12}. \tag{23}$$

The square root with three free parameters α, β and γ is

$$\begin{aligned}\sqrt{a} = &\pm\frac{\sqrt{-p^2 + 4p\alpha\beta + 4\gamma^2(s + \beta^2 + \gamma^2) - 4\alpha^2(\beta^2 + \gamma^2)}}{2\gamma}\,\mathbf{e}_1\\ &- \frac{p - 2\alpha\beta}{2\gamma}\mathbf{e}_2 + \alpha\mathbf{e}_3 + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13}.\end{aligned} \tag{24}$$

The square root with four free parameters α, β, δ and γ is

$$\begin{aligned}\sqrt{a} &= \frac{\pm w\gamma + (p - 2\alpha\beta)\delta}{2(\gamma^2 + \delta^2)}\mathbf{e}_1 + \frac{\pm w\delta - (p - 2\alpha\beta)\gamma}{2(\gamma^2 + \delta^2)}\mathbf{e}_2\\ &\quad + \alpha\mathbf{e}_3 + \beta\mathbf{e}_{12} + \gamma\mathbf{e}_{13} + \delta\mathbf{e}_{23},\\ w &= \sqrt{-p^2 + 4p\alpha\beta + 4s\bigl(\gamma^2 + \delta^2\bigr) + 4\bigl(-\alpha^2 + \gamma^2 + \delta^2\bigr)\bigl(\beta^2 + \gamma^2 + \delta^2\bigr)}.\end{aligned} \tag{25}$$

The four-parameter formula depicts a hypersurface in five dimensional parameter space, where infinite number of roots exists. After squaring of the right-hand side of (25), all parameters in the formula cancel out to yield $a = s + pI$. If $\alpha = \beta = \gamma = 0$, formula (25) reduces to

$$\sqrt{a} = \frac{p\mathbf{e}_1 \pm \sqrt{-p^2 + 4\delta^2(s + \delta^2)}\,\mathbf{e}_2}{2\delta} + \delta\mathbf{e}_{23},$$

where, as always, the expression under the square root must be positive. The formula has singularity at $\delta = 0$. After squaring, the singularity disappears. The presence of vector $\mathbf{e}_1$ and bivector $\mathbf{e}_{23}$ ensures that after squaring of the right-hand side the elementary pseudoscalar $I = \mathbf{e}_{123}$ will appear in $a = s + pI$.

6 Numerical methods

6.1 Numerical solution of Eq. (13)

The pair of coupled equations in (13) may be useful in numerical calculations of MV square roots by standard numerical programs. For example, if coefficients in B have the following values $b_0 = 0.6$, $b_1 = 1.4$, $b_2 = 2.1$, $b_3 = 3.1$, $b_{12} = 1.2$, $b_{13} = 1.3$, $b_{23} = 2.3$, $b_{123} = 2.2$, then the equations in (13) simplify to

$$\begin{aligned} s^2 - S^2 + \frac{1.89s^2 + 4.21sS - 1.89S^2}{(s^2 + S^2)^2} &= 0.6,\\ 2sS + \frac{2.105s^2 - 3.78sS - 2.105S^2}{(s^2 + S^2)^2} &= 2.2. \end{aligned}$$

Solution of this system by *Mathematica* gives four real roots, $r_{1,2} = \pm\{s \approx 0.918303, S \approx -0.371171\}$ and $r_{3,4} = \pm\{s \approx 1.17858, S \approx 1.22253\}$. The remaining roots generated by *Mathematica* are complex conjugate pairs and, consequently, do not represent MV coefficients. The vectors $\mathbf{v}$ and $\mathbf{V}$ can be found from Eqs. (10)–(12). Finally, the plus/minus multivectors related with roots $r_{1,2}$ are

$$\sqrt{\mathsf{B}} = \pm(0.918303 + 0.220137\mathbf{e}_1 + 1.22877\mathbf{e}_2 + 1.22386\mathbf{e}_3 + 1.14805\mathbf{e}_{12} + 0.211169\mathbf{e}_{13} + 1.34129\mathbf{e}_{23} - 0.371171I).$$

The remaining roots $r_{3,4}$ also give two plus/minus multivectors,

$$\sqrt{\mathsf{B}} = \pm(1.17858 + 0.773648\mathbf{e}_1 + 0.153581\mathbf{e}_2 + 0.887881\mathbf{e}_3 - 0.4119\mathbf{e}_{12} + 0.710817\mathbf{e}_{13} + 0.173255\mathbf{e}_{23} + 1.22253I).$$

After squaring of roots, the initial MV $\mathsf{B} = 0.6 + 1.4\mathbf{e}_1 + 2.1\mathbf{e}_2 + 3.1\mathbf{e}_3 + 1.2\mathbf{e}_{12} + 1.3\mathbf{e}_{13} + 2.3\mathbf{e}_{23} + 2.2I$ is recovered.

6.2 Square root by series expansion

The standard series formula of square root is

$$\sqrt{1 \pm x} = 1 \pm \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots, \quad x^2 \leqslant 1, \tag{26}$$

or, in short,

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n-3)!!}{2n!!}x^n, \qquad \sqrt{1-x} = 1 - \sum_{n=1}^{\infty}\frac{(2n-3)!!}{2n!!}x^n, \tag{27}$$

where $k!! = k(k-2)(k-4)\cdots$. We shall generalize (26) and (27) to MV case. Now, 1 and x are replaced by, respectively, scalar and remaining part of MV. If A is the multivector, then in the Clifford algebra, formula (26) becomes

$$\sqrt{\mathsf{A}} = \langle\mathsf{A}\rangle^{1/2}\sqrt{1 + \left(\frac{\mathsf{A}}{\langle\mathsf{A}\rangle} - 1\right)} \equiv \langle\mathsf{A}\rangle^{1/2}\sqrt{1+\mathsf{B}},$$

where $\langle\mathsf{A}\rangle$ is the scalar part, $\mathsf{B} = \mathsf{A}/\langle\mathsf{A}\rangle - 1$ and $\sqrt{1+\mathsf{B}}$ is MV series similar to (26),

$$\sqrt{1+\mathsf{B}} = 1 + \frac{1}{2}\mathsf{B} - \frac{1\cdot 1}{2\cdot 4}\mathsf{B}^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\mathsf{B}^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\mathsf{B}^4 + \cdots, \quad \|\mathsf{B}\|^2 \leqslant 1. \tag{28}$$

The condition $x^2 \leqslant 1$ now must be replaced by condition for norm $\|\mathsf{B}\| \leqslant 1$. For example, let us assume that $\mathsf{A} = 1 + \mathbf{e}_1/2 + \mathbf{e}_{12}/2 + \mathbf{e}_{123}/3$. If series (28) is summed up to B^{10}, then in $Cl_{3,0}$, we find that

$$\begin{aligned}\sqrt{\mathsf{A}} &= \sqrt{1 + \frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_{12} + \frac{1}{3}\mathbf{e}_{123}} \\ &= 1.01343 + 0.240357\mathbf{e}_1 + 0.03900\mathbf{e}_3 + 0.240357\mathbf{e}_{12} \\ &\quad - 0.03900\mathbf{e}_{23} + 0.164458\mathbf{e}_{123}.\end{aligned} \tag{29}$$

After squaring the right-hand side of (29), we have

$$\begin{aligned}\mathsf{A} \approx 1. &+ 0.500001\mathbf{e}_1 + 0.500001\mathbf{e}_{12} + 1.95\cdot 10^{-7}\mathbf{e}_3\\ &- 1.95\cdot 10^{-7}\mathbf{e}_{23} + 0.333333\mathbf{e}_{123},\end{aligned}$$

which shows that the error is in the seventh digit, and therefore, the terms with $\mathbf{e}_3$ and $\mathbf{e}_{23}$ must be rejected. The norm of MV under square root is $\sqrt{20/18} \approx 1.27$, which is somewhat larger than unity. To reduce the total number of MV multiplications, it may be useful to rewrite the MV series in the Horner form:

$$\sqrt{1+\mathsf{B}} = 1 + \frac{1}{2}\mathsf{B}\biggl(1 - \frac{1}{4}\mathsf{B}\biggl(1 - \frac{3}{6}\mathsf{B}\biggl(1 - \frac{5}{8}\mathsf{B}(1 - \cdots)\biggr)\biggr)\biggr).$$

6.3 Newton's iteration method

As known, the square root of a real number a can be effectively calculated by Newton's iteration formula, $\sqrt{a} \to x_{k+1} = (x_k + a/x_k)/2$. The iteration frequently starts with $x_0 = a$. This formula works in matrix form as well, albeit with some modifications to improve the convergence [11]. Here we present the first attempt to see whether, in principle, the Newton's method is applicable to CA. If multiplication is replaced by MV multiplication, the Newton's formula can be written in CA in the following form:

$$\mathsf{X}_{k+1} = \frac{1}{2}\bigl(\mathsf{X}_k + \mathsf{X}_k^{-1}\mathsf{A}\bigr), \quad \mathsf{X}_0 \approx \mathsf{A}. \tag{30}$$

X_0 is an initial MV, $k = 0$, in the iteration. Also, the term $\mathsf{X}_k^{-1}\mathsf{A}$ may be replaced by $\mathsf{A}\mathsf{X}_k^{-1}$. Numerical experiments show that Newton–Clifford formula (30) works as well. To avoid divergence during iteration, it is recommended the scaling, which consists of division of all MV coefficients by the largest coefficient c and multiplication of the final result by $\sqrt{c}$. As an example, let us find the square root of

$$\mathsf{A} = \mathbf{e}_1 + 2\mathbf{e}_3 + 2\mathbf{e}_{12} + \mathbf{e}_{23}, \quad \mathsf{A} \in Cl_{3,0}.$$

After 12 iterations, we obtain $\sqrt{\mathsf{A}} \approx 1.16172 + 0.215199\mathbf{e}_1 + 0.430397\mathbf{e}_3 + 1.03907\mathbf{e}_{12} + 0.519536\mathbf{e}_{23} - 0.481199\mathbf{e}_{123}$, the squaring of which returns back the MV A with a high precision. However, the Newton's algorithm comes to a vicious cycle if one tries to find the square root of a pure vector. To break the circle, we have added a small seed bivector to initial MV X_0. For example, we have assumed that $\mathsf{X}_0 = \mathsf{A} + \mathbf{e}_{12}/5$, where now A is the vector $\mathsf{A} = \mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3$. Then, the Newton's algorithm after 12 iterations and with a scaling factor 4 (division of A by 4 and multiplication of $\sqrt{\mathsf{A}}$ by 2) gives the following approximate root:

$$\begin{aligned}\sqrt{\mathsf{A}} \approx 0.9672 &+ 0.2487\mathbf{e}_1 + 0.7722\mathbf{e}_2 + 0.5267\mathbf{e}_3 + 0.5085\mathbf{e}_{12}\\ &- 0.7839\mathbf{e}_{13} + 0.2501\mathbf{e}_{23} - 0.9672\mathbf{e}_{123}.\end{aligned}$$

Damped oscillation were observed during iteration. For a comparison, the exact value calculated with the first formula in Table 1 gives,

$$\sqrt{\mathsf{A}} \approx 0.9672 + 0.2585\mathbf{e}_1 + 0.7755\mathbf{e}_2 + 0.5170\mathbf{e}_3 + 0.5170\mathbf{e}_{12} - 0.7755\mathbf{e}_{13} + 0.2585\mathbf{e}_{23} - 0.9672\mathbf{e}_{123}.$$

Thus, after 12 Newton's iterations, the error spreads out from the third significant digit. Further investigations are needed to understand the stability and convergence properties of the Newton–Clifford iterator. Also, one should remember that the MV may have multiple roots. The investigations of convergence of roots in matrix theory [11] are mainly concerned with matrices having special spectrum, which must lie in the real half of a complex plane of eigenvalues to guarantee the fast convergence [11]. On the other hand, the matrices that represent Clifford algebra are totally different. Their properties are described by 8-periodicity table [18] rather than matrix spectrum and, as a consequence, the totally different methods that take into account the periodicity are required to investigate convergence properties of Newton–Clifford algorithm.

7 Application: Algebraic Riccati equation

The matrix Riccati equation is frequently addressed in control and systems theory [1, 16, 17]. Solution of quaternionic ($Cl_{0,2}$) Riccati equation has been presented in [16]. Here we shall consider the Riccati–Clifford equation where matrices are replaced by MVs and matrix product by geometric (Clifford) product.

The simplest quadratic CA equations are

$$\mathsf{X}^2 = \mathsf{B} \quad \text{and} \quad \mathsf{XAX} = \mathsf{B}, \tag{31}$$

where A and B are known and X is unknown MV. The solutions of (31) can be expressed through the square root and inversion of MV,

$$\mathsf{X} = \pm\sqrt{\mathsf{B}} \quad \text{and} \quad \mathsf{X} = \pm\sqrt{\mathsf{BA}}\,\mathsf{A}^{-1} = \pm\mathsf{A}^{-1}\sqrt{\mathsf{AB}}.$$

The validity of the second solution can be easily verified in the following way: $\mathsf{XAX} = (\pm\sqrt{\mathsf{BA}}\,\mathsf{A}^{-1})\mathsf{A}(\pm\sqrt{\mathsf{BA}}\,\mathsf{A}^{-1} = \sqrt{\mathsf{BA}}\sqrt{\mathsf{BA}}\,\mathsf{A}^{-1} = \mathsf{BA}\,\mathsf{A}^{-1} = \mathsf{B}$.

A simple Riccati–Clifford equation with a linear term added,

$$\mathsf{XAX} + p\mathsf{X} = \mathsf{B}, \quad p \in \mathbb{R}, \tag{32}$$

can be solved in a similar manner. After multiplication by A from right and addition of $(p/2)^2$ to both sides, Eq. (32) reduces to

$$\left(\mathsf{C} + \frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 + \mathsf{BA}, \quad \mathsf{C} = \mathsf{XA},$$

the solution of which is

$$\mathsf{X} = \left(-\frac{p}{2} \pm \sqrt{\mathsf{BA} + \left(\frac{p}{2}\right)^2}\right)\mathsf{A}^{-1}.$$

A slightly general Riccati–Clifford equation

$$\mathsf{XAX}+\mathsf{CX}+\mathsf{XC}=\mathsf{B} \tag{33}$$

can be solved too if MV C belongs to the center of geometric algebra, i.e., C commutes with X. For example, in $Cl_{3,0}$ algebra, the center is a sum of scalar and pseudoscalar, $\mathsf{C}=\alpha+\beta I$, $\alpha,\beta\in\mathbb{R}$. Then, one can check that the solution of Eq. (33) is

$$\mathsf{X}_{1,2}=\bigl(-\mathsf{C}\pm\sqrt{\mathsf{BA}+\mathsf{C}^2}\,\bigr)\mathsf{A}^{-1}. \tag{34}$$

If MVs are replaced by scalars ($\mathsf{A}\to a$, $\mathsf{B}\to c$ and $\mathsf{C}\to b/2$) and geometric product by standard product, then (34) reduces to the well-known scalar quadratic equation for roots, $x_{1,2}=(-b\pm\sqrt{b^2+4ac})/2a$. Thus, to solve the Clifford–Riccati equation one must know MV square root. How to calculate the inverse MV A^{-1} in (34), recently it was described in [2]. It should be noted that the solution (34) remains valid for an arbitrary CA if C is the center of algebra.

Finally, let us illustrate the solution (34) when all known MVs in Riccati–Clifford equation (33) consist of scalar and pseudoscalar: $\mathsf{A}=1+2I$, $\mathsf{B}=2+3I$, $\mathsf{C}=3+4I$. Then $\mathsf{A}^{-1}=(1-2I)/5$. Since under square root the sum of scalar and pseudoscalar appears, Eq. (22) (for example, with plus sign) can be applied,

$$\mathsf{X}=\frac{53\sqrt{2}+4\sqrt{541}-22\sqrt{11+\sqrt{1082}}+(-51\sqrt{2}+2\sqrt{541}+4\sqrt{11+\sqrt{1082}})I}{10\sqrt{11+\sqrt{1082}}}.$$

Insertion of X back into (33) shows that the Riccati–Clifford equation is satisfied indeed.

A more interesting situation arises when free parameters are allowed in the square root. For example, if we assume that $\mu=0$ in (23), then we have the following square root with a single parameter β: $\sqrt{s+pI}=+\sqrt{s-(p/2\beta)^2+\beta^2}\mathbf{e}_1+(p/2\beta)\mathbf{e}_3+\beta\mathbf{e}_{12}$. Then, the solution of Riccati–Clifford equation (34) is

$$\mathsf{X}_\beta=-\frac{11}{5}+\frac{1}{10}\biggl(\sqrt{-44+4\beta^2-\frac{961}{\beta^2}}\,\mathbf{e}_1+\biggl(4\beta+\frac{31}{\beta}\biggr)\mathbf{e}_3\biggr)$$
$$+\frac{1}{5}\biggl(\biggl(\beta-\frac{31}{\beta}\biggr)\mathbf{e}_{12}-\sqrt{-44+4\beta^2-\frac{961}{\beta^2}}\,\mathbf{e}_{23}\biggr)+\frac{2}{5}I.$$

This one-parameter solution also satisfies the Riccati–Clifford equation (33). For a solution to exist, one must ensure that the MV coefficients are real, i.e., the free parameter must satisfy $-44+4\beta^2-961/\beta^2\geqslant 0$ or $\beta\geqslant(11/2+\sqrt{541/2}\,)^{1/2}$. Similarly, using Eqs. (22)–(25), one can find the solutions of algebraic Riccati–Clifford equation with two, three, or even with four free parameters.

8 Conclusions

In the present article, the Gröbner-basis algorithm was applied to extract the square root from MV in real algebras (CA) $Cl_{3,0}$, $Cl_{2,1}$, $Cl_{1,2}$ and $Cl_{0,3}$. It is shown that the square root of general MV can be expressed in radicals, and there are four isolated roots in mentioned algebras. In this article, the main attention, however, was concentrated on square roots of individual grades of MV or their combinations. A number of concrete square root formulas for scalars, vectors, bivectors and pseudoscalars as well as scalar-vector and scalar-pseudoscalar combinations are presented in a basis-free form. It is shown that roots of graded MVs may accommodate a number free parameters that bring a continuum of square roots on respective parameter hypersurfaces and therefore become an important ingredient in the nonlinear problems. Concrete expressions that contain up to four free parameters are presented. The free parameters appear in those MVs, which belong to the center of algebra, and therefore, the MVs make up an equivalence class in the considered algebra. Also, a number of numerical methods to calculate square roots in CAs are demonstrated. However, more investigations are required to optimize and adapt them in calculations of multiple roots. Conversely to matrix square roots [11], where the spectrum of matrix must be positive for a square root to exist, in CA where the MV can be represented by matrices too (although of special form as follows from the 8-periodicity table [18]), no such requirement is needed. The Riccati–Clifford equation was solved in CA terms, which is expected to pave the way to control theory based on more-and-more popular CA.

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