THE MATHEMATICAL DEFINITION OF THE BUBBLES AND CRASHES

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Introduction

Since the seminal work of D. Sornette et al. [3] that found evidence of log-periodic structures in several crashes in a variety of markets, an abundance of research has found similar structure in financial markets. D. Sornette [4] argued that a bubble leading to a crash can be quantified in terms of a log-periodic power law in order to predict when a stock market will crash. Sornette et al. [5] have given a mathematical description of the end of the bubbles as the spontaneous singularity assuming that a crash follows after rapid growth of economic indicators faster than an exponential. Although their method is practically useful for prediction of the end of a bubble, it says nothing about the start of a bubble. K. Watanabe et al. [6] proposed the method how to predict the start of the bubble. In this article we use this latter method to investigate the real estate market in Lithuania during the bubble and crash.

In recent years the existence of bubbles and crashes in the world's property markets has become the main Focus for both policy makers and investors. From a macroeconomic point of view, the property sector is the aspect of paramount importance of holistic performance of economy. In fact, over the past few decades the real estate industry has been a target government fiscal and monetary policies aimed at achieving balanced economic growth, low inflation and low unemployment. However, policy-makers seem to have no way in restraining bubbles in real estate markets. Indeed, their adherence to orthodox policies (easy monetary policy or low interest rates to avoid recession) may be making the problems worse.

A large and growing numbers of papers propose methods to detect asset bubbles. The debate on house price bubbles mostly focuses on charts of house prices to income or rent ratios.

There has been a long tradition of studying the theory and models of price bubbles in general stock markets, and these models are also utilised to detect bubbles in property markets. The bubble detection methodologies can de divided into two generations. The first generation includes methods which are developed to compare the volatility between the assumed fundamental housing price and the actual housing price. If the volatility of the actual housing price is significantly larger than that of the assumed fundamental housing price, the test will claim that housing price bubbles are being detected indirectly.

The second generation bubble tests are direct tests for the no-bubble hypothesis by econometrics method (unit root test or co-integration test). If the house price to income or rent ration is stationary or the house price is co-integrated with fundamental price, there no bubble hypothesis can be rejected.

These empirical studies can provide an integrated conclusion regarding the presence of property bubbles to some extent. The difficulty of accepting a bubble lies in the specification of the underlying fundamental economic models.

The recent crisis caught the economists in a misplaced consensus of "one instrument one target". The nominal interest rate was the instrument, and the monetary policy focused only on an implicit inflation target. Stock market bubbles were dismissed as unimportant. After the bursting of the housing bubble, the collapse of the theory prompted a rebirth of old-style Keynesianism, along with talks of an elusive "macroprudental" regulation. However, such approaches fail to recognise the basic fact that stock markets are complex systems.

Prior to the financial crisis on one side of the controversy, central banks should respond to housing price movements. The main problem here is that the regulation of a complex system such as the real estate market cannot be achieved using conventional policy tools. Although, economists are hardly prepared to recognise this fact, here one has resort to the control theory of self-organised systems. Collective behaviour based on self-organisation has been shown on animals living in groups as well as humans.

The definition of "bubble" given by Kindleberger [2] now is widely accepted: "A bubble may be defined loosely as a sharp rise in price of an asset… in a continuous process, with the initial rise generating expectations of further rises and attracting new buyers – generally speculators interested in profits from trading in the asset rather than its use of earning capacity. The rise is usually followed by a reversal of expectations and a sharp decline in price often resulting in financial crisis". The bubble is a huge rise in prices and then a crash follows. Speculative bubbles have the special characteristic: increasing deviations of prices from their fundamental values and we may observe this phenomenon as systematic and persistent.

The Mathematical Definition of the Bubbles and Crashes

Suppose the price at time t is P(t), $P_0(T_i)$ and $w(T_i)$are uniquely determined parameters from the past $T_i$data points for which root-mean-square of error F(t) achieves minimum (1).

$$P(t)-P(t-1)= \\ (w(T_i)-1)(P(t-1)-P_0(T_i))+F(t) \quad (1)$$

The prices behave in three manners due to the value of $w(T_i)$:

1. If $w(T_i)>1$, then the price is exponentially increasing or decreasing and $P_0(T_i)$ is base line of exponential divergence;
2. If $w(T_i)=1$, then price follows random walk;
3. If $w(T_i)<1$, then the price converges to $P_0(T_i)$ (base line).

Our purpose is to apply equation (1) to real estate market data in Lithuania from 1 May 2000 until 31 December 2009. The data in this period illustrates the bubble and the crash. The data is presented in Table 1.

We must find the optimal period of observation, i.e. fix T_i. For this we will investigate an ordinary autoregression model and will estimate parameters a_j and error term $f(t)$:

$$P(t)=\sum_{j=1}^{N} a_j P(t-j)+f(t) \quad (2)$$

The optimal period of observation is the minimum of time scale in which we cannot observe $w(T_i)>1$ in (1) for this time series (2) with $N=5$. For data presented in Table 1 the value of T_i is 12 months, in Figure 1 we plotted the exponential trend curve (3) for each convergent and divergent periods.

$$P_{trend}(t)=w(T_i)P_{trend}(t-1)+(1-w(T_i))\,P_0(T_i) \quad (3)$$

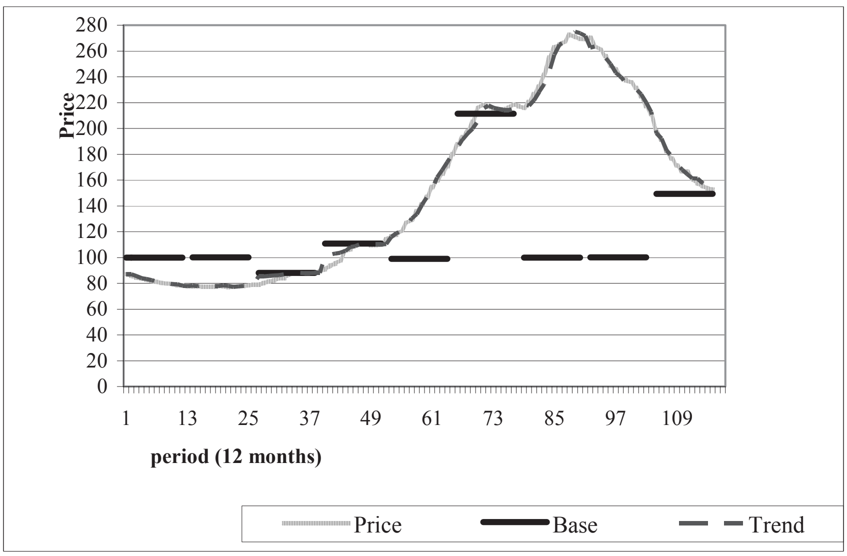

Fig. 1 The price and exponential trend curve and $P_0(T_i)$ of each period (Δt = 12 *months*)

As we can see from Figure 1, there are convergent intervals (3, 4, 6, 9) and divergence intervals (1, 2, 5, 7, 8). The results obtained here are coincident with the results in [4]. We may identify the bubble and crash with divergence intervals.

The Mathematical Definition of the Bubbles and Crashes in Case When Two Kinds of Agents Exist

In this paper we intend to investigate how this approach proposed in [4] and certified above with real data may be useful when we have two different kinds of agents, i.e. chartists and fundamentalists.

Fundamentalists: There are NF fundamentalists who believe that the price will eventually converge to fundamental value.

Chartists: There are NS chartists who behave according the trend of price.

As we will explore the same data we assume that T_i is equal to 12 and $w(T_i)$, $P_0(T_i)$ have the same meanings as above. Then we modify (1) and we have (4). N=5 as above.

$$P(t) - P(t-1) = \frac{NS}{NS + NF} \sum_{h=1}^{NS} \left(\sum_{j=t-1}^{N-t-1} w(T_i)[P(j) - P(j-1)] \right) + \\ + \frac{NF}{NS + NF} \sum_{h=1}^{NF} [P(t-1) - P_0(T_i)]\, q(T_i) + F(t) \tag{4}$$

Here $q(T_i)$ and $w(T_i)$ are adjustment speeds for fundamentalists and chartists. When we introduce two kinds of agents and estimate (4) in the same manner as (1) above, then we take into account some market structure. In analogous way we define the trend by (5).

$$P_{trend}(t) = \\ w(T_i)P_{trend}(t-1) + (1 - q(T_i)\; P_{trend}(t-1) \tag{5}$$

Here $P_0(T_i)$, $q(T_i)$, $w(T_i)$, SN and FN are uniquely determined parameters from the past T_i data points for which root-mean-square of error $F(t)$ achieve minimum (4). We note that $w(T_i) + (1 - q(T_i)$ is equal to 1.0 ($w(T_i)$ or $q(T_i)$ here may be negative).

Hui E. C. M. et al. [1] proposed another approach to bubbles and crashes by adding $P(t-2)$ to the model. Similarly to us in their model there are two parameters w_1 and w_2 and they analyse whether $w_1 + w_2$ is less, equal or greater than 1 instead of one parameter w.

In Figure 2 we plotted the exponential trend curve (5) for each period. Here we have a different picture as in Figure 1.

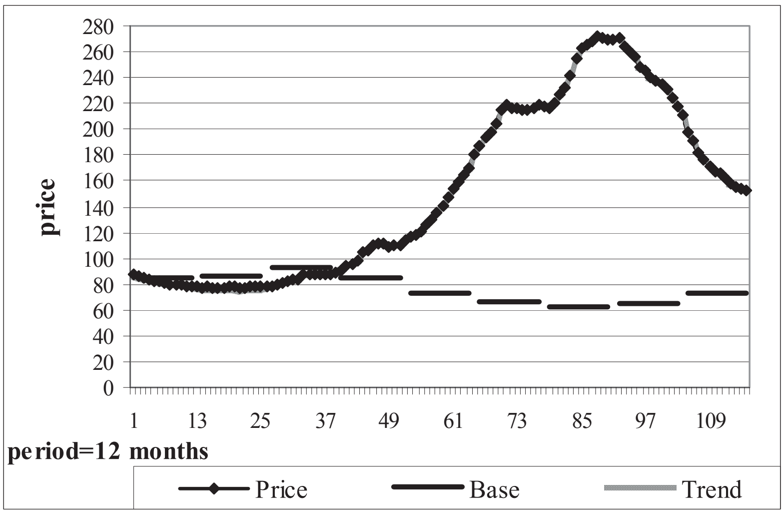

Fig. 2 The price and exponential trend curve and $P_0(T_i)$ of each period (Δt = 12 *months*)

In Figure 3, we presented the ratio SN / FN. We can see that the ratio SN / FN is changing stepwise when the price is changing.

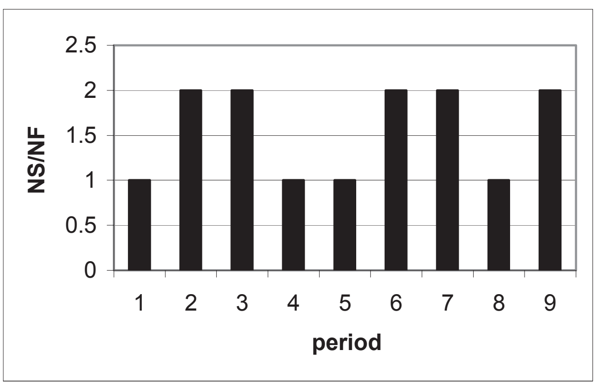

Fig. 3 The ratio SN / FN for each period (12 months)

Our analysis suggests further investigation. Let us suppose in equation (4) P=100 (the Index of the price was calculated assuming that the primary price of 1994 01 was equal to 100) and let it is the fundamental value of real estate in normal sense. Also let us suppose w=q=0.0001. Now we will assess the number of fundamentalists and chartists by equation (4). The period as above is equal to 12 months.

In Figure 4, we show the ratio of the price at the end of the period to the price at the start of the period and ratio of the number of chartists to the number of fundamentalists.

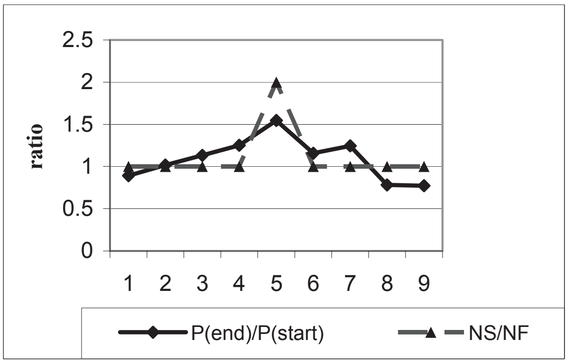

Fig. 4 The ratio of the price at the end of the period to the price at the start of the period and the ratio of the number of chartists to the number of fundamentalists.

Because our period is comparatively long, thus, ratio of the number of chartists to the number of fundamentalists is slightly changing. But if we take the period shortly, this change will be more significant. If we take the moving period, i. e. by changing the start and end of the period by some integer, then ratio of the number of chartists to the number of fundamentalists naturally will be changing not so sharply. As the number of chartists and number of fundamentalists are changing we may think that the agent of one kind becomes the agent of another kind. But our model doesn't cover the case of such change. In addition, every agent may change one's belief about the fundamental value, but our model doesn't deal with this case.

The Simple Heterogeneous Agents Model (HAM)

In this section, we describe the traditional heterogeneous agents model (HAM) [7] and analyse the behaviour of speculators.

The main assumption of this model is that there are two groups of speculators, i.e. fundamentalists and chartists with heterogeneous expectations.

The demand for fundamentalists is based on the difference between the price at time t and the expected price at time $t+1$

$$D_t^F = a^F(E_t^F(P_{t+1}) - P_t) \quad (6)$$

Here a^F is a positive parameter and E is an expectation operator. Fundamentalists' demand will increase as they expect the future price to be higher than the current price and vice versa.

The fundamentalists expect prices of over-valued assets to decrease and prices of undervalued assets to increase until the price of the asset reflects the fundamental value.

$$E_t^F(P_{t+1}) = P_t + b_1^F(P_t - F_t)^+ + b_2^F(P_t - F_t)^- \quad (7)$$

In (7) F_t is the fundamental price in period t and $(P_t - F_t)^+ = P_t - F_t$ if $P_t - F_t \ge 0$ and 0 otherwise, $(P_t - F_t)^- = P_t - F_t$ if $P_t - F_t < 0$ and zero otherwise.

We define the demand of the chartists as

$$D_t^F = a^F(E_t^F(P_{t+1}) - P_t) \quad (8)$$

where

$$E_t^C(P_{t+1}) = P_t + b_1^C(P_t - P_{t-1})^+ + b_2^C(P_t - P_{t-1})^- \quad (9)$$

Here the fundamentalists' and chartists' approach can be seen as different strategies that agents use to evaluate the market and to make investment decisions. Agents choose their strategy based on the performance of a certain forecasting strategy, i.e.

$$A_t^F = -\sum_{k=1}^{K}[\ E_{t-k-1}^F(P_{t-k}) - P_{t-k}]^2 \quad (10)$$

$$A_t^C = -\sum_{k=1}^{K}[\ E_{t-k-1}^C(P_{t-k}) - P_{t-k}]^2 \quad (11)$$

(here K=4 months).

Where A_t^F and A_t^C is fundamentalist and chartist approach accordingly.

From (10) and (11) we define the switching rule by

$$W_t = [1 + \exp(-\gamma[\frac{A_t^F - A_t^C}{A_t^F + A_t^C}]^{-1} \quad (12)$$

The real demand is the function of an exogenous component and an endogenous component which is a negative function of the price

$$D_t^R = a^R - b^R P_t \quad (13)$$

The supply is assumed to be a linear function of the price

$$S_t = a^S + b^S P_t \quad (14)$$

Total market demand is given by

$$D_t^M = D_t^R + W_t D_t^F + (1 - W_t) D_t^C \tag{15}$$

The price changes according to (16)

$$P_{t+1} = P_t + \delta(D_t^M - S_t) \tag{16}$$

Here the fundamental price is moving average with $M = 3$.

We obtain the following (17) by substitution.

$$\begin{aligned} \Delta P_{t+1} &= a + bP_t + W_t(\alpha_1 (P_t - F_t)^+ \\ &+ \alpha_2 (P_t - F_t)^-) + (1 - W_t) * \\ &* (\beta_1 (P_t - P_{t-1})^+ + \beta_2 (P_t - P_{t-1})^-) \end{aligned} \tag{17}$$

Here $a = \delta(a^R - a^S)$, $b = \delta(b^R - b^S)$, $\alpha_1 = \delta a^F b_1^F$, $\alpha_2 = \delta a^F b_2^F$ and $\beta_1 = \delta a^C b_1^C$, $\beta_2 = \delta a^C b_2^C$. $\alpha_1, \alpha_2, \beta_1, \beta_2$ represent the price impact of fundamentalists and chartists.

By the least square method we find that $\alpha_1, \alpha_2, \beta_1, \beta_2$ and a, b, γ fit with real data and are $\gamma = 0.625$, $\alpha_1 = -0.012$, $\alpha_2 = -3.275$ $\beta_1 = 0.003646$, $\beta_2 = 0.0002336$, $a = 0.0002173$, $b = 0.954$.

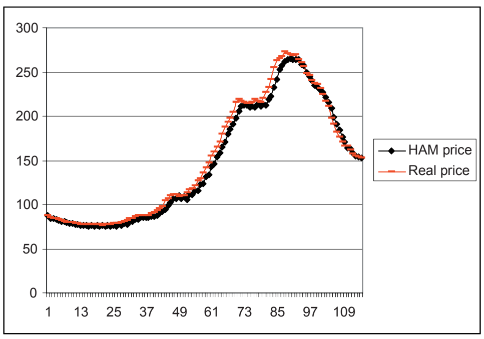

Fig. 5 The price by HAM and the real price

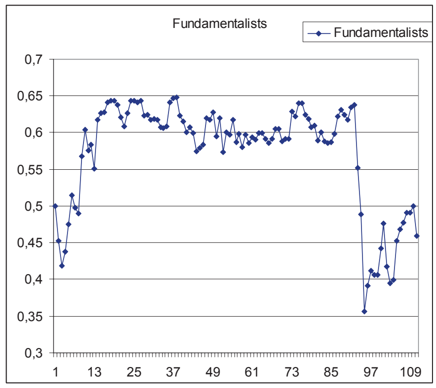

Fig. 6 The fraction of the fundamentalists

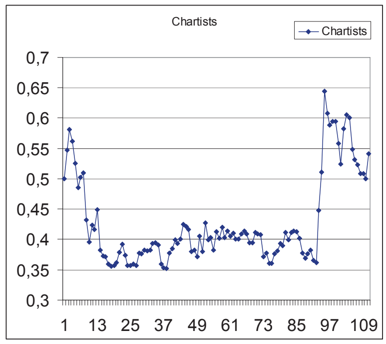

Fig. 7 The fraction of the chartists

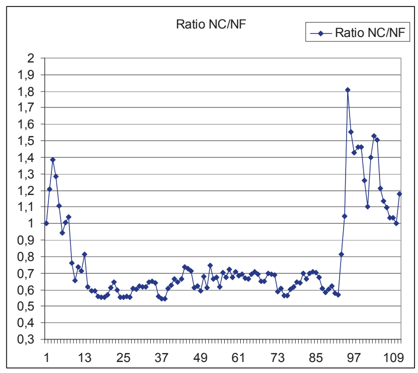

Fig. 8 The ratio of the chartists and fundamentalists

Table 1. ***The real estate market data in Lithuania***

Month / Year	01	02	03	04	05	06	07	08	09	10	11	12
2000					87.2	85.9	84.3	83.8	82.8	81.7	81.1	80.2
2001	79.8	79.7	78.5	78.4	77.9	77.4	77.9	77.5	77.5	77.4	78	77.7
2002	76.9	77.3	77.7	77.8	78.4	78.8	78.9	80.1	81.4	82.2	84.2	84.2
2003	87	87.8	87.5	87.5	87.9	88	89.5	90.9	93.6	95.1	98	104.9
2004	106.4	109.9	111.1	110.9	109.3	109.6	110	113.9	116.5	117.7	121.2	126.6
2005	129.5	135.8	141.1	146.7	154.5	159.7	165	170.5	180.1	187.5	193.1	197.6
2006	204.9	215.6	219	216.7	215.8		214.7	215.9	219	217.2	216	220.2
2007	226.5	232.3	241.6	255	262.6	264.9	267.5	272.5	271.2	269.3	269.1	270.1
2008	264.3	260.7	255.9	248.8	245.6	240.1	237.4	235.5	231.4	224.9	217.1	211.1
2009	197.6	191.1	182.1	176.5	171.2	166.7	166.5	161.3	157.6	155.4	153.5	152.7

Conclusions

In this paper we investigated the method of detection of the financial bubble and crash which was proposed by K. Watanabe and others by applying this method to data of real estate prices of Lithuania within the period from May 2000 until December 2009 [8]. We showed that this method is fitted well to real data. We modified the equation of K. Watanabe and analysed the case when there are two kinds of agents (fundamentalists and chartists). We demonstrated that this method was suitable for identifying the number of fundamentalists and chartists.

In addition, we analysed the HAM model with the switching rule between fundamentalists and chartists with the same data.

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Summary

We will use the method of detecting financial bubbles or crashes proposed in [6]. This method is based on data fitting with an exponential function. We will apply this method for real estate market data in Lithuania during bubble and crash. The data of real estate prices used here is from the company “Ober-Haus” [8]. We will introduce two kinds of agents, i. e. fundamentalists and chartists, and modify the equation which was proposed in [6], we will try to identify the number of the agents of these groups.

Key words: financial bubble, crash, fundamentalists, chartists.

FINANSINIŲ BURBULŲ IR GRIŪTIES MATEMATINIS NUSTATYMAS

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Santrauka

Tyrimui taikytos K. Watanala, H. Takayosu, M. Talayosu (2007) pasiūlytas finansinių burbulų ir griūties aptikimo metodas. Šio metodo esmė: duomenys priderinami pagal eksponentinę funkciją. Metodas taikytas Lietuvos nekilnojamo turto duomenims analizuoti burbulo ir griūties metu. Duomenis pateikė firma „Ober – Haus“. Analizės metu įvesti dviejų rūšių agentai, t. y. fundamentalistai ir čartistai. Be to, pakeista aukščiau minėtų moksininkų lygtis; pabandyta identifikuoti tų grupių agentų skaičių. Tirtas heterogeninių agentų modelis (HAM) su persijungimo tarp fundamentalistų ir čartistų taisykle prie tų pačių duomenų.

Įteikta 2012-09-24