MODIFIED DISTRIBUTION OF VALUES OF THE MATSUMOTO ZETA-FUNCTION

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1. Introduction

Let **N**, **Z**, **R** and **C** be the sets of all positive integers, integers, real numbers and complex numbers, respectively, and $s = \sigma + it$ be a complex variable. For any $m \in \mathbf{N}$, we define a positive integer $g(m)$. Let $a_m^{(j)}$ be complex numbers, and $f(j,m)$, $1 \le j \le g(m)$ be positive integers. Define the polynomials

$$A_m(x) = \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} x^{f(j,m)}\right)$$

of degree $f(1, m) + ... + f(g(m), m)$. Then the Matsumoto zeta-function $\varphi(s)$ is defined by

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}\left(p_m^{-s}\right), \qquad (1)$$

where p_m denotes the m-th prime number. In [9], it is assumed that the conditions

$$g(m) = Bp_m^{\alpha}, \quad \left|a_m^{(j)}\right| \le p_m^{\beta}, \qquad (2)$$

hold with non-negative constants α and β, and where B is a quantity bounded by a constant. Then the infinite product in (1) converges absolutely in the half-plane $\sigma > \alpha + \beta + 1$, and defines there a holomorphic function $\varphi(s)$ without zeros. It is not difficult to see that the Matsumoto zeta-function in the half-plane can be defined by an absolutely convergent Dirichlet series

$$\sum_{m=1}^{\infty} \frac{b_m}{m^s}$$

with coefficients b_m satisfying the estimate $b_m = Bm^{\alpha+\beta+\varepsilon}$ for every $\varepsilon > 0$. The function $\varphi(s)$ $\varphi(s)$ is a generalization of several classical zeta-functions with Euler product. For example, Dirichlet L-functions $L(s, \chi)$, Dedekind zeta-function of the algebraic number field and so on.

The limit distribution in the sense of weak convergence of probability measures of the function $\varphi(s)$ was investigated by K. Matsumoto and A. Laurinčikas, see, for example, [5, 6], [8, 9]. The discrete limit theorems for the Matsumoto zeta-function were proved by A. Laurinčikas and the first author, see, for example, [2, 3]. Let $meas\{A\}$ stand for the Lebesgue measure of a measurable set $A \subset \mathbf{R}$, and denote by $\mathbf{B}(\mathbf{S})$ the class of Borel sets of the space S. For example, the first theorem from [9] considers the case $\sigma > \alpha + \beta + 1$. Let R be a closed rectangle in **C** with edges parallel to the coordinate axes, and, for $\sigma_0 > \alpha + \beta + 1$, $\log\varphi(\sigma_0 + it)$ is the sum of principal values, i.e.,

$$\log \varphi(\sigma_0 + it) = -\sum_{m=1}^{\infty} \sum_{j=1}^{g(m)} Log\left(1 - a_m^{(j)} p_m^{-f(j,m)(\sigma_0 + it)}\right).$$

Theorem 1 ([9]). *Let $\sigma_0 > \alpha + \beta + 1$, and hypotheses (2) are satisfied. Then the following limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{ t \in [0,T] : \log\varphi(\sigma_0 + it) \in R \right\} \text{ exists.}$$

The aim of this paper is to obtain more complex limit theorems for the Matsumoto zeta-function $\varphi(s)$.

Let, for $T > 0$,

$$\nu_T(...) = \frac{1}{T^2} meas_2\left\{(t_1, t_2) \in [0,T]^2 : ...\right\},$$

where in the place of dots a condition satisfied by pair (t_1, t_2) is to be written, and $meas_2\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbf{R}^2$. On $(\mathbf{C}, \mathbf{B}(\mathbf{C}))$, define a probability measure

$$P_T(A) = \nu_T\left\{\varphi(\sigma + it_1 + it_2) \in A\right\}, \quad A \in \mathbf{B}(\mathbf{C}).$$

Theorem 2. *Let $\sigma > \alpha + \beta + \frac{1}{2}$. Then on $(\mathbf{C}, \mathbf{B}(\mathbf{C}))$ there exists a probability measure P_σ such that P_T converges weakly to P_σ as $T \to \infty$.*

2. Auxiliary results

For a proof of Theorem 2, we need the well-known continuity theorem for the probability measures on (R^2, $B(R^2)$). But at first we recall the definition of the weak convergence of probability measures. Let P_n and P be probability measures on (S, B(S)). We say that P_n converges weakly to P as n tends to infinity if for all bounded continuous functions $f: S \to \mathbf{R}$

$$\lim_{n\to\infty} \int_S f dP_n = \int_S f dP .$$

Lemma 1. *Let* $\{P_n\}$ *be a sequence of probability measures on* (R^k, $B(R^k)$) *and let* $\{f_n(\tau_1, \tau_2, ..., \tau_k)\}$ *be a sequence of corresponding characteristic functions,* $n \in \mathbf{N}$. *Suppose that*

$$\lim_{n\to\infty} f_n(\tau_1, \tau_2, ..., \tau_k) = f(\tau_1, \tau_2, ..., \tau_k),$$

for all $(\tau_1, \tau_2, ..., \tau_k) \in \mathbf{R}^k$, *and that* $f(\tau_1, \tau_2, ..., \tau_k)$ *is continuous at the point* $(0,0,...,0)$. *Then there exists a probability measure P on* (R^k, $B(R^k)$) *such that* P_n *converges weakly to P as* $n \to \infty$. *In this case,* $f(\tau_1, \tau_2, ..., \tau_k)$ *is the characteristic function of P.*

The proof of the lemma can be found, for example, in [1]. In our case, $k = 2$.

To prove the weak convergence of the probability measure P_T, we can consider the measure

$$Q_T(A) = \nu_T\big((\Re\varphi(\sigma + it_1 + it_2), \Im\varphi(\sigma + it_1 + it_2)) \in A\big), \ A \in B(\mathbf{R}^2).$$

Let $N \in \mathbf{N}$,

$$p_N(\sigma + it) = \sum_{j=1}^{N} a_j e^{it\lambda_j}, \ a_j \in \mathbf{C}, \ \lambda_j \in \mathbf{R},$$

be an arbitrary Dirichlet polynomial. Let $J_k(x)$ stand for the Bessel function, for any fixed real k, defined by the series

$$J_k(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+k}}{m!\,\Gamma(k+m+1)}.$$

Here $\Gamma(k)$ is the gamma-function. Define the characteristic function

$$f_{p_N}(\tau_1, \tau_2) = \sum{}^{*} \prod_{j=1}^{N} J_{k_j}(|a_j|\tau_1) J_{l_j}(|a_j|\tau_2) \times$$
$$\times \exp\left\{ i\left(\frac{\pi}{2}\right)\sum_{j=1}^{N} k_j + \sum_{j=1}^{N} (k_j + l_j)\lambda_j \right\}, \quad (3)$$

where the asterisk means the summation over all integers k_j and l_j, $j = 1,...,N$, satisfying

$$\sum_{j=1}^{n} (k_j + l_j)\lambda_j = 0 .$$

Lemma 2. *The probability measure*
$P_{T,p_N}(A) = \nu_T\big((\Re\varphi(\sigma + it_1 + it_2), \Im\varphi(\sigma + it_1 + it_2)) \in A\big)$, $A \in B(\mathbf{R}^2)$ converges weakly to the measure on (R^2, $B(R^2)$) defined by the characteristic function $f_{p_N}(\tau_1, \tau_2)$ as $T \to \infty$.

Proof. To prove the weak convergence of the probability measure P_{T,p_N}, we will study the asymptotic behaviour of its characteristic function

$$f_{T,p_N}(\tau_1, \tau_2) = \int_{\mathbf{R}^2} \exp\{i(\tau_1 x_1 + \tau_2 x_2)\} dP_{T,p_N}$$
$$= \frac{1}{T^2} \int_0^T \int_0^T \exp\{i\tau_1 \Re p_N(\sigma + it_1 + it_2) + i\tau_2 \Im p_N(\sigma + it_1 + it_2)\} dt_1 dt_2. \quad (4)$$

It is easily seen that

$$\Re p_N(\sigma + it) = \sum_{\substack{j=1 \\ a_j \neq 0}}^{N} |a_j| \cos(\lambda_j t + \eta_j),$$

$$\Im p_N(\sigma + it) = \sum_{\substack{j=1 \\ a_j \neq 0}}^{N} |a_j| \sin(\lambda_j t + \eta_j),$$

where $\eta_j = \arg a_j$.

By Theorem 2.4.1 from [7], for real x and θ,

$$e^{ix\sin\theta} = \sum_{k=-\infty}^{\infty} J_k(x) e^{ik\theta},$$

$$e^{ix\cos\theta} = \sum_{k=-\infty}^{\infty} i^k J_k(x) e^{ik\theta}.$$

Then we have that

$$\exp\{i\tau_1 \Re p_N(\sigma+it_1+it_2)\}$$
$$=\sum_{k_1,\dots,k_N=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)..J_{k_N}(|a_n|\tau_1)\times\exp\left\{i\left(\frac{\pi}{2}\sum_{j=1}^{N}k_j+(t_1+t_2)\sum_{j=1}^{N}k_j\lambda_j+\sum_{j=1}^{N}k_j\eta_j\right)\right\}$$
$$=\sum_{k_1,\dots,k_N=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)..J_{k_N}(|a_n|\tau_1)\times\exp\left\{i\left(\sum_{j=1}^{N}\left(\frac{\pi}{2}+\eta_j\right)k_j+(t_1+t_2)\sum_{j=1}^{N}k_j\lambda_j\right)\right\} \tag{5}$$

and

$$\exp\{i\tau_2 \Im p_N(\sigma+it_1+it_2)\}$$
$$=\sum_{l_1,\dots,l_N=-\infty}^{\infty} J_{l_1}(|a_1|\tau_2)..J_{l_N}(|a_n|\tau_2)\times\exp\left\{i\left(\sum_{j=1}^{N}l_j\eta_j+(t_1+t_2)\sum_{j=1}^{N}l_j\lambda_j\right)\right\} \tag{6}$$

In view of (3), and (5)–(6), from (4) we find that

$$f_{T,p_N}(\tau_1,\tau_2)=f_{p_N}(\tau_1,\tau_2)+\sum{}^{**}\prod_{j=1}^{N}J_{k_j}(|a_j|\tau_1)J_{l_j}(|a_j|\tau_2)$$
$$\times\exp\left\{i\left(\frac{\pi}{2}\sum_{j=1}^{N}k_j+\sum_{j=1}^{N}(k_j+l_j)\eta_j\right)\right\} \tag{7}$$
$$\times\frac{1}{T^2}\int_0^T\int_0^T\exp\left\{i(t_1+t_2)\sum_{j=1}^{N}(k_j+l_j)\lambda_j\right\}dt_1dt_2,$$

where ** means the summation over all integers k_j and l_j, $j=1,\dots,N$, satisfying

$$\sum_{j=1}^{N}(k_j+l_j)\lambda_j\neq 0 .$$

Obviously, if $\sum_{j=1}^{N}(k_j+l_j)\lambda_j\neq 0$, then

$$\int_0^T\int_0^T\exp\left\{i(t_1+t_2)\sum_{j=1}^{N}(k_j+l_j)\lambda_j\right\}dt_1dt_2=$$
$$=\left(\frac{1-\exp\{iT\sum_{j=1}^{N}(k_j+l_j)\lambda_j\}}{1-\exp\{i\sum_{j=1}^{N}(k_j+l_j)\lambda_j\}}\right)^2$$

It is known that, for the Bessel functions, the estimate

$$J_k(x)=\frac{B_1^{|k|}}{|k|!},\ |x|\le c_1,$$

is valid, where c_1 is an arbitrary constant [10]. Thus, taking an arbitrary $\varepsilon>0$, we find $K=K(\varepsilon)$ such that

$$\left|\sum_{(|k_1|+|l_1|+\dots+|k_N|+|l_N|)>K(\varepsilon)}^{**}\prod_{J=1}^{N}J_{k_j}(|a_j|\tau_1)J_{l_j}(|a_j|\tau_2)\exp\left\{i\left(\frac{\pi}{2}\sum_{j=1}^{N}k_j+\sum_{j=1}^{N}(k_j+l_j)\eta_j\right)\right\}\right.$$
$$\left.\times\frac{1}{T^2}\left(\frac{1-\exp\{iT\sum_{j=1}^{N}(k_j+l_j)\lambda_j\}}{1-\exp\{i\sum_{j=1}^{N}(k_j+l_j)\lambda_j\}}\right)^2\right|<\frac{\varepsilon}{2} \tag{8}$$

for all $|\tau_1|\le c_2$, $|\tau_2|\le c_3$ (c_2 and c_3 are positive constants). Now we choose $N_0=N_0(\varepsilon)$ such that, for the remainder term in the sum of (7) the inequality $N\ge N_0$,

$$\left|\sum_{(|k_1|+|l_1|+\dots+|k_N|+|l_N|)\le K(\varepsilon)}^{*}\prod_{J=1}^{N}J_{k_j}(|a_j|\tau_1)J_{l_j}(|a_j|\tau_2)\exp\left\{i\left(\frac{\pi}{2}\sum_{j=1}^{N}k_j+\sum_{j=1}^{N}(k_j+l_j)\eta_j\right)\right\}\right.$$
$$\left.\times\frac{1}{T^2}\left(\frac{1-\exp\{iT\sum_{j=1}^{N}(k_j+l_j)\lambda_j\}}{1-\exp\{i\sum_{j=1}^{N}(k_j+l_j)\lambda_j\}}\right)^2\right|<\frac{\varepsilon}{2}$$

holds. Consequently, from (7)–(8), we deduce that

$$\lim_{T\to\infty} f_{T,p_N}(\tau_1,\tau_2) = f_{p_N}(\tau_1,\tau_2),$$

i.e., $f_{T,p_N}(\tau_1,\tau_2)$ converges to $f_{p_n}(\tau_1,\tau_2)$ as $T\to\infty$ uniformly on τ_1 and τ_2 in every finite interval. In view of Lemma 1, we obtain the weak convergence of probability measure P_{T,p_N} to the measure on (R^2, B(R^2)) defined by the characteristic function $f_{p_N}(\tau_1,\tau_2)$. The lemma is proven.

3. Proof of Theorem 2

Let $f_T(\tau_1,\tau_2)$ be the characteristic function of the measure P_T. Then

$$\begin{aligned}
&\left|f_T(\tau_1,\tau_2) - f_{T,p_N}(\tau_1,\tau_2)\right| \\
&<< \frac{|\tau_1|}{T^2}\int_0^T\int_0^T \left|\Re\varphi(\sigma+it_1+it_2) - p_N(\sigma+it_1+it_2)\right| dt_1 dt_2 \\
&+ \frac{|\tau_2|}{T^2}\int_0^T\int_0^T \left|\Im\varphi(\sigma+it_1+it_2) - p_N(\sigma+it_1+it_2)\right| dt_1 dt_2 \\
&\le \frac{|\tau_1|+|\tau_2|}{T^2}\int_0^T\int_0^T \left|\varphi(\sigma+it_1+it_2) - p_N(\sigma+it_1+it_2)\right| dt_1 dt_2 .
\end{aligned} \tag{10}$$

In similar way as in Lemma 4 of [4] and Lemma 2, there exists a positive number $T_{01} = T_{01}(\sigma, c_2, c_3, \varepsilon)$ such that, for $T > T_{01}$,

$$\left|f_{T,p_N}(\tau_1,\tau_2) - f_{p_N}(\tau_1,\tau_2)\right| < \frac{\varepsilon}{4} \tag{11}$$

uniformly in $|\tau_1| \le c_2$, $|\tau_2| \le c_3$, where $\varepsilon, \tau_1, \tau_2$ are arbitrary positive numbers. Then from (10) it follows that there exist positive numbers $N = N(\sigma,\varepsilon)$ and $T_{02} = T_{02}(\sigma, c_2, c_3, \varepsilon)$ such that, for $T \ge T_{02}$,

$$\left|f_T(\tau_1,\tau_2) - f_{T,p_N}(\tau_1,\tau_2)\right| < \frac{\varepsilon}{2} \tag{12}$$

is satisfied uniformly in $|\tau_1| \le c_2$, $|\tau_2| \le c_3$. Let N be a fixed number, and $T_0 = \max(T_{10}, T_{02})$. In view of (11) and (12), for T_1, $T_2 > T_0$, we find that

$$\left|f_{T_1}(\tau_1,\tau_2) - f_{T_2}(\tau_1,\tau_2)\right| < \varepsilon$$

uniformly in $|\tau_1| \le c_2$, $|\tau_2| \le c_3$. From this we deduce that the function $f_T(\tau_1,\tau_2)$ converges uniformly by τ_1 and τ_2 in every finite interval to some function $f(\tau_1,\tau_2)$. But in view of Lemma 1, the function $f(\tau_1,\tau_2)$ is continuous at the point $(0,0)$. Therefore, the theorem is proven as a consequence of Lemma 1.

Remark. The Theorem 2 gives only the existence of the limit measure.

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Summary

In the paper we investigate a modified distribution of values of the Matsumoto zeta-function on the complex plane. We obtain a limit theorem in the sense of the weak convergence of probability measures for the function $\varphi(s)$. The main theorem gives only the existence of the limit measure.

Keywords: Dirichlet series, probability measure, weak convergence, zeta-function.

MODIFIKUOTAS MATSUMOTO DZETA FUNKCIJOS REIKŠMIŲ PASISKIRSTYMAS

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Santrauka

Straipsnyje nagrinėjama Matsumoto dzeta funkcija. Ji apibendrina klasikines dzeta funkcijas, išreiškiamas ne tik Dirichlė eilutėmis, bet ir Oilerio sandauga. Įprasta, kad yra nagrinėjamas dzeta funkcijų reikšmių pasiskirstymas, kai kompleksinio kintamojo reikšmės kinta tolydžiai postūmių atžvilgiu. Darbe tiriamas sudėtingesnis atvejis – kai tuo pat metu atsiranda du skirtingi postūmiai. Įrodyta modifikuota ribinė teorema Matsumoto dzeta funkcijai silpno tikimybinių matų konvergavimo prasme kompleksinėje plokštumoje.

Prasminiai žodžiai: Dirichlė eilutės, dzeta funkcija, silpnas konvergavimas, tikimybinis matas.

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