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ON SOME INEQUALITIES OF PROBABILISTIC NUMBER THEORY

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Abstract. Let Q_+ denote the set of positive rational numbers. We define discrete probability measures ν_x on the σ-algebra of subsets of Q_+. We introduce additive functions $f\colon Q_+ \to G$ and obtain a bound for $\nu_x(f(r) \notin X + X - X)$ using a probability related to some independent random variables. This inequality is an analogue to that proved by I. Ruzsa for additive arithmetical functions.

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1. INEQUALITIES FOR ADDITIVE FUNCTIONS

Parallelism between the value distributions of arithmetical functions and the distributions related to systems of independent random variables is the essential idea of probabilistic number theory.

Let $f\colon N \to G$ be an arithmetical additive function taking values in some Abelian group G, and let ν_x $(x \geqslant 1)$ be the discrete probabilistic measure assigning the weights

$$\nu_x(n) = \begin{cases} \frac{1}{[x]}, & \text{if } n \leqslant x, \\ 0, & \text{if } n > x, \end{cases}$$

to positive integers. Then the accompanying independent random variables indexed by prime numbers are specified by

$$P(\xi_p = a) = \sum_{f(p^\alpha)=a} \frac{1}{p^\alpha}\Big(1 - \frac{1}{p}\Big).$$

If G consists of real (or complex) numbers, the moments of an additive function and random variables can be considered. The inequality proved by Kubilius in 1955 for a complex-valued additive function f is, in fact, the statement about the second central moments:

$$E_x|f - E_x f|^2 \ll \sum_{p\leqslant x} E|\xi_p^x - E\xi_p^x|^2,$$

where, for $g\colon N \to C$,

$$E_x g = \int g \, \mathrm{d}\nu_x = \frac{1}{[x]} \sum_{n \leqslant x} g(n),$$

and by ξ_p^x we denote the independent random variables associated with "truncated" additive function f_x, defined by $f_x(p^\alpha) = f(p^\alpha)$ for $p^\alpha \leqslant x$ and $f_x(p^\alpha) = 0$ for $p^\alpha > x$. Efforts were made to prove similar inequalities for other moments and functionals of additive functions. Various aspects of research in this field are reviewed in [1]–[3].

In 1984, I. Ruzsa proved a surprisingly general inequality. In [4], it is proved that if G is an Abelian group and $f\colon N \to G$ an additive function, then, for each subset $X \subset G$,

$$\nu_x(f(n) \notin X + X - X) \ll P\Big(\sum_{p \leqslant x} \xi_p \notin X\Big). \tag{1}$$

This inequality can be used as a tool to transfer the problem of estimating some moments of f to that of estimating moments related to independent random variables ξ_p. The reader should see [4] for details.

2. ADDITIVE FUNCTIONS IN THE DOMAIN OF RATIONAL NUMBERS

Let Q_+ be the set of positive rational numbers $\frac{m}{n}$; we always suppose that $(m, n) = 1$. For $q = \frac{m}{n}$ and $q' = \frac{m'}{n'}$, we denote

$$(q, q') = \frac{(m, m')}{(n, n')}.$$

Definition. Let G be an arbitrary Abelian group. We call a function $f\colon Q_+ \to G$ additive if

$$f(q \cdot q') = f(q) + f(q')$$

for all $q, q' \in Q_+$ such that $(q, q') = (q^{-1}, q') = 1$.

For $\frac{m}{n} \in Q_+$ and a prime number p such that $p|mn$, we define

$$\alpha_p\Big(\frac{m}{n}\Big) = \begin{cases} \alpha, & \text{if } p^\alpha \parallel m, \\ -\alpha, & \text{if } p^\alpha \parallel n. \end{cases}$$

If $p \nmid mn$, we set $\alpha_p(\frac{m}{n}) = 0$. Then, for an additive function $f\colon Q_+ \to G$,

$$f\Big(\frac{m}{n}\Big) = \sum_p f\Big(p^{\alpha_p(m/n)}\Big).$$

There are many possibilities to assign weights to $m/n \in Q_+$ and to define discrete measures on Q_+. In this paper, we confine ourselves to the numbers of interval (0; 1) and define, for $A \subset Q_+$,

$$\nu_x(A) = \frac{\#A \cap \mathcal{F}_x}{\#\mathcal{F}_x}, \quad \text{where} \quad \mathcal{F}_x = \left\{\frac{m}{n}\colon\, m < n \leqslant x, (m,n) = 1\right\}, \quad x > 1. \quad (2)$$

We associate with an additive function $f\colon Q_+ \to G$ two systems of independent random variables $\xi_p^{(1)}$ and $\xi_p^{(2)}$ (p is a prime number) such that

$$\begin{aligned} P(\xi_p^{(1)} = a) &= \sum_{\substack{\alpha \in Z \\ f(p^{\max(\alpha,0)}) = a}} \frac{1}{p^{|\alpha|}}\Big(1 - \frac{2}{p+1}\Big), \\ P(\xi_p^{(2)} = a) &= \sum_{\substack{\alpha \in Z \\ f(p^{\min(\alpha,0)}) = a}} \frac{1}{p^{|\alpha|}}\Big(1 - \frac{2}{p+1}\Big). \end{aligned} \quad (3)$$

The aim of this paper is to prove Ruzsa's inequality (1) in the setting just defined.

THEOREM. *Let $f\colon Q_+ \to G$ be an additive function with values in Abelian group G; $X^{(1)}, X^{(2)} \subset G, X = X^{(1)} + X^{(2)}$. Then, for the measure ν_x defined by* (2) *and independent random variables* (3)*, the inequality*

$$\nu_x\Big(f\Big(\frac{m}{n}\Big) \notin X + X - X\Big) \ll P\Big(\sum_{p \leqslant x} \xi_p^{(1)} \notin X^{(1)}\Big) + P\Big(\sum_{p \leqslant x} \xi_p^{(2)} \notin X^{(2)}\Big) \quad (4)$$

holds with the constant in $\ll$ independent of $X^{(1)}, X^{(2)}$, and f.

In [4], it is explained how from the inequality like (4) the bounds for the moments can be derived. For example, let $f(m/n)$ be a complex-valued additive function such that $f(p^\alpha) = 0$ for $p^{|\alpha|} > x$. We denote

$$\begin{aligned} A_x^{(1)} &= \sum_{p \leqslant x} E\xi_p^{(1)}, \quad A_x^{(2)} = \sum_{p \leqslant x} E\xi_p^{(2)}, \\ A_x &= A_x^{(1)} + A_x^{(2)} = \sum_{\substack{p \leqslant x \\ \alpha \in Z}} \frac{f(p^\alpha)}{p^{|\alpha|}}\Big(1 - \frac{2}{p+1}\Big). \end{aligned}$$

Then with $X^{(1)} = \{z\colon |z - A_x^{(1)}| < u/6\}$ and $X^{(2)} = \{z\colon |z - A_x^{(2)}| < u/6\}$ we have

$$X^{(i)} + X^{(i)} - X^{(i)} = \{z\colon |z - A_x^{(i)}| < u/2\}, \quad X + X - X = \{z\colon |z - A_x| < u\},$$

and inequality (4) can be rewritten as

$$\nu_x\Big(\Big|f\Big(\frac{m}{n}\Big)-A_x\Big|\geqslant u\Big)\ll P\Big(\Big|\sum_{p\leqslant x}(\xi_p^{(1)}-E\xi_p^{(1)})\Big|\geqslant u/6\Big) + P\Big(\Big|\sum_{p\leqslant x}(\xi_p^{(2)}-E\xi_p^{(2)})\Big|\geqslant u/6\Big). \tag{5}$$

Using the equality

$$E|\xi|^2 = 2\int_0^\infty uP(|\xi|\geqslant u)\,\mathrm{d}u,$$

we have

$$E_x|f-A_x|^2=\frac{1}{\#\mathcal{F}_x}\sum_{m/n\in\mathcal{F}_x}\Big|f\Big(\frac{m}{n}\Big)-A_x\Big|^2=2\int_0^\infty u\nu_x\Big(\Big|f\Big(\frac{m}{n}\Big)-A_x\Big|\geqslant u\Big)\,\mathrm{d}u,$$

$$E\Big|\sum_{p\leqslant x}(\xi_p^{(i)}-E\xi_p^{(i)})\Big|^2=2\int_0^\infty uP\Big(\Big|\sum_{p\leqslant x}(\xi_p^{(i)}-E\xi_p^{(i)})\Big|\geqslant u\Big)\,\mathrm{d}u$$
$$=\frac{1}{18}\int_0^\infty uP\Big(\Big|\sum_{p\leqslant x}(\xi_p^{(i)}-E\xi_p^{(i)})\Big|\geqslant \frac{u}{6}\Big)\,\mathrm{d}u.$$

Hence, from (5) we can derive that

$$E_x|f-A_x|^2\ll E\Big|\sum_{p\leqslant x}(\xi_p^{(1)}-E\xi_p^{(1)})\Big|^2+E\Big|\sum_{p\leqslant x}(\xi_p^{(2)}-E\xi_p^{(2)})\Big|^2,$$

which after some calculations can be reduced to the Kubilius-type number-theoretic inequality:

$$\sum_{m/n\in\mathcal{F}_x}\Big|f\Big(\frac{m}{n}\Big)-A_x\Big|^2\ll\#\mathcal{F}_x\cdot\sum_{\substack{p^{|\alpha|}\leqslant x\\ \alpha\in Z}}\frac{|f(p^\alpha)|^2}{p^{|\alpha|}}.$$

Note that similar inequalities were proved in [5] with

$$\mathcal{F}_x=\Big\{\frac{m}{n}\colon\ (m,n)=1,\,n\leqslant x\Big\}\cap(\alpha,\beta),$$

using the large sieve inequalities, approach initiated by Elliott (see, [1], [2]).

PROOF OF THE THEOREM

First note that the independent random variables $\xi_p^{(i)}$, $p \leqslant x$, with distributions (3) can be realized as functions $\xi_p^{(i)}\colon Q_+ \to G$:

$$\xi_p^{(1)}\Big(\frac{m}{n}\Big) = f\Big(p^{\max(\alpha_p(m/n),0)}\Big), \quad \xi_p^{(2)}\Big(\frac{m}{n}\Big) = f\Big(p^{\min(\alpha_p(m/n),0)}\Big),$$

if we define the discrete measure $P = \lambda_x$ on subsets $A \subset Q_+$ by

$$\lambda_x(A) = \Big(\sum_{m/n\in Q_x} \frac{1}{mn}\Big)^{-1} \sum_{m/n\in Q_x\cap A} \frac{1}{mn},$$

where

$$Q_x = \Big\{\frac{m}{n}\colon\ p^+(mn) \leqslant x\Big\},$$

and $p^+(k)$ denotes the greatest prime divisor of k. Then

$$f\Big(\frac{m}{n}\Big) = f^{(1)}\Big(\frac{m}{n}\Big) + f^{(2)}\Big(\frac{m}{n}\Big), \quad f^{(i)}\Big(\frac{m}{n}\Big) = \sum_p \xi_p^{(i)}\Big(\frac{m}{n}\Big).$$

Let

$$\mathcal{F}_x^* = \Big\{\frac{m}{n}\colon\ (m,n) = 1, m, n \leqslant x\Big\}.$$

Evidently, $\mathcal{F}_x \subset \mathcal{F}_x^* \subset Q_x$, $\#\mathcal{F}_x^* = 2\#\mathcal{F}_x$, and

$$\#\mathcal{F}_x^* = 2\sum_{m\leqslant x} \varphi(m) \sim \frac{6}{\pi^2}x^2 \quad \text{as } x \to \infty, \tag{6}$$

where $\varphi(m)$ is the Euler function. For $A \subset Q_+$, we define

$$\nu_x^*(A) = \frac{\#A \cap \mathcal{F}_x^*}{\#\mathcal{F}_x^*}.$$

Then

$$\nu_x(A) = \frac{\#A \cap \mathcal{F}_x}{\#\mathcal{F}_x} \leqslant \frac{\#\mathcal{F}_x^*}{\#\mathcal{F}_x} \cdot \frac{\#A \cap \mathcal{F}_x^*}{\#\mathcal{F}_x^*} \leqslant 2\nu_x^*(A),$$

and it is sufficient to prove inequality (4) with ν_x^* instead of ν_x.

If $f^{(i)}(\frac{m}{n}) \in X^{(i)} + X^{(i)} - X^{(i)}$ for $i = 1, 2$, then $f(\frac{m}{n}) \in X + X - X$, hence,

$$\nu_x^*\Big(f\Big(\frac{m}{n}\Big) \notin X + X - X\Big) \ll \nu_x^*\Big(f^{(1)}\Big(\frac{m}{n}\Big) \notin X^{(1)} + X^{(1)} - X^{(1)}\Big)$$
$$+ \nu_x^*\Big(f^{(2)}\Big(\frac{m}{n}\Big) \notin X^{(2)} + X^{(2)} - X^{(2)}\Big).$$

Then it suffices to prove the inequalities

$$\nu_x^*\Big(f^{(i)}\Big(\frac{m}{n}\Big) \notin X^{(i)} + X^{(i)} - X^{(i)}\Big) \ll \lambda_x\Big(f^{(i)}\Big(\frac{m}{n}\Big) \notin X^{(i)}\Big), \quad i = 1, 2. \tag{7}$$

We proceed with the proof for $i = 1$; the proof for $i = 2$ is almost identical.

For a nonempty set of positive integers $W \subset N$, we denote

$$W^Q = \Big\{\frac{m}{n}\colon m \in W, (m, n) = 1\Big\}.$$

Then taking

$$U = \{m\colon f(m) \in X^{(1)}\}, \quad V = \{m\colon f(m) \in X^{(1)} + X^{(1)} - X^{(1)}\},$$

we can rewrite inequality (7) for $i = 1$ as

$$\nu_x^*(\overline{V^Q}) \ll \lambda_x(\overline{U^Q}), \tag{8}$$

where $\overline{B}$ denotes the complement of a set B. We need some results about λ_x.

Lemma. *For any set $W \subset N$, we have*

$$\lambda_x(W^Q) = \prod_{p\leqslant x}\Big(1 - \frac{1}{p+1}\Big) \sum_{m\in W\cap Q_x} \frac{1}{m}\prod_{p|m}\Big(1 - \frac{1}{p}\Big)$$
$$= \Big(\sum_{m\in Q_x}\frac{\varphi(m)}{m^2}\Big)^{-1} \sum_{m\in W\cap Q_x}\frac{\varphi(m)}{m^2}. \tag{9}$$

Let

$$E = \{p^k\colon p \leqslant x; m \in W, (m, p) = 1 \Rightarrow mp^k \notin W\}.$$

If $\lambda_x(W^Q) \geqslant 99/100$, then

$$\sum_{p^k\in E}\frac{1}{p^k} \leqslant 5\lambda_x(\overline{W^Q}). \tag{10}$$

Proof of Lemma. Equalities (9) can be established by a straightforward calculation. If we denote

$$P_x = \prod_{p\leqslant x}\Big(1 - \frac{1}{p+1}\Big) = \Big(\sum_{m\in Q_x}\frac{\varphi(m)}{m^2}\Big)^{-1},$$

then

$$\lambda_x(W^Q) = P_x \sum_{m \in W \cap Q_x} \frac{1}{m} \prod_{p|m} \Big(1 - \frac{1}{p}\Big)$$
$$= P_x \sum_{m \in W \cap Q_x} \frac{\varphi(m)}{m^2}, \quad P_x \sim c\Big(\log x\Big)^{-1} \text{ as } x \to \infty.$$

Let $k \in Q_x$ be a positive integer, i.e., $p^+(k) \leqslant x$. By $A_k \subset N$ we denote the set of multiples of k. Using the obvious inequalities

$$\varphi(l)\varphi(m) \leqslant \varphi(lm) \leqslant l\varphi(m),$$

from (9) we obtain that

$$\frac{\varphi(k)}{k^2} \leqslant \lambda_x(A_k^Q) \leqslant \frac{1}{k}.$$

We have

$$\lambda_x(W^Q) = P_x \sum_{m \in W \cap Q_x} \frac{\varphi(m)}{m^2} = 1 - \epsilon, \quad \epsilon \leqslant \frac{1}{100}.$$

For $q = p^k$, where p is prime, denote

$$W_q = \big\{qm\colon (m, p) = 1, m \in W\big\}.$$

If $q \in E$, then $W_q \subset \overline{W}$, hence,

$$\epsilon \geqslant \lambda_x(W_q^Q) = \frac{1}{q}\Big(1 - \frac{1}{p}\Big) \cdot P_x \cdot \sum_{\substack{m \in W \cap Q_x \\ (m,p)=1}} \frac{\varphi(m)}{m^2}$$
$$\geqslant \frac{1}{q}\Big(1 - \frac{1}{p}\Big)\Big(\lambda_x(W^Q) - \lambda_x(A_p^Q)\Big) \geqslant \frac{1}{q}\Big(1 - \frac{1}{p}\Big)\Big(1 - \epsilon - \frac{1}{p}\Big) \geqslant \frac{49}{200} \cdot \frac{1}{q}. \quad (11)$$

Since $\cup_{q \in E} W_q \subset \overline{W}$, we have

$$\epsilon \geqslant \lambda_x\Big(\bigcup_{q \in E} W_q^Q\Big) \geqslant \sum_{q \in E} \lambda_x(W_q^Q) - \sum_{q_1, q_2 \in E} \lambda_x(W_{q_1}^Q \cap W_{q_2}^Q). \quad (12)$$

If q_1 and q_2 are powers of the same prime number, then $W_{q_1}^Q \cap W_{q_2}^Q = \emptyset$. We denote

$$\sigma = \sum_{q \in E} \frac{1}{q}.$$

Using in (12) the inequalities

$$\lambda_x(W_q^Q) \geqslant \frac{49}{200}\frac{1}{q}, \quad \lambda_x(W_{q_1}^Q \cap W_{q_2}^Q) \leqslant \lambda_x(A_{q_1 q_2}^Q) \leqslant \frac{1}{q_1 q_2},$$

we obtain

$$\epsilon \geqslant c\sigma - \sigma^2, \quad c = \frac{49}{200}.$$

If $\sigma \leqslant c/2$, then

$$\epsilon \geqslant c\sigma - \sigma^2 \geqslant \frac{1}{2}c\sigma, \quad \sigma \leqslant \frac{2\epsilon}{c} = \frac{400}{49}\epsilon < 5\epsilon,$$

and the statement is proved. We show that $\sigma > c/2$ can never occur.

Let $\sigma > c/2$. Because of (11), for each $q \in E$, we have

$$\frac{1}{q} \leqslant \frac{200}{49}\epsilon = \frac{\epsilon}{c}.$$

Then it is possible to choose a subset E' such that

$$\sigma' = \sum_{p \in E'} \frac{1}{q}, \quad \frac{c}{2} \geqslant \sigma' > \frac{c}{2} - \frac{\epsilon}{c}.$$

Having in mind what is already proved, we obtain

$$\sigma' \leqslant \frac{2\epsilon}{c}.$$

Now we conclude that there must be

$$\frac{2\epsilon}{c} \geqslant \frac{c}{2} - \frac{\epsilon}{c}, \quad \epsilon \geqslant \frac{c^2}{6}.$$

This is not true with $c = \frac{49}{200}$, and $\epsilon \leqslant \frac{1}{100}$ since $c^2/6 > 0,0010004$. The proof of lemma is complete.

Let us return to the proof of (8). It is sufficient to prove this inequality for

$$U = \{m\colon\ f(m) \in X^{(1)}\}, \quad V = \{m\colon\ f(m) \in X^{(1)} + X^{(1)} - X^{(1)}\} \quad (U \subset V)$$

with $\lambda_x(U^Q) \geqslant \frac{99}{100}$. Let

$$T = \sum_{m/n \in \overline{V^Q} \cap \mathcal{F}_x^*} \log\frac{x}{m} = \log x \cdot \#(\overline{V^Q} \cap \mathcal{F}_x^*) - S, \quad S = \sum_{m/n \in \overline{V^Q} \cap \mathcal{F}_x^*} \log m.$$

We establish the bounds

$$T \ll x^2 \log x \cdot \lambda_x(\overline{U^Q}), \quad S \ll x^2 \log x \cdot \lambda_x(\overline{U^Q}); \tag{13}$$

using them and (6), we obtain inequality (8) from

$$\log x \cdot \#(\overline{V^Q} \cap \mathcal{F}_x^*) \ll T + S.$$

In what follows, we use the simple asymptotics

$$\sum_{\substack{n \leqslant x \\ (n,m)=1}} 1 = x \prod_{p|m} \Bigl(1 - \frac{1}{p}\Bigr) + \mathrm{O}\Bigl(2^{\omega(m)}\Bigr).$$

We have

$$T = \sum_{m/n \in \overline{V^Q} \cap \mathcal{F}_x^*} \log \frac{x}{m} \ll x \sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{1}{m} \sum_{\substack{n \leqslant x \\ (n,m)=1}} 1 = x^2 \sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{1}{m} \prod_{p|m} \Bigl(1 - \frac{1}{p}\Bigr)$$
$$+ \mathrm{O}\Bigl(x \sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{2^{\omega(m)}}{m}\Bigr) \ll x^2 \log x \cdot \lambda_x(\overline{V^Q}) + \mathrm{O}\Bigl(x \sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{2^{\omega(m)}}{m}\Bigr).$$

Now we get

$$\sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{2^{\omega(m)}}{m} = \sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{d_m}{m} \prod_{p|m} \Bigl(1 - \frac{1}{p}\Bigr), \quad d_m = 2^{\omega(m)} \prod_{p|m} \Bigl(1 + \frac{1}{p-1}\Bigr) \leqslant 4^{\omega(m)}.$$

Since $\omega(m) \ll \log m / \log\log m$, we have $d_m \ll m^{\epsilon}$ ($\epsilon > 0$) and

$$\sum_{\substack{m \in \overline{V} \\ m \leqslant x}} \frac{2^{\omega(m)}}{m} \ll x \log x \cdot \lambda_x(\overline{V^Q}),$$
$$T \ll x^2 \log x \cdot \lambda_x(\overline{V^Q}) \ll x^2 \log x \cdot \lambda_x(\overline{U^Q}).$$

The first inequality of (13) is established.

We define the set E as in Lemma with $W = U$. If $p^k \notin E$, then there exists $l \in U$, $(l, p) = 1$, such that $p^k \cdot l \in U$. The important observation of I. Ruzsa [4] is that, for $m \in U$, $p^k \notin E$, $(m, p) = 1$, we have $p^k m = \frac{p^k l}{l} \cdot m$ and

$$f(p^k m) = f(p^k l) + f(m) - f(l) \in V.$$

We now proceed with S:

$$S = \sum_{m/n \in \overline{V^Q} \cap \mathcal{F}_x^*} \log m = \sum_{p^k \leqslant x} \log p^k \sum_{\substack{\frac{p^k m}{n} \in \overline{V^Q} \cap \mathcal{F}_x^* \\ (m,p)=1}} 1 = S_1 + S_2,$$

$$S_1 = \sum_{\substack{p^k \leqslant x \\ p^k \in E}} \log p^k \sum_{\substack{\frac{p^k m}{n} \in \overline{V^Q} \cap \mathcal{F}_x^* \\ (m,p)=1}} 1, \quad S_2 = \sum_{\substack{p^k \leqslant x \\ p^k \notin E}} \log p^k \sum_{\substack{\frac{p^k m}{n} \in \overline{V^Q} \cap \mathcal{F}_x^* \\ (m,p)=1}} 1.$$

For S_1, we apply the statement of lemma:

$$S_1 \ll x^2 \log x \sum_{p^k \in E} \frac{1}{p^k} \ll x^2 \log x \cdot \lambda_x(\overline{U^Q}).$$

Estimating S_2, we use the arguments for $p^k \notin E$ explained above:

$$S_2 \leqslant \sum_{p^k \leqslant x} \log p^k \sum_{\substack{m \in \overline{U}, (m,p)=1 \\ \frac{mp^k}{n} \in \mathcal{F}_x^*}} 1 \leqslant \sum_{\substack{m \in \overline{U} \\ m \leqslant x}} \sum_{p^k \leqslant \frac{x}{m}} \log p^k \sum_{\substack{n \leqslant x \\ (n,m)=1}} 1$$

$$\ll \sum_{\substack{m \in \overline{U} \\ m \leqslant x}} \left(x \prod_{p|m} \left(1 - \frac{1}{p}\right) + 2^{\omega(m)} \right) \cdot \sum_{p^k \leqslant \frac{x}{m}} \log p^k$$

$$\ll x^2 \sum_{\substack{m \in \overline{U} \\ m \leqslant x}} \frac{1}{m} x \prod_{p|m} \left(1 - \frac{1}{p}\right) + x \sum_{\substack{m \in \overline{U} \\ m \leqslant x}} \frac{2^{\omega(m)}}{m} \ll x^2 \log x \cdot \lambda_x(\overline{U^Q}).$$

Hence,

$$S \ll x^2 \log x \cdot \lambda_x(\overline{U^Q}),$$

and the proof of the theorem is complete.

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REZIUMĖ

V. Stakėnas. Tikimybinės skaičių teorijos nelygybės

Įrodyti tikimybinės skaičių teorijos nelygybių adityvių funkcijų dažniams analogai, kai adityviosios funkcijos nagrinėjamos racionaliųjų skaičių aibėje. Naudojantis šiomis nelygybėmis galima gauti adityviųjų funkcijų momentų įverčius.