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ANOTHER APPROACH TO ASYMPTOTIC EXPANSIONS FOR EULER'S APPROXIMATIONS OF SEMIGROUPS

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Abstract. In [2], optimal bounds for the remainder terms in asymptotic expansions for Euler's approximations of semigroups were derived. The approach was based on applications of the Fourier–Laplace transforms, which allowed one to reduce the problem to estimation of error terms in the Law of Large Numbers. In this paper, we propose an alternative (direct) approach based on application of certain integro-differential identities (so-called multiplicative representations of differences). Such identities were introduced by Bentkus in [3] and applied (see Bentkus and Paulauskas [4]) to derive the optimal convergence rates in Chernoff-type lemmas and Euler's approximations of semigroups.

Keywords: Euler's approximations, semigroups, asymptotic expansions, holomorphic semigroups, convergence rate.

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1. INTRODUCTION AND RESULTS

Let A be a possibly unbounded linear operator on a complex Banach space X and $R(\lambda) = (I + \lambda A)^{-1}$ be the resolvent of A. Let $E(t) = \exp\{-tA\}$, $t \geqslant 0$, be a semigroup of operators with generator A. We consider the functions $t \mapsto R^n(t/n) = (I + tA/n)^{-n}$, $n \in \mathbb{N}$, which are called the Euler's approximations of the semigroup $E(t)$.

We consider asymptotic expansions of Euler's approximations

$$(I + tA/n)^{-n} = \exp\{-tA\} + \frac{a_1}{n} + \ldots + \frac{a_k}{n^k} + \mathrm{o}\Big(\frac{1}{n^k}\Big) \quad \text{as } n \to \infty, \tag{1.1}$$

with coefficients a_k depending on $E(t)$ and independent of n. We also obtain asymptotic expansions of the semigroup

$$\exp\{-tA\} = (I + tA/n)^{-n} + \frac{b_1}{n} + \ldots + \frac{b_k}{n^k} + \mathrm{o}\Big(\frac{1}{n^k}\Big) \quad \text{as } n \to \infty, \tag{1.2}$$

with b_k which are linear combinations of functions $t \mapsto (tA)^m R^n(t/n)$ with coefficients independent of n. We provide explicit and optimal bounds for the remainder terms in (1.1) and (1.2).

To obtain asymptotic expansions we use a decomposition of Euler's approximations which is an integro-differential identity. Such an approach was introduced by Bentkus in [3] for analysis of errors in the Central Limit Theorem and in approximations by accompanying laws and applied by Bentkus and Paulauskas in [4] to derive optimal convergence rates in Chernoff-type lemmas and Euler's approximations of semigroups. It consists of a choice of a curve, say $\gamma(\tau)$, $0 \leqslant \tau \leqslant 1$, connecting two close objects, say a and b, such that $\gamma(0) = a$ and $\gamma(1) = b$. In our case, we have $a = \exp\{-tA\}$, $b = R^n(t/n)$ and $\gamma(\tau) = R^n(\tau t/n)\exp\{-(1-\tau)tA\}$ (see Section 4 in Bentkus ([3]). Then we apply the mean-value theorem (or Newton-Leibnitz formula) along the curve, that is, $b - a = \int_0^1 \gamma'(\tau)\,\mathrm{d}\tau$. In our case, we have

$$R^n(t/n) - \exp\{-tA\} = \frac{1}{n}\int_0^1 \tau(tA)^2 R^{n+1}(\tau t/n)E(t(1-\tau))\,\mathrm{d}\tau. \tag{1.3}$$

Iterative applications of this identity lead to asymptotic expansions of $R^n(t/n)$ and $E(t)$.

Using this approach, Bentkus and Paulauskas in [4] provided a rather short and simple proof of $\Delta_n = \mathrm{O}(n^{-1})$, where $\Delta_n = \|R^n(t/n) - E(t)\|$, in the case of bounded holomorphic semigroups. For semigroups in Hilbert spaces generated by m-sectorial operators, Cachia and Zagrebnov in [6] obtained the bound $\Delta_n = \mathrm{O}(n^{-1}\ln n)$. Paulauskas in [9] improved the bound to the optimal $\mathrm{O}(n^{-1})$ using a new method based on the results and methods of probability theory related to the Central Limit Theorem. Cachia in [5] extended the bound $\mathrm{O}(n^{-1}\ln n)$ to the case of bounded holomorphic semigroups and noticed that the Paulauskas argument can be applied to improve the bound to $\mathrm{O}(n^{-1})$. Proofs in [6], [9] and [5] are rather complicated, and it is not clear how to extend such proofs in order to obtain asymptotic expansions. Bentkus in [2] obtained asymptotic expansions for Euler's approximations of semigroups and optimal error bounds using another approach based on applications of the Fourier–Laplace transforms and a reduction of the problem to the convergence rates and asymptotic expansions in the Law of Large Numbers. In particular, for bounded differentiable semigroups, the error bound $\Delta_n = \mathrm{O}(n^{-1})$ was obtained. It covers and refines all known related results obtained for semigroups of operators in Banach spaces. An alternative (direct) approach we use in this paper does not involve the Laplace transforms. The proofs are much simpler and shorter here; besides, this method also gives us a constructive description of the terms of the expansion.

To make the exposition more comprehensible, we first formulate the results in the special case of short expansions, i.e., expansions with remainders $\mathrm{O}(n^{-2})$. We start with a related integro-differential identity.

THEOREM 1.1. *We have*

$$(I + tA/n)^{-n} = \exp\{-tA\} + \frac{(tA)^2}{2n}\exp\{-tA\} + D_1, \tag{1.4}$$

where the remainder term D_1 *is given by*

$$D_1 = \frac{1}{n^2}(D_{1,1} + D_{1,2}) \tag{1.5}$$

with

$$D_{1,1} = -\int_0^1 \tau^2 (tA)^3 R^{n+1}(\tau t/n) E(t(1-\tau))\, \mathrm{d}\tau$$

and

$$D_{1,2} = \int_0^1 \int_0^1 \tau_1^3 \tau_2 (tA)^4 R^{n+1}(\tau_1\tau_2 t/n) E(t(1-\tau_1\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2.$$

To estimate the remainder term D_1 in expansion (1.4), we use the same conditions as in Theorem 1.3 of Bentkus and Paulauskas in [4]. These conditions are satisfied by bounded holomorphic semigroups (see Bentkus and Paulauskas [4]).

THEOREM 1.2. *Assume that there exists a constant* K *independent of* n *and* t *such that*

$$n\|tA(I + tA)^{-n}\| \leqslant K \tag{1.6}$$

and

$$\|\exp\{-tA\}\| \leqslant K, \quad \|tA\exp\{-tA\}\| \leqslant K \tag{1.7}$$

for all $n = 1, 2, \dots$ *and* $t \geqslant 0$. *Then the remainder term* D_1 *in asymptotic expansion* (1.4) *satisfies*

$$\|D_1\| \leqslant \frac{C_1}{n^2} K^4,$$

where C_1 *is an absolute positive constant.*

Now we consider the so-called inverse expansions, i.e., expansions where the exponent $E(t)$ is approximated by $R^n(t/n)$. Again we start with a short expansion with the remainder term $\mathrm{O}(n^{-2})$.

THEOREM 1.3. *We have*

$$\exp\{-tA\} = (I + tA/n)^{-n} + \frac{b_1}{n} + \Delta_1, \tag{1.8}$$

where $b_1 = -\frac{(tA)^2}{2}(I + tA/n)^{-n}$. *The remainder term* Δ_1 *is given by*

$$\Delta_1 = -D_1 - \frac{(tA)^2}{2n}\Delta_0, \tag{1.9}$$

where D_1 is given by (1.5), *and*

$$\Delta_0 = -\frac{1}{n}\int_0^1 \tau(tA)^2 R^{n+1}(\tau t/n)E(t(1-\tau))\,\mathrm{d}\tau.$$

To estimate the remainder term Δ_1 in expansion (1.8), we use the same conditions as in Theorem 1.2.

THEOREM 1.4. *Assume that there exists a constant K independent of n and t such that conditions* (1.6) *and* (1.7) *are satisfied for all $n = 1, 2, \ldots$ and $t \geqslant 0$. Then the remainder term Δ_1 in asymptotic expansion* (1.8) *satisfies*

$$\|\Delta_1\| \leqslant \frac{C_1}{n^2}K^5,$$

where C_1 is an absolute positive constant.

Now we generalize the results of Theorems 1.1–1.4 for asymptotic expansions of any given length k. We first need some additional notation. Henceforth, $\sum_\alpha$ means summation over all integer components $\alpha_1, \ldots, \alpha_k$ of vectors $\alpha = (\alpha_1, \ldots, \alpha_k)$ which satisfy certain conditions. Write

$$c_{m,1} = \frac{1}{m+1} \quad \text{and} \quad c_{m,j} = \frac{1}{m+j}\sum_i \frac{1}{i_2 i_3 \ldots i_j} \quad \text{for } j = 2, \ldots, m, \tag{1.10}$$

where $i = (i_2, i_3, \ldots, i_j)$ satisfy $2 \leqslant i_j \leqslant m - j + 2$ and $i_{n+1} + 2 \leqslant i_n \leqslant m + j - 2(n-1)$ for $n = 2, 3, \ldots, j-1$. Then the coefficients a_m in (1.1) are given by

$$a_m = \sum_{j=1}^{m} c_{m,j}(-tA)^{m+j}\exp\{-tA\}. \tag{1.11}$$

For example, we have

$$a_1 = \frac{(tA)^2}{2}\exp\{-tA\},$$
$$a_2 = -\frac{(tA)^3}{3}\exp\{-tA\} + \frac{(tA)^4}{8}\exp\{-tA\},$$
$$a_3 = \frac{(tA)^4}{4}\exp\{-tA\} - \frac{(tA)^5}{6}\exp\{-tA\} + \frac{(tA)^6}{48}\exp\{-tA\}.$$

We note that an alternative form of the coefficients $c_{m,j}$ is

$$c_{m,j} = \frac{1}{j!}\sum_{i_1+\ldots+i_j=m+j}\frac{1}{i_1 i_2 \ldots i_j},$$

where $i_1, i_2, \dots, i_j \geqslant 2$ and $1 \leqslant j \leqslant m$. Here $\sum_{i_1+\dots+i_n=k}$ means summation over all distinct ordered n-tuples of positive integers $i_1, \dots, i_n$ whose elements sum to k.

We also define the functions

$$\sigma_{k,j} = \sigma_{k,j}(\tau_1, \dots, \tau_j) = \tau_1^{k+j} \sum_i \tau_2^{i_2} \dots \tau_j^{i_j} \quad \text{for } j = 2, \dots, k+1, \tag{1.12}$$

where $i = (i_2, i_3, \dots, i_j)$ satisfy $1 \leqslant i_j \leqslant k - j + 2$ and $i_{n+1} + 2 \leqslant i_n \leqslant k + j - 2(n-1)$ for $n = 2, 3, \dots, j-1$. When $j = 1$, $\sigma_{k,1} = \tau_1^{k+1}$.

To shorten the notation for multiple integrals we use a sequence of independent identically distributed random variables $\tau, \tau_1, \tau_2, \dots$ uniformly distributed in the interval [0, 1]. Then we can write $\int_0^1 f(\tau)\,\mathrm{d}\tau = \mathbf{E}\, f(\tau)$ for any function f. In the case where f is a function of k variables, we write $\mathbf{E}\, f(\tau_1, \dots, \tau_k)$ instead of a k-tuple integral.

We also introduce the indicator functions

$$\mathbb{I}_{1,k} = \mathbb{I}\{\tau_2 \geqslant 1/2, \dots, \tau_k \geqslant 1/2\}, \quad k \geqslant 2,$$
$$\mathbb{I}_{m,k} = \mathbb{I}\{\tau_m \leqslant 1/2, \tau_{m+1} \geqslant 1/2, \dots, \tau_k \geqslant 1/2\}, \quad k \geqslant 3,$$

for $m = 2, \dots, k-1$, and

$$\mathbb{I}_{k,k} = \mathbb{I}\{\tau_k \leqslant 1/2\}, \quad k \geqslant 2.$$

THEOREM 1.5. *We have*

$$(I + tA/n)^{-n} = \exp\{-tA\} + \frac{a_1}{n} + \dots + \frac{a_k}{n^k} + D_k, \quad k = 1, 2, \dots, \tag{1.13}$$

where a_m are given by (1.11). *The remainder term is given by*

$$D_k = \frac{1}{n^{k+1}} \sum_{j=1}^{k+1} D_{k,j}, \tag{1.14}$$

where

$$D_{k,j} = \mathbf{E}\, \sigma_{k,j} (-tA)^{k+j+1} R^{n+1}(\tau_1 \dots \tau_j t/n) E(t(1 - \tau_1 \dots \tau_j)) \tag{1.15}$$

with $\sigma_{k,j}$ given by (1.12).

THEOREM 1.6. *Assume that there exists a constant K independent of n and t such that conditions* (1.6) *and* (1.7) *are satisfied for all $n = 1, 2, \dots$ and $t \geqslant 0$. Then the remainder term D_k in asymptotic expansion* (1.13) *satisfies*

$$\|D_k\| \leqslant \frac{C_k}{n^{k+1}} K^{2k+2}, \quad k = 1, 2, \dots \tag{1.16}$$

with a positive constant C_k depending only on k.

Now we provide asymptotic expansions of the semigroup $E(t)$ via Euler's approximations $R^n(t/n)$. First, we define

$$d_m = \sum_{i=1}^{m} c_{m,i}(-tA)^{m+i}, \tag{1.17}$$

where $c_{m,i}$ are given by (1.10). Then the coefficients b_m in (1.2) are given by

$$b_m = -\sum_{j=1}^{m} d_j b_{m-j}, \quad m = 1, 2, \ldots, \qquad b_0 = (I + tA/n)^{-n}. \tag{1.18}$$

We note that b_m can be represented as

$$b_m = h_m (I + tA/n)^{-n},$$

where

$$h_m = (-1)^m \sum_{i=1}^{m} c_{m,i}(tA)^{m+i}. \tag{1.19}$$

For example, we have

$$\begin{aligned}
b_1 &= -\frac{(tA)^2}{2}\Big(I + \frac{tA}{n}\Big)^{-n},\\
b_2 &= \frac{(tA)^3}{3}\Big(I + \frac{tA}{n}\Big)^{-n} + \frac{(tA)^4}{8}\Big(I + \frac{tA}{n}\Big)^{-n},\\
b_3 &= -\frac{(tA)^4}{4}\Big(I + \frac{tA}{n}\Big)^{-n} - \frac{(tA)^5}{6}\Big(I + \frac{tA}{n}\Big)^{-n} - \frac{(tA)^6}{48}\Big(I + \frac{tA}{n}\Big)^{-n}.
\end{aligned}$$

THEOREM 1.7. *We have*

$$\exp\{-tA\} = (I + tA/n)^{-n} + \frac{b_1}{n} + \ldots + \frac{b_k}{n^k} + \Delta_k, \quad k = 1, 2, \ldots, \tag{1.20}$$

where b_m are given by (1.18). *The remainder term is given by*

$$\Delta_k = -D_k - \frac{d_1}{n}\Delta_{k-1} - \ldots - \frac{d_k}{n^k}\Delta_0, \tag{1.21}$$

where D_k is given by (1.14).

THEOREM 1.8. *Assume that there exists a constant K independent of n and t such that conditions* (1.6) *and* (1.7) *are satisfied for all $n = 1, 2, \dots$ and $t \geqslant 0$. Then the remainder term Δ_s in asymptotic expansion* (1.20) *satisfies*

$$\|\Delta_s\| \leqslant \frac{C_s}{n^{s+1}} K^{2s+3}, \quad s = 1, 2, \dots \tag{1.22}$$

with a positive constant C_s depending only on s.

2. PROOFS

Proof of Theorem 1.1. From (1.3) we have

$$(I + tA/n)^{-n} = \exp\{-tA\} + D_0, \tag{2.1}$$

where

$$D_0 = \frac{1}{n} \int_0^1 \tau_1 (tA)^2 R^{n+1}(\tau_1 t/n) E(t(1-\tau_1))\, \mathrm{d}\tau_1 = \frac{1}{n} D_{0,1} \tag{2.2}$$

(see (4.2) in Bentkus [3]).

It is easy to check the algebraic identity

$$R^{n+1}(\tau_1 t/n) = R^n(\tau_1 t/n) - \frac{1}{n}\tau_1 t A R^{n+1}(\tau_1 t/n). \tag{2.3}$$

Also, from (2.1) and (2.2) we obtain that

$$R^n(\tau_1 t/n) = \exp\{-\tau_1 t A\} + \frac{1}{n}\int_0^1 \tau_1^2 \tau_2 (tA)^2 R^{n+1}(\tau_1\tau_2 t/n) E(\tau_1 t(1-\tau_2))\, \mathrm{d}\tau_2. \tag{2.4}$$

Substituting (2.3) and then (2.4) into expression of $D_{0,1}$, we get

$$D_{0,1} = \frac{1}{n} D_{1,1} + \frac{(tA)^2 \exp\{-tA\}}{2} + \frac{1}{n} D_{1,2}, \tag{2.5}$$

where

$$D_{1,1} = -\int_0^1 \tau_1^2 (tA)^3 R^{n+1}(\tau_1 t/n) E(t(1-\tau_1))\, \mathrm{d}\tau_1$$

and

$$D_{1,2} = \int_0^1 \int_0^1 \tau_1^3 \tau_2 (tA)^4 R^{n+1}(\tau_1\tau_2 t/n) E(t(1-\tau_1\tau_2))\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2.$$

Substituting expression (2.5) into (2.2), we obtain the asymptotic expansion

$$(I + tA/n)^{-n} = \exp\{-tA\} + \frac{a_1}{n} + D_1,$$

where

$$a_1 = \frac{(tA)^2 \exp\{-tA\}}{2}$$

and

$$D_1 = \frac{1}{n^2}(D_{1,1} + D_{1,2}).$$

In order to prove Theorem 1.2, we first prove the following:

LEMMA 2.1. *Assume that there exists a constant K independent of n and t such that*

$$n\|tA(I + tA)^{-n}\| \leqslant K$$

and

$$\|tA \exp\{-tA\}\| \leqslant K$$

for all $n = 1, 2, \ldots$ and $t \geqslant 0$. Then

$$n^m \|(tA)^m (I + tA)^{-n}\| \leqslant m^m K^m \tag{2.6}$$

and

$$\|(tA)^m \exp\{-tA\}\| \leqslant m^m K^m \tag{2.7}$$

for all $m = 1, 2, \ldots$.

Proof. We first prove (2.6). We consider only the case where $n = ms$, since in the cases $n = ms + 1, n = ms + 2, \ldots, n = ms + m - 1$, the proof is similar. So, for $n = ms$, we have

$$n^m \|(tA)^m (I + tA)^{-n}\| \leqslant n^m \|tA(I + tA)^{-s}\|^m \leqslant \frac{n^m}{s^m} K^m = m^m K^m.$$

Now let us prove (2.7). Using the semigroup property $\exp\{-(t + s)A\} = \exp\{-tA\}\exp\{-sA\}$, we obtain

$$\|(tA)^m \exp\{-tA\}\| \leqslant m^m \|tA/m \exp\{-tA/m\}\|^m \leqslant m^m K^m.$$

Proof of Theorem 1.2. From (1.5) we have

$$\|D_1\| \leqslant \frac{1}{n^2}\big(\|D_{1,1}\| + \|D_{1,2}\|\big).$$

Let us first estimate $\|D_{1,1}\|$. It is clear that $\|D_{1,1}\| \leqslant \theta_{1,1} + \theta_{1,2}$, where

$$\theta_{1,1} = \int_{1/2}^{1} \big\|\tau^2(tA)^3 R^{n+1}(\tau t/n)E(t(1-\tau))\big\|\,\mathrm{d}\tau$$

and

$$\theta_{1,2} = \int_{0}^{1/2} \big\|\tau^2(tA)^3 R^{n+1}(\tau t/n)E(t(1-\tau))\big\|\,\mathrm{d}\tau.$$

Let $\varrho_{1,1} = \|(\tau tA)^3 R^{n+1}(\tau t/n)\|$. Then

$$\theta_{1,1} = \int_{1/2}^{1} \varrho_{1,1}\frac{\|E(t(1-\tau))\|}{\tau}\,\mathrm{d}\tau.$$

By Lemma 2.1 we have $\varrho_{1,1} \leqslant 27K^3$ and $\|E(t(1-\tau))\| \leqslant K$. Integrating over the interval $[1/2, 1]$, we get

$$\theta_{1,1} \leqslant 27\ln 2K^4.$$

Let us estimate $\theta_{1,2}$. Write

$$\varrho_{1,2} = \|(\tau tA)^2 R^{n+1}(\tau t/n)\|, \quad \varrho_{1,3} = \|(1-\tau)tAE(t(1-\tau))\|.$$

Then

$$\theta_{1,2} = \int_{0}^{1/2} \varrho_{1,2}\varrho_{1,3}\frac{1}{(1-\tau)}\,\mathrm{d}\tau.$$

By Lemma 2.1 we have $\varrho_{1,2} \leqslant 4K^2$ and $\varrho_{1,3} \leqslant K$. Integrating over the interval $[0, 1/2]$, we get

$$\theta_{1,2} \leqslant 4\ln 2K^3$$

and (note that $K \geqslant 1$)

$$\|D_{1,1}\| \leqslant C_{1,1}K^4,$$

where $C_{1,1}$ is an absolute positive constant.

Now we estimate $\|D_{1,2}\|$. It is clear that $\|D_{1,1}\| \leqslant \theta_{2,1} + \theta_{2,2}$, where

$$\theta_{2,1} = \int_0^1 \int_{1/2}^1 \left\| \tau_1^3 \tau_2 (tA)^4 R^{n+1}(\tau_1\tau_2 t/n) E(t(1-\tau_1\tau_2)) \right\| \mathrm{d}\tau_1 \, \mathrm{d}\tau_2$$

and

$$\theta_{2,2} = \int_0^1 \int_0^{1/2} \left\| \tau_1^3 \tau_2 (tA)^4 R^{n+1}(\tau_1\tau_2 t/n) E(t(1-\tau_1\tau_2)) \right\| \mathrm{d}\tau_1 \, \mathrm{d}\tau_2.$$

Write

$$\varrho_{2,1} = \left\| (\tau_1\tau_2 tA)^3 R^{n+1}(\tau_1\tau_2 t/n) \right\|, \quad \varrho_{2,2} = \left\| (1-\tau_1\tau_2) tAE(t(1-\tau_1\tau_2)) \right\|.$$

Then

$$\theta_{2,1} = \int_0^1 \int_{1/2}^1 \varrho_{2,1}\varrho_{2,2} \frac{1}{\tau_2^2(1-\tau_1\tau_2)} \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2.$$

By Lemma 2.1 we have $\varrho_{2,1} \leqslant 27K^3$ and $\varrho_{2,2} \leqslant K$. Integrating, we get

$$\theta_{2,1} \leqslant 27(1/2 + 2\ln 2)K^4.$$

It remains to estimate $\theta_{2,2}$. Let

$$\varrho_{2,3} = \left\| \tau_1\tau_2 tAR^{n+1}(\tau_1\tau_2 t/n) \right\|, \quad \varrho_{2,4} = \left\| (1-\tau_1\tau_2)^3 (tA)^3 E(t(1-\tau_1\tau_2)) \right\|.$$

Then

$$\theta_{2,2} = \int_0^1 \int_0^{1/2} \varrho_{2,3}\varrho_{2,4} \frac{\tau_1^2}{(1-\tau_1\tau_2)^3} \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2.$$

By Lemma 2.1 we have $\varrho_{2,3} \leqslant K$ and $\varrho_{2,4} \leqslant 27K^3$. Integrating, we get

$$\theta_{2,2} \leqslant 27(7/4 - 2\ln 2)K^4$$

and

$$\|D_{1,2}\| \leqslant C_{1,2}K^4,$$

where $C_{1,2}$ is an absolute positive constant. Then

$$\|D_1\| \leqslant \frac{1}{n^2}\big(\|D_{1,1}\| + \|D_{1,2}\|\big) \leqslant \frac{C_1}{n^2} K^4.$$

Proof of Theorem 1.3. From Theorem 1.1 we have

$$\exp\{-tA\} = (I + tA/n)^{-n} - \frac{a_1}{n} - D_1,$$

where $a_1 = \frac{(tA)^2}{2}\exp\{-tA\}$ and

$$\exp\{-tA\} = (I + tA/n)^{-n} + \Delta_0, \tag{2.8}$$

where $\Delta_0 = -D_0$. Substituting (2.8) into expression of a_1, we obtain

$$\exp\{-tA\} = (I + tA/n)^{-n} - \frac{(tA)^2}{2n}\big((I + tA/n)^{-n} + \Delta_0\big) - D_1. \tag{2.9}$$

Regrouping the terms in (2.9), we obtain asymptotic expansion (1.8).

Proof of Theorem 1.4. From Theorem 1.2 we have

$$\|D_1\| \leqslant \frac{c_1}{n^2}K^4.$$

Let us estimate $\|(tA)^2\Delta_0\|$. It is clear that $\|(tA)^2\Delta_0\| \leqslant \frac{1}{n}(\theta_1 + \theta_2)$, where

$$\theta_1 = \int_{1/2}^{1} \big\|\tau(tA)^4 R^{n+1}(\tau t/n)E(t(1-\tau))\big\|\,\mathrm{d}\tau$$

and

$$\theta_2 = \int_{0}^{1/2} \big\|\tau(tA)^4 R^{n+1}(\tau t/n)E(t(1-\tau))\big\|\,\mathrm{d}\tau.$$

Write $\varrho_1 = \|(\tau tA)^4 R^{n+1}(\tau t/n)\|$. Then

$$\theta_1 = \int_{1/2}^{1} \varrho_1 \frac{\|E(t(1-\tau))\|}{\tau^3}\,\mathrm{d}\tau.$$

By Lemma 2.1 we have $\varrho_1 \leqslant 4^4K^4$ and $\|E(t(1-\tau))\| \leqslant K$. Integrating over the interval $[1/2, 1]$, we get

$$\theta_1 \leqslant 384K^5.$$

Now we estimate θ_2. Let

$$\varrho_2 = \big\|\tau tAR^{n+1}(\tau t/n)\big\|, \quad \varrho_3 = \big\|(1-\tau)^3(tA)^3E(t(1-\tau))\big\|.$$

Then

$$\theta_2 = \int_0^{1/2} \varrho_2 \varrho_3 \frac{1}{(1-\tau)^3}\, \mathrm{d}\tau.$$

From (1.6) we have $\varrho_2 \leqslant K$ and by Lemma 2.1 we have $\varrho_3 \leqslant 27K^3$. Integrating over the interval $[0, 1/2]$, we get

$$\theta_2 \leqslant 81K^4/2$$

and (note that $K \geqslant 1$)

$$\|(tA)^2 \Delta_0\| \leqslant \frac{C_0}{n} K^5,$$

where C_0 is an absolute positive constant. Then

$$\|\Delta_1\| \leqslant \|D_1\| + \frac{\|(tA)^2 \Delta_0\|}{2n} \leqslant \frac{C_1}{n^2} K^5.$$

Proof of Theorem 1.5. We prove the theorem using induction with respect to k. In the case $k = 0$, we have

$$(I + tA/n)^{-n} = \exp\{-tA\} + D_0, \tag{2.10}$$

where

$$D_0 = \frac{1}{n} \int_0^1 \tau_1 (tA)^2 R^{n+1}(\tau_1 t/n) E(t(1-\tau_1))\, \mathrm{d}\tau_1 = \frac{1}{n} D_{0,1} \tag{2.11}$$

(see (4.2) in Bentkus [3]). The case $k = 1$ was proved in Theorem 1.1 (see also Bentkus [3]).

Assume that (1.13) holds for $0, 1, \dots, k-1$. Let us show that (1.13) holds for k as well. To this end we have to show that

$$D_{k-1} = \frac{a_k}{n^k} + D_k.$$

From (1.14) we have

$$D_{k-1} = \frac{1}{n^k} \sum_{j=1}^{k} D_{k-1,j}, \tag{2.12}$$

where

$$D_{k-1,j} = \mathbf{E}\, \sigma_{k-1,j} (-tA)^{k+j} R^{n+1}(\tau_1 \dots \tau_j t/n) E(t(1 - \tau_1 \dots \tau_j)). \tag{2.13}$$

It is easy to show that the following algebraic identity holds:

$$R^{n+1}(\tau_1\dots\tau_j t/n) = R^n(\tau_1\dots\tau_j t/n) - \frac{1}{n}\tau_1\dots\tau_j t A R^{n+1}(\tau_1\dots\tau_j t/n). \quad (2.14)$$

Also, from (2.10) and (2.11) we obtain

$$R^n(\tau_1\tau_2\dots\tau_j t/n) = \exp\{-\tau_1\tau_2\dots\tau_j tA\} + \frac{1}{n}\int_0^1 \tau_{j+1}(\tau_1\tau_2\dots\tau_j tA)^2 \times R^{n+1}(\tau_1\tau_2\dots\tau_j\tau_{j+1}t/n)E(\tau_1\tau_2\dots\tau_j t(1-\tau_{j+1}))\,\mathrm{d}\tau_{j+1}. \quad (2.15)$$

Substituting expressions (2.14) and (2.15) into (2.13), we get

$$D_{k-1,j} = I_{k-1,j,1} + I_{k-1,j,2} + I_{k-1,j,3},$$

where

$$I_{k-1,j,1} = \frac{1}{n}\mathbf{E}\,\sigma_{k-1,j}\tau_1\dots\tau_j(-tA)^{k+j+1}R^{n+1}(\tau_1\dots\tau_j t/n)E(t(1-\tau_1\dots\tau_j)),$$
$$I_{k-1,j,2} = \mathbf{E}\,\sigma_{k-1,j}(-tA)^{k+j}\exp\{-tA\},$$

and

$$I_{k-1,j,3} = \frac{1}{n}\mathbf{E}\,\sigma_{k-1,j}\tau_{j+1}\tau_1^2\dots\tau_j^2(-tA)^{k+j+2}R^{n+1}(\tau_1\dots\tau_{j+1}t/n)E(t(1-\tau_1\dots\tau_{j+1})).$$

Integrating $I_{k-1,j,2}$ (note that $\mathbf{E}\,\sigma_{k-1,j} = c_{k,j}$), we get

$$I_{k-1,j,2} = c_{k,j}(-tA)^{k+j}\exp\{-tA\},$$

where $c_{k,j}$ is given by (1.10). Taking the sum, we obtain that $a_k = \sum_{j=1}^k I_{k-1,j,2}$.

It is easy to check that

$$\sigma_{k-1,j}\tau_1\dots\tau_j + \sigma_{k-1,j-1}\tau_1^2\dots\tau_{j-1}^2\tau_j = \sigma_{k,j}.$$

From this equality it follows that

$$D_{k,j} = n(I_{k-1,j,1} + I_{k-1,j-1,3}) \quad \text{for } j = 2,\dots,k,$$
$$D_{k,1} = nI_{k-1,1,1}, \quad \text{and} \quad D_{k,k+1} = nI_{k-1,k,3}.$$

Finally, we get

$$D_{k-1} = \frac{1}{n^k}\sum_{j=1}^k (I_{k-1,j,1} + I_{k-1,j,2} + I_{k-1,j,3}) = \frac{a_k}{n^k} + D_k$$

with a_k given by (1.11) and D_k given by (1.14).

Proof of Theorem 1.6. From (1.14) we have

$$\|D_k\| \leqslant \frac{1}{n^{k+1}} \sum_{j=1}^{k+1} \|D_{k,j}\|. \tag{2.16}$$

Each $D_{k,j}$ is the sum of j-tuple integrals of the type

$$J_i = \mathbf{E}\, \tau_1^{k+j} \tau_2^{i_2} \dots \tau_j^{i_j} (-tA)^{k+j+1} R^{n+1}(\tau_1 \dots \tau_j t/n) E(t(1-\tau_1 \dots \tau_j)),$$

where the sum is taken over all integer components of $i = (i_2, i_3, \dots, i_j)$ such that $1 \leqslant i_j \leqslant k - j + 2$ and $i_{n+1} + 2 \leqslant i_n \leqslant k + j - 2(n-1)$ for $n = 2, 3, \dots, j-1$.

We rewrite each J_i as the sum of $j \geqslant 2$ integrals

$$J_i = J_{i,1} + \dots + J_{i,j},$$

where

$$J_{i,m} = \mathbf{E}\, \mathbb{I}_{m,j} \tau_1^{k+j} \tau_2^{i_2} \dots \tau_j^{i_j} (-tA)^{k+j+1} R^{n+1}(\tau_1 \dots \tau_j t/n) E(t(1-\tau_1 \dots \tau_j))$$

for all $m = 1, \dots, j$. Then

$$\|J_i\| \leqslant \|J_{i,1}\| + \dots + \|J_{i,j}\| \leqslant \theta_{i,1} + \dots + \theta_{i,j},$$

where

$$\theta_{i,m} = \mathbf{E}\, \mathbb{I}_{m,j} \big\| \tau_1^{k+j} \tau_2^{i_2} \dots \tau_j^{i_j} (tA)^{k+j+1} R^{n+1}(\tau_1 \dots \tau_j t/n) E(t(1-\tau_1 \dots \tau_j)) \big\|$$

for $m = 1, ..., j$. Write

$$\varrho_{i,m,1} = \|(\tau_1 \dots \tau_j t A)^{i_m} R^{n+1}(\tau_1 \dots \tau_j t/n)\|$$

and

$$\varrho_{i,m,2} = \big\| (t(1-\tau_1 \dots \tau_j)A)^{k+j+1-i_m} E(t(1-\tau_1 \dots \tau_j)) \big\|.$$

Then we have

$$\theta_{i,m} = \mathbf{E}\, \mathbb{I}_{m,j} \varrho_{i,m,1} \varrho_{i,m,2} |g_{i,m}(\tau_1, \dots, \tau_j)|,$$

where

$$g_{i,m}(\tau_1, \dots, \tau_j) = \frac{\tau_1^{k+j-i_m} \tau_2^{i_2-i_m} \dots \tau_{m-1}^{i_{m-1}-i_m}}{\tau_{m+1}^{i_m-i_{m+1}} \dots \tau_j^{i_m-i_j} (1-\tau_1 \dots \tau_j)^{k+j+1-i_m}}.$$

The function $g_{i,m}(\tau_1, \dots, \tau_j)$, $m = 2, \dots, j$, is bounded for $\tau_1, \dots, \tau_{m-1} \in [0, 1]$, $\tau_m \in [0, 1/2]$, and $\tau_{m+1}, \dots, \tau_j \in [1/2, 1]$. By Lemma 2.1 we have

$$\varrho_{i,m,1} \leqslant i_m^{i_m} K^{i_m}, \quad \varrho_{i,m,2} \leqslant (k + j + 1 - i_m)^{k+j+1-i_m} K^{k+j+1-i_m},$$

and, integrating, we get

$$\theta_{i,m} \leqslant C_{i,m,k,j} K^{k+j+1},$$

where $C_{i,m,k,j}$ is a positive constant depending only on k, j, m, and $i_1, \dots, i_j$.

In the case $m = 1$, we have

$$\varrho_{i,1,1} = \|(\tau_1 \dots \tau_j tA)^{k+j} R^{n+1}(\tau_1 \dots \tau_j t/n)\|$$

and

$$\varrho_{i,1,2} = \big\|(1 - \tau_1 \dots \tau_j)tAE(t(1 - \tau_1 \dots \tau_j))\big\|.$$

Then

$$\theta_{i,1} = \mathbf{E}\,\mathbb{I}_{1,j}\varrho_{i,1,1}\varrho_{i,1,2}|g_{i,1}(\tau_1, \dots, \tau_j)|,$$

where

$$g_{i,1}(\tau_1, \dots, \tau_j) = \frac{1}{\tau_2^{k+j-i_2} \dots \tau_j^{k+j-i_j}(1 - \tau_1 \dots \tau_j)}.$$

We note that, for $\tau_2, \dots, \tau_j \in [1/2, 1]$,

$$g_{i,1}(\tau_1, \dots, \tau_j) \leqslant \frac{2^N}{(1 - \tau_1\tau_2)},$$

where $N = i_2 + \dots + i_j - (k + j)(j - 1)$. Integrating over $\tau_3, \dots, \tau_j \in [1/2, 1]$, we have

$$\theta_{i,1} \leqslant 2^{N-j+2}(k + j)^{k+j} K^{k+j+1} \int_0^1 \int_{1/2}^1 \frac{1}{(1 - \tau_1\tau_2)}\, d\tau_1\, d\tau_2.$$

This integral converges, and from this it follows that

$$\theta_{i,1} \leqslant C_{i,1,k,j} K^{k+j+1},$$

where $C_{i,1,k,j}$ is a positive constant depending only on k, j, and $i_2, \dots, i_j$. Taking the sums over all m and $i_2, \dots, i_j$, we obtain

$$\|D_{k,j}\| \leqslant C_{k,j} K^{k+j+1}, \quad j = 2, \dots, k + 1, \tag{2.17}$$

where $C_{k,j}$ is a positive constant depending only on k and j.

It remains to prove the case where $j = 1$. Then

$$D_{k,1} = \mathbf{E}\,\tau_1^{k+1}(-tA)^{k+2}R^{n+1}(\tau_1 t/n)E(t(1-\tau_1)).$$

Write

$$\theta_1 = \mathbf{E}\,\mathbb{I}\{\tau_1 > 1/2\}\big\|\tau_1^{k+1}(-tA)^{k+2}R^{n+1}(\tau_1 t/n)E(t(1-\tau_1))\big\|$$

and

$$\theta_2 = \mathbf{E}\,\mathbb{I}\{\tau_1 \leqslant 1/2\}\big\|\tau_1^{k+1}(-tA)^{k+2}R^{n+1}(\tau_1 t/n)E(t(1-\tau_1))\big\|,$$

where $\mathbb{I}$ is the indicator function. Then we have $\|D_{k,1}\| \leqslant \theta_1 + \theta_2$.

Let $\varrho_{1,1} = \|(\tau_1 tA)^{k+2}R^{n+1}(\tau_1 t/n)\|$. Then

$$\theta_1 = \mathbf{E}\,\mathbb{I}\{\tau_1 > 1/2\}\varrho_{1,1}\|E(t(1-\tau_1))\|/\tau_1.$$

By Lemma 2.1 we have $\|\varrho_{1,1}\| \leqslant (k+2)^{k+2}K^{k+2}$ and $\|E(t(1-\tau_1))\| \leqslant K$. Integrating over the interval [1/2, 1], we get

$$\theta_1 \leqslant C_{1,k,1}K^{k+3}.$$

Now we estimate θ_2. Let $\varrho_{2,1} = \|(\tau_1 tA)^{k+1}R^{n+1}(\tau_1 t/n)\|$ and $\varrho_{2,2} = \|(1-\tau_1)(tA) \times E(t(1-\tau_1))\|$. Then

$$\theta_2 = \mathbf{E}\,\mathbb{I}\{\tau_1 \leqslant 1/2\}\varrho_{2,1}\varrho_{2,2}/(1-\tau_1).$$

By Lemma 2.1 we have $\|\varrho_{2,1}\| \leqslant (k+1)^{k+1}K^{k+1}$ and $\|\varrho_{2,2}\| \leqslant K$. Integrating over the interval [0, 1/2], we obtain

$$\theta_2 \leqslant C_{2,k,1}K^{k+2}.$$

Then (note that $K \geqslant 1$) we have

$$\|D_{k,1}\| \leqslant C_{k,1}K^{k+3}. \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16), we get

$$\|D_k\| \leqslant \frac{C_k}{n^{k+1}}K^{2k+2}.$$

Proof of Theorem 1.7. We prove the theorem using induction with respect to k. In the case $k = 0$, we have

$$\exp\{-tA\} = (I + tA/n)^{-n} + \Delta_0,$$

where $\Delta_0 = -D_0$. The case where $k = 1$ was proved in Theorem 1.3. Assume that (1.20) and (1.21) hold for $0, 1, \dots, k-1$. Let us prove that (1.20) and (1.21) hold for k as well. From Theorem 1.5 we have

$$\exp\{-tA\} = (I + tA/n)^{-n} - \frac{a_1}{n} - \dots - \frac{a_k}{n^k} - D_k, \tag{2.19}$$

where

$$\frac{a_m}{n^m} = \frac{d_m}{n^m} \exp\{-tA\} \tag{2.20}$$

for $m = 1, \dots, k$. Substituting expression (1.20) (with expansion length $k - m$) into expression (2.20), we get

$$\frac{a_m}{n^m} = \frac{d_m}{n^m}\Big((I + tA/n)^{-n} + \frac{b_1}{n} + \dots + \frac{b_{k-m}}{n^{k-m}} + \Delta_{k-m}\Big) \tag{2.21}$$

for $m = 1, \dots, k$.

Substituting (2.21) into (2.19), then collecting terms with the same powers of n and moving terms containing the remainder terms into Δ_k, we obtain expressions (1.20) and (1.21).

Proof of Theorem 1.8. From (1.21) we obtain another expression for Δ_s

$$\Delta_s = -D_s - \sum_{k=0}^{s-1} h_{s-k} \frac{D_k}{n^{s-k}},$$

where h_m are given by (1.19). From Theorem 1.6 we have

$$\|D_s\| \leqslant \frac{c_s}{n^{s+1}} K^{2s+2}, \quad s = 1, 2, \dots,$$

where c_s is a positive constant depending only on s. For $s = 0$, we have $\|D_0\| \leqslant 4K^3/n$ by Theorem 1.3 in [4].

Then we note that h_{s-k} are linear combinations of $(tA)^{s-k+1}, \dots, (tA)^{2s-2k}$ with some numerical coefficients depending only on k and s. So, in order to prove the theorem we have to show that

$$\|(tA)^p D_k\| \leqslant \frac{C_{p,k}}{n^{k+1}} K^{2s+3}, \quad k = 0, 1, \dots, s-1,$$

where $p = s - k + 1, \dots, 2s - 2k$ and $C_{p,k}$ is a positive constant depending only on p and k. The proof is similar to the proofs of Theorems 1.6 and 1.4. In the case $k = 1, \dots, s-1$, we obtain $\|(tA)^p D_k\| \leqslant \frac{C_{p,k}}{n^{k+1}} K^{2s+2}$ and, in the case $k = 0$, we have $\|(tA)^p D_0\| \leqslant \frac{C_{p,0}}{n} K^{2s+3}$. We omit the proof here.

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REZIUMĖ

M. Vilkienė. Pusgrupių Eulerio aproksimacijos ir asimptotiniai skleidiniai

Straipsnyje gauti pusgrupių Eulerio aproksimacijų asimptotiniai skleidiniai ir liekanų optimalūs įverčiai. Buvo naudojamas metodas, pateiktas Bentkaus [3] straipsnyje.